1. (15 points total; 5 points each part)
   Inside $G := GL_n(\mathbb{Z}/17\mathbb{Z})$, consider the subset $H$ consisting of those matrices $A$ having $\det A \in \{\pm 1, \pm 4\}$.
   (a) Show $H$ is a normal subgroup of $G$.
   (b) Identify the group $G/H$ up to isomorphism.
   (c) Compute $|H|$ as a function of $n$.

2. (15 points total) Let $H, K$ be two subgroups and $g$ any element, in a finite group $G$.
   (a) (5 points) Prove
      $$|g^{-1}Hg \cap K| = |H \cap gKg^{-1}|$$
   (b) (10 points) Prove
      $$|HgK| = \frac{|H||K|}{|g^{-1}Hg \cap K|} \quad \left(= \frac{|H||K|}{|H \cap gKg^{-1}|} \text{ by part (a)} \right).$$

3. (30 point total; 5 points each part) Prove or disprove:
   (i) $D_{12} \cong \mathbb{Z}/2\mathbb{Z} \times D_6$ as groups.
   (ii) $D_{16} \cong \mathbb{Z}/2\mathbb{Z} \times D_8$ as groups.
   (iii) $\mathbb{R}^+ \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{R}^+$ as groups.
   (iv) For $g, h$ in a finite group, the order of the product $gh$ divides the product of the orders of $g$ and $h$.
   (v) This group is simple:
      $$SL_7(\mathbb{Z}/120\mathbb{Z}) := \{ A \in (\mathbb{Z}/120\mathbb{Z})^{7 \times 7} : \det A = 1 \}.$$
   (vi) This group is simple:
      $$SL_7(\mathbb{Z}/121\mathbb{Z}) := \{ A \in (\mathbb{Z}/121\mathbb{Z})^{7 \times 7} : \det A = 1 \}.$$

4. (10 points) Let let $H$ be a normal subgroup of a finite $G$, and assume that $|H| = p$ is the smallest prime number dividing $|G|$. Show that $H \leq Z(G)$, the center of $G$.
   (Hint: Consider the action of $G$ on $H$ via conjugation, that is, $g$ sends $h$ to $ghg^{-1}$. Also note that the identity $e$ in $H$ is fixed under this action by every $g$ in $G$.)
5. (15 points total) Let $G$ be a finite group $G$ acting on a finite set $A$.
(a) (5 points) Count in two ways the cardinality $|\{(g, a) \in G \times A : g(a) = a\}|$ to prove
$$|G| \cdot |\{G\text{-orbits } O \text{ on } A\}| = \sum_{g \in G} |\{a \in A : g(a) = a\}|.$$ 
(b) (10 points) Use (a) to compute the number of orbits of the dihedral group $D_{10}$ of cardinality 10 on the set $A$ of cardinality $|A| = k^{10}$ consisting of all colorings with $k$ colors of these dihedrally symmetric points:

For example with $k = 2$ colors black and white, among the $2^{10}$ colorings, these three lie in the same orbit since the first two differ by reflecting across a vertical line, and the last two differ by $\frac{2\pi}{5}$ rotation.

(Hint: your answer should end up being a polynomial function of $k$.)

6. (15 points total) Let $G$ be a group.
(a) (10 points) For $H \leq G$ a subgroup with finite index $n = [G : H]$, show that $H$ contains a subgroup $N$ which is normal in $G$, that is, $N \triangleleft G$, and has index $[G : N]$ dividing $n!$.
(Hint: let $G$ act on $G/H$ by left-translation, that is, $g$ sends the coset $aH$ to the coset $gaH$)
(b) (5 points) For subgroups $H_1, H_2$ of $G$, both of finite index in $G$, show $H_1 \cap H_2$ also has finite index in $G$. 