Instructions: This is an open book, library, notes, web, take-home exam, but you are not to collaborate. The instructor is the only human source you are allowed to consult. Indicate outside sources used.

1. (20 points total; 5 points each part) Prove or disprove:

(a) (5 points) If there exists an element of order \( n \) in a quotient group \( G/N \) of a finite group \( G \), then there will also exist an element of order \( n \) in \( G \).

(b) (5 points) A vector space \( V \) over a field can be isomorphic to one of its own proper subspaces \( U \subsetneq V \).

(c) (5 points) A subset \( U \subset V \) of an \( \mathbb{R} \)-vector space \( V \) is an \( \mathbb{R} \)-subspace if and only if \( (U,+):=(V,+). \)

(d) (5 points) If \( \mathbb{F}_p \) denotes the finite field \( \mathbb{Z}/p\mathbb{Z} \) for a prime \( p \), then a subset \( U \subset V \) of an \( \mathbb{F}_p \)-vector space \( V \) is an \( \mathbb{F}_p \)-subspace if and only if \( (U,+):=(V,+). \)

2. (15 points total; 5 points each part)

(a) (5 points) Given an exact sequence of finite-dimensional vector spaces over a field \( \mathbb{F} \)

\[
0 \longrightarrow V_0 \overset{f_0}{\longrightarrow} V_1 \overset{f_1}{\longrightarrow} V_2 \overset{f_2}{\longrightarrow} \cdot \cdot \cdot \overset{f_{\ell-1}}{\longrightarrow} V_{\ell-1} \overset{f_{\ell-1}}{\longrightarrow} V_{\ell} \overset{f_{\ell}}{\longrightarrow} V_0 \longrightarrow 0
\]

prove that

\[
dim_{\mathbb{F}} V_0 - dim_{\mathbb{F}} V_1 + \cdot \cdot \cdot + (-1)^\ell \dim_{\mathbb{F}} V_\ell = 0
\]

(b) (5 points) Given a short exact sequence of finite groups \( 1 \longrightarrow A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C \longrightarrow 1 \), prove \( |B|=|A||C| \).

(c) (5 points) Given an exact sequence of finite groups

\[
1 \longrightarrow G_\ell \overset{f_\ell}{\longrightarrow} G_{\ell-1} \overset{f_{\ell-1}}{\longrightarrow} \cdot \cdot \cdot \overset{f_2}{\longrightarrow} G_2 \overset{f_2}{\longrightarrow} G_1 \overset{f_1}{\longrightarrow} G_0 \longrightarrow 1
\]

prove that

\[
|G_1||G_3||G_5|\cdots = |G_0||G_2||G_4|\cdots
\]

3. (10 points; 5 points each part) For a field \( \mathbb{F} \) and a linear operator \( \varphi : V \rightarrow V \) on a finite-dimensional \( \mathbb{F} \)-vector space \( V \), define the trace \( \text{Tr}_V(\varphi) \) as follows: make a choice of an ordered basis \( (v_1, \ldots, v_n) \) for \( V \) in which to express \( \varphi \) by an \( n \times n \) matrix \( A = (a_{ij})_{i,j=1,2,\ldots,n} \), and then set

\[
\text{Tr}(\varphi) = \text{Tr}(A) := \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}.
\]

(a) Show \( \text{Tr}(\varphi) \) is well-defined, that is, independent of the choice of the ordered basis \( (v_1, \ldots, v_n) \) for \( V \).

(b) Given an \( \mathbb{F} \)-linear subspace \( W \subseteq V \) with \( \varphi(W) \subset W \), show that the restriction map \( \varphi_W \) on \( W \) and the induced map \( \varphi_{V/W} \) on the quotient \( V/W \) satisfy

\[
\text{Tr}(\varphi) = \text{Tr}(\varphi_W) + \text{Tr}(\varphi_{V/W}).
\]
4. (20 points total; 5 points each) Let $\mathbb{F}_q$ be a finite field with $q$ elements, e.g., if $q$ is prime then $\mathbb{F}_q = \mathbb{Z}/q\mathbb{Z}$. Let $V = \mathbb{F}_q^n$ considered as an $n$-dimensional vector space over $\mathbb{F}_q$. Fixing $k$ in the range $0 \leq k \leq n$, let $G(k, V) = G(k, \mathbb{F}_q^n)$ denote the set of all $k$-dimensional $\mathbb{F}_q$-linear subspaces of $V$.

(a) (5 points) Show that when then group $GL(V) = GL_n(\mathbb{F}_q)$ acts on $V$, it takes an $\mathbb{F}_q$-subspace to another $\mathbb{F}_q$-subspace, preserving dimension, so that it acts on the set $G(k, V)$.

(b) (5 points) Show that this action on $G(k, V)$ is transitive.

(c) (5 points) Let $P_k$ be the subgroup $P$ of $GL(V)$ which is the stabilizer of the particular $k$-dimensional subspace of $V = \mathbb{F}_q^n$ spanned by the first $k$ standard basis vectors $\{e_1, \ldots, e_k\}$. Writing elements of $GL(V)$ as $n \times n$ matrices in block form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A \in \mathbb{F}^k \times k, B \in \mathbb{F}^{k \times (n-k)}, C \in \mathbb{F}^{(n-k) \times k}, D \in \mathbb{F}^{(n-k) \times (n-k)}$, identify the elements of $P_k$ by saying what are the conditions on $A, B, C, D$ for this matrix to lie in $P_k$.

(d) (5 points) Find the cardinality of $G(k, V)$, as a function of $k, n$ and $q$.

5. (15 points) For two simple groups $G_1, G_2$ and a normal subgroup $N \trianglelefteq G_1 \times G_2$, show that either

- $N = \{e\}$, or
- $N = G_1 \times G_2$, or
- $N$ is isomorphic to one of $G_1$ or $G_2$.

6. (20 points) Let $G$ be a finite group,

- with $|G| = pqr$ for primes $p < q < r$,
- with $q$ not dividing $r - 1$, and
- containing a normal subgroup $N \trianglelefteq G$ having $|N| = p$.

Prove that $G$ is cyclic.