1. Recall from lecture that a (combinatorial) projective geometry $(P, L)$ was defined by 4 axioms PG1, PG2, PG3, PG4, and that its dimension was defined to be 1 less than the rank of its lattice of flats.

(a) Show that a projective plane, that is a projective geometry of dimension 2, has the following equivalent axiomatization:

PP1. Every two distinct points lie on a unique line.
PP2. Every two distinct lines have a unique point in common.
PP3. Every line contains at least 3 points.
PP4. There exist 3 non-collinear points.

(b) Show that in a finite projective plane, all lines have the same number of points, and call this number $q + 1$.

(c) Show that each point lies on $q + 1$ lines, and $|P| = |L| = q^2 + q + 1$.

2. Given $G = (V, E)$ be a bipartite graph with bipartition $V = A \sqcup B$, let $\mathbb{F} \subseteq \mathbb{K}$ be a field extension in which there exist elements $\{c_{a,b} : \{a,b\} \in E\}$ of $\mathbb{K}$ which are algebraically independent over $\mathbb{F}$, i.e. there are no polynomials in variables $x_{a,b}$ with coefficients in $\mathbb{F}$ which vanish when one plugs in $x_{a,b} = c_{a,b}$. Define vectors $\{v_a : a \in A\}$ in $\mathbb{K}^B$ by

$$v_a := \sum_{b \in B : \{a,b\} \in E} c_{a,b} \epsilon_b$$

where $\epsilon_b$ is a standard basis vector in $\mathbb{K}^B$.

Show that a subset $A' \subset A$ can be matched along edges in $E$ to distinct elements of $B$ if and only if the subset $\{v_a : a \in A'\}$ is $\mathbb{K}$-linearly independent. In other words, partial matchings of $A$ into $B$ form the independent sets of a matroid that is representable over $\mathbb{K}$. Such matroids are called transversal matroids.

(Hint: Consider the $|A'| \times |B|$ matrix having $\{v_a : a \in A'\}$ as its columns. Under what circumstances does the square submatrix with rows indexed by some subset $B' \subseteq B$ with $|B'| = |A'|$ have non-zero determinant? What does it mean for there to exist such a $B'$?)
3. Show that the following axiom systems are equivalent to the axiomizations of finite matroids given in lecture (by an exchange closure and/or independent sets):

(a) (Basis axioms) A family $B \subseteq 2^E$ forms the set of bases of a matroid $M$ on the finite set $E$ if

B1. All sets $B$ in $B$ have the same cardinality (called the rank of $M$).

B2. Given $B, B' \in B$, and $e \in B$, there exists some $e' \in B'$ with $B - \{e\} \cup \{e'\} \in B$.

(Hint: The bases are supposed to model the maximal independent sets.)

(b) (Circuit axioms) A family $C \subseteq 2^E$ forms the set of circuits of a matroid $M$ on the finite set $E$ if

C1. The sets in $C$ form an antichain under inclusion.

C2. Given $C, C' \in C$, with $C \neq C'$ and $e \in C \cap C'$, there exists some $C'' \in C$ with $C'' \subseteq C \cup C' - \{e\}$.

(Hint: The circuits are supposed to model the minimal dependent sets.)

4. Given a graph $G = (V, E)$ with loops and multiple edges allowed, show that for any field $\mathbb{F}$, the matroid associated with the vector configuration in $\mathbb{F}^V$ defined by

$$\{v_e = \epsilon_i - \epsilon_j : e = \{i, j\} \in E(G)\}$$

satisfies the following.

(a) the closure $\bar{A}$ of a subset $A \subset E$ consists of all edges $e \in E$ for which there exists a path from the endpoints of $e$ in $G$ using only edges from $A$.

(b) its independent sets are the subforests of $G$, that is, the subsets of edges containing no cycles.

(c) its bases are the spanning subforests of $G$, that is, the subforests which consist of one spanning tree in each connected component of $G$ (here spanning means connecting all vertices).

(d) its circuits are the simple cycles of $G$, that is, sequences of edges $e_1, \ldots, e_k$ in $E$ with the property that there are $k$ distinct vertices $v_1, \ldots, v_k$ for which $e_i = \{v_i, v_{i+1}\}$ (and the subscripts on $v_j$’s are taken modulo $k$).
5. Let $M$ be a matroid on $E$, and choose a linear order $e_1, e_2, \ldots, e_n$ for the elements of $E$. Given a circuit $C$ of $M$, with minimum element $c$ in this order, call $C - \{c\}$ a broken circuit. Say that a subset $A \subseteq E$ is NBC if it contains no broken circuits $C - \{c\}$.

(a) Show that for any flat $F$ in the geometric lattice of flats $L(M)$, one has
$$\mu_{L(M)}(\emptyset, F) = (-1)^{r(F)}|\{ \text{NBC sets } A \subseteq E : \bar{A} = F \}|.$$  

(Hint: Show the right-hand side satisfies the proper identity that defines $\mu_{L(M)}(\emptyset, F)$, via a sign-reversing involution).

(b) The linear ordering on $E$ gives an ordering on the join-irreducibles (atoms) of the upper-semimodular lattice $L(M)$, and hence induces an $R$-labelling of $L(M)$ as explained in lecture. Show why the Möbius function calculation this $R$-labelling provides agrees with part (a), by exhibiting a bijection between NBC bases for $M$ and maximal chains in $L(M)$ whose label set is decreasing.

6. (a) Explain why the partition lattice $\Pi_n$ is the lattice of flats for the matroid associated with the complete graph $K_n$ on $n$ vertices.

(b) Indexing the atoms $E$ of $\Pi_n$ by pairs $\{i, j\}$ (i.e. edges of $K_n$), pick any linear ordering of $E$ in which $\min \{i, j\} > \min \{i', j'\}$ implies that $\{i, j\}$ comes before $\{i', j'\}$. Show that for every triple $i < j < k$, the pair of edges $\{i, j\}, \{i, k\}$ forms a broken circuit. Show furthermore that every broken circuit contains at least one such pair.

(c) Use part (b) and Problem 8(a) to prove that
$$\mu_{\Pi_n}(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!.$$  

7. Let $M$ be a matroid on ground set $E$, and $c : E \to \mathbb{R}$ any assignment of costs $c(e) \in \mathbb{R}$ to each $e$ in $E$. Show that the following “greedy” algorithm for finding a basis $B$ of $M$ with minimum total cost $\sum_{e \in B} c(e)$ always works, that is, it will always terminate with a basis $B$ for $M$, and $B$ achieves the minimum:

Start at stage 0 with $I_0 = \emptyset$, an independent set. At stage $j$, given the independent set $I_{j-1}$, choose an edge $e_j \in E$ with lowest cost among those such that $I_j := I_{j-1} \cup \{e_j\}$ remains independent. Repeat.

When $M = M(G)$ is a graphic matroid, this is called Kruskal’s algorithm for finding a minimum cost spanning tree.
8. (a) Let \( G \) be a planar graph with a chosen planar embedding, and \( G^\perp \) its planar dual with respect to this embedding. Show that \( M(G)^\perp = M(G^\perp) \).

(b) For any orientation \( \omega \) of the edges \( E(G) \), let \( \omega^\perp \) be the induced orientation of the dual edges \( E(G^\perp) \) defined by the right-hand rule: if you place the origin at the crossing of some pair of dual edges \( e, e^\perp \) in \( E(G), E(G^\perp) \) respectively, then the pair of tangent vectors to those edges pointing in the directions of the edges should form a right-handed coordinate system in the plane (like the usual positive \( x \)-axis, positive \( y \)-axis). Show that \( \omega \) is acyclic if and only if \( \omega^\perp \) is totally cyclic.

9. Prove the following Tutte polynomial evaluation for graphic matroids: if \( G \) is a graph with \( c(G) \) connected components, and \( p, q \) are positive integers, then
\[
T_{M(G)}(1-p, 1-q) = (-p)^{-c(G)}(-1)^{|V(G)|} \sum_{(x,y)} (-1)^{|\text{supp}(y)|}
\]
where \((x,y)\) runs over all pairs in which
- \( x \) is a vertex \( p \)-coloring,
- \( y \) is a \( \mathbb{Z}/q\mathbb{Z} \)-valued flow, and
- for every edge \( e \in E(G) \), one has \( y_e \neq 0 \) if and only if \( x \) colors \( e \) improperly, i.e. \( x_v = x_{v'} \) where \( e = \{v, v'\} \).
Here \(|\text{supp}(y)|\) is the number of edges \( e \) with \( y_e \neq 0 \) or equivalently, the number of edges that are improperly colored by \( x \).

10. (Character theory warm-up) Given two finite groups \( G, G' \) and complex representations
\[
\rho : G \to GL(V) \\
\rho' : G' \to GL(V')
\]
define a new representation
\[
\rho \otimes \rho' : G \times G' \to GL(V \otimes V')
\]
by
\[
(\rho \otimes \rho')(g, g')(v \otimes v') = \rho(g)v \otimes \rho'(g')v'.
\]
(a) Show \( \chi_{\rho \otimes \rho'}(g, g') = \chi_{\rho}(g) \cdot \chi_{\rho'}(g') \).
(b) Show that \( \rho \otimes \rho' \) is irreducible for \( G \times G' \) if and only if both \( \rho, \rho' \) are irreducibles for \( G, G' \).
(c) If \( \{\rho_i\}_{i \in I}, \{\rho'_i\}_{i' \in I'} \) are complete sets of representatives of the (equivalence classes of) irreducible representations of \( G, G' \), respectively,
show that \( \{\rho_i \otimes \rho_{i'}\}_{(i,i') \in I \times I'} \) gives a complete set of representatives for the irreducibles of \( G \times G' \).

11. If \( G \) is a finite group acting on \([n]\), say that the action is

- **transitive** if there is only one \( G \)-orbit on \([n]\),
- **doubly transitive** if it is transitive on ordered pairs, that is, for every pair \( i \neq j \) and \( i' \neq j' \) in \([n]\) there exists \( g \in G \) with \( g(i) = i' \), \( g(j) = j' \).

Let \( \chi \) be the permutation representation/character associated with the \( G \)-action.

(a) Show that the action is transitive if and only if \( \langle \chi, \chi_{\text{trivial}} \rangle = 1 \).

(b) Show that the action is doubly transitive if and only if \( \chi - \chi_{\text{trivial}} \) is irreducible.