1. Say that a sequence \( a_0, a_1, \ldots, a_n \) of non-negative real numbers is \textit{unimodal} if there exists an index \( k \) for which

\[
a_0 \leq a_1 \leq \cdots \leq a_k \geq \cdots \geq a_{n-1} \geq a_n.
\]

Say that it is \textit{log-concave} if for each \( k \in \{2, 3, \ldots, n-1\} \) one has

\[
a_k^2 \geq a_{k-1}a_{k+1}.
\]

(a) Assuming \( a_k > 0 \) for all \( k \), show that log-concave implies unimodal, but not conversely.

(b) Show that whenever the \( a_k \) are non-negative, any real root of the polynomial

\[
A(x) := \sum_{k=0}^{n} a_k x^k
\]

must be non-positive.

(c) Show that if \( A(x) \) has \textit{only real roots} and \textit{positive} coefficients \( a_k \), then this coefficient sequence is log-concave, and hence unimodal.

(Hint: there is more than one proof of this. One way factors \( A(x) \) into linear factors according to its roots and proceeds combinatorially. Another way applies Rolle’s Theorem to deduce that the quadratic polynomial

\[
\frac{d^{n-k-1}}{dx^{n-k-1}} \left( x^{n-k+1} \cdot \left[ \frac{d^{k-1}}{dx^{k-1}} A(x) \right]_{x \to x^{-1}} \right)
\]

also has only real roots, so one can look at its discriminant, giving an even stronger inequality than log-concavity).

(d) Recall that the signless Stirling number of the 1st kind \( c(n, k) \) is the number of permutations in \( S_n \) with \( k \) cycles, and the Stirling number of the 2nd kind \( S(n, k) \) is the number of partitions of the set \([n]\) into \( k \) blocks. Show that both sequences \( (c(n, k))_{k=1}^{n}, (S(n, k))_{k=1}^{n} \) are log-concave, and hence unimodal.

(Hint: \( \sum_k s(n, k)x^k \) has a simple factored expression that shows it has real roots. For \( A_n(x) = \sum_k S(n, k)x^k \), show that \( A_n(x) = xA_{n-1}(x) + xA'_{n-1}(x) \) and use this to give a proof of real-rootedness by induction on \( n \) involving the \textit{interlacing} of the roots of \( A_n(x), A_{n-1}(x) \) (that is, between every pair of roots of \( A_{n-1}(x) \) there is one for \( A_{n}(x) \), and then two more on the extreme right and extreme left).

2. Define two infinite upper-triangular matrices \( E, H \) by

\[
E_{i,j} = (-1)^{j-i} e_{j-i}
\]

\[
H_{i,j} = h_{j-i}
\]
where the elementary and complete symmetric functions $e_r, h_r$ are both taken to be 1 for $r = 0$, and 0 for $r < 0$.

(a) Explain why the matrices $E, H$ are inverse to each other.

(b) Specialize the $e_i(x_1, x_2, \ldots), h_i(x_1, x_2, \ldots)$ to $x_1 = x_2 = \cdots = x_n = 1$ and $x_i = 0$ for $i > n$. What do $e_k, h_k$ specialize to, and what identity results from $E, H$ being inverse to each other?

(c) Generalize part (b) by answering the same questions for the specialization $x_i = q^{i-1}$ for $i \leq n$, $x_i = 0$ for $i > n$.

(d) Answer the same questions for the specialization $x_i = i - 1$ for $i \leq n$, $x_i = 0$ for $i > n$.

(Hint for part (d): Stirling numbers are relevant.)

3. Let $b = (b_1, b_2, \ldots, b_{mn+1})$ be a sequence of length $mn + 1$ in the alphabet $\{1, 2, \ldots\}$. Show that $b$ contains either a weakly increasing subsequence of length $m+1$, or a strictly decreasing subsequence of length $n+1$.

4. Write down explicitly the entire character table for the symmetric group $\mathfrak{S}_5$ using the Murnaghan-Nakayama rule.

5. Show that the irreducible character $\chi^{\lambda}$ of the symmetric group $\mathfrak{S}_n$ has $\chi^{\lambda}(w) = 0$ whenever the side-length of $\lambda$’s Durfee square (the largest square contained inside the Ferrers diagram of $\lambda$) is larger than the number of cycles of $w$.

6. Prove that the character table for $\mathfrak{S}_n$ has determinant

$$\pm \prod_{\lambda \vdash n} \prod_{i=1}^{\ell(\lambda)} \lambda_i.$$