Math 1271. Lecture 060 (V. Reiner)  Midterm Exam II  
Thursday, October 29, 2009

This is a 50 minute exam. No books, notes, calculators, cell phones or other electronic devices are allowed. There are a total of 100 points. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. Do all of your calculations on this test paper.

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Problem 1. (24 points total) Compute the following derivatives. Do not worry about simplifying your answer after the derivative has been computed.

a. (8 points) \( \frac{d}{dx} \left( \sqrt[3]{x^5} \cdot e^{x^2} \right) \)

b. (8 points) \( \frac{d}{dx} (\arcsin(x)^{2x}) \)
c. (8 points total; 2 points each) Assuming that
\[ f(10) = 1, \ g(10) = 2, \ f'(10) = 3, \ g'(10) = 4, \ f'(2) = 5, \]
compute \( h'(10) \) for each of these functions \( h(x) \):
\[ h(x) = f(x) + 100g(x) \]
\[ h(x) = f(x) \cdot g(x) \]
\[ h(x) = \frac{f(x)}{g(x)} \]
\[ h(x) = f(g(x)) \]
Problem 2. (25 points) The sides of a cube are growing in such a way that it always maintains a cubical shape (so all sidelengths are equal), but its volume is increasing by 2 cubic centimeters per second. At what rate is its sidelength growing when the volume is 1000 cubic centimeters?
Problem 3. (26 points) Consider the curve in the plane defined by the equation

$$(x - 1)^4 + y^4 = 16.$$ 

a. (10 points) Write an expression for the slope $\frac{dy}{dx}$ of the tangent line to a point $(x, y)$ on this curve, as a function of $x$ and $y$.

b. (8 points) Find the $(x, y)$ coordinates of both points on this curve where the tangent line to the curve is horizontal.
c. (8 points) Find the equation of the tangent line to the curve at the point \((x, y) = (1 + \sqrt[3]{8}, \sqrt[4]{8})\).
Problem 4. (25 points) Carbon-16 is radioactive, and decays at a rate proportional to how much is present, with a half-life of about 5730 years. How many kilograms will remain after a 1 kilogram sample has been allowed to decay for 1000 years?

Note: Since you are not allowed to use a calculator, do not provide an answer in decimals; rather you may leave functions like exponentials and logarithms in your answer.
Brief solutions.

1. a. (8 points)

\[
\frac{d}{dx} \left( \sqrt[3]{x^5} \cdot e^{x^2} \right) = \frac{d}{dx} \left( x^{\frac{5}{3}} \right) \cdot e^{x^2} + x^{\frac{5}{3}} \cdot \frac{d}{dx} e^{x^2} = \frac{5}{3} x^{\frac{2}{3}} \cdot e^{x^2} + x^{\frac{5}{3}} \cdot e^{x^2} \cdot 2x.
\]

b. (8 points) In computing \( \frac{d}{dx} (\arcsin(x)^{2x}) \), there were two ways that people interpreted \( \arcsin(x)^{2x} \), and I accepted either answer, if they differentiated it correctly.

The first one is the way I intended:

\[
y = (\arcsin(x))^{2x}
\]

\[
\ln(y) = \ln(\arcsin(x))^{2x} = 2x \ln(\arcsin(x))
\]

\[
\frac{d}{dx} \ln(y) = \frac{d}{dx} (2x \ln(\arcsin(x)))
\]

\[
\frac{1}{y} \frac{dy}{dx} = 2 \ln(\arcsin(x)) + 2x \cdot \frac{1}{\arcsin(x)} \frac{1}{\sqrt{1-x^2}}
\]

\[
\frac{dy}{dx} = y \left( 2 \ln(\arcsin(x)) + 2x \cdot \frac{1}{\arcsin(x)} \frac{1}{\sqrt{1-x^2}} \right)
\]

\[
= (\arcsin(x))^{2x} \left( 2 \ln(\arcsin(x)) + 2x \cdot \frac{1}{\arcsin(x)} \frac{1}{\sqrt{1-x^2}} \right)
\]

The second way went like this:

\[
\frac{d}{dx} \arcsin(x^{2x}) = \frac{1}{\sqrt{1-(x^{2x})^2}} \cdot \frac{d}{dx} (x^{2x})
\]

and one uses logarithmic differentiation to compute that for \( y = x^{2x} \), one has

\[
\ln(y) = \ln(x^{2x}) = 2x \ln(x)
\]

\[
\frac{d}{dx} \ln(y) = \frac{d}{dx} (2x \ln(x))
\]

\[
\frac{1}{y} \frac{dy}{dx} = 2 \ln(x) + 2x \cdot \frac{1}{x} = 2 \ln(x) + 2
\]

\[
\frac{dy}{dx} = y \left( 2 \ln(x) + 2 \right)
\]

\[
\frac{dy}{dx} = 2x^{2x} (\ln(x) + 1)
\]

so plugging into the above calculation gives

\[
\frac{d}{dx} \arcsin(x^{2x}) = \frac{1}{\sqrt{1-(x^{2x})^2}} \cdot 2x^{2x} (\ln(x) + 1).
\]
c. (8 points total; 2 points each) Assuming that
\[ f(10) = 1, \ g(10) = 2, \ f'(10) = 3, \ g'(10) = 4, \ f'(2) = 5, \]
compute \( h'(10) \) for each of these functions \( h(x) \):
\[
\begin{align*}
  h(x) &= f(x) + 100g(x) \\
  h'(10) &= f'(10) + 100g'(10) = 3 + 100 \cdot 4 = 403 \\
  h(x) &= f(x) \cdot g(x) \\
  h'(10) &= f'(10)g(10) + f(10)g'(10) = 3 \cdot 2 + 1 \cdot 4 = 10 \\
  h(x) &= \frac{f(x)}{g(x)} \\
  h'(10) &= \frac{g(10)f'(10) - f(10)g'(10)}{g(10)^2} = \frac{2 \cdot 3 - 1 \cdot 4}{2^2} = \frac{1}{2} \\
  h(x) &= f(g(x)) \\
  h'(10) &= f'(g(10))g'(10) = f'(2)g'(10) = 5 \cdot 4 = 20
\end{align*}
\]

2. (25 points) The sides of a cube are growing in such a way that it always maintains a cubical shape (so all sidelengths are equal), but its volume is increasing by 2 cubic centimeters per second. At what rate is its sidelength growing when the volume is 1000 cubic centimeters?

Letting \( V(t) \) be the volume of the cube, and \( s(t) \) its sidelength, one is given that \( \frac{dV}{dt} = 2 \, \text{cm}^3/\text{sec} \) and they are asking for \( \frac{ds}{dt} \) at the moment when \( V = 1000 \, \text{cm}^3 \) (so it must be that \( s = \sqrt[3]{1000} \, \text{cm} = 10 \, \text{cm} \) at that moment).

Starting with \( V = s^3 \) and implicitly taking \( \frac{dV}{dt} \) gives
\[
\frac{dV}{dt} = 3s^2 \frac{ds}{dt} \text{ at all times, so}
\]
\[
2 \, \text{cm}^3/\text{sec} = 3(10 \, \text{cm})^2 \frac{ds}{dt} \text{ at that moment, and}
\]
\[
\frac{ds}{dt} = \frac{2 \, \text{cm}^3/\text{sec}}{3(10 \, \text{cm})^2} = \frac{1}{150} \, \text{cm/sec}
\]
3. (26 points total) Consider the curve in the plane defined by the equation 

\[(x - 1)^4 + y^4 = 16.\]

a. (10 points) Write an expression for the slope \(\frac{dy}{dx}\) of the tangent line to a point \((x, y)\) on this curve, as a function of \(x\) and \(y\).

Taking \(\frac{d}{dx}\) of the above equation implicitly gives

\[4(x - 1)^3 + 4y^3 \frac{dy}{dx} = 0\]

and solving for \(\frac{dy}{dx}\) gives

\[\frac{dy}{dx} = -\frac{(x - 1)^3}{y^3}\]

b. (8 points) Find the \((x, y)\) coordinates of both points on this curve where the tangent line to the curve is horizontal.

A horizontal tangent means that

\[0 = \frac{dy}{dx} = -\frac{(x - 1)^3}{y^3}\]

which forces the \((x - 1)^3 = 0\), so that \(x = 1\). The points on the curve having \(x = 1\) are found by plugging \(x = 1\) back into the equation, and solving for their \(y\) coordinates:

\[(1 - 1)^4 + y^4 = 16\]

so \(y^4 = 16\) and \(y = \sqrt[4]{16} = \pm 2\). Thus the two points are \((x, y) = (1, 2)\) and \((1, -2)\).

c. (8 points) Find the equation of the tangent line to the curve at the point \((x, y) = (1 + \sqrt{8}, \sqrt{8})\).

The slope will be

\[\left[\frac{dy}{dx}\right]_{(x,y)=(1+\sqrt{8},\sqrt{8})} = -\frac{(1 + \sqrt{8} - 1)^3}{(\sqrt{8})^3} = -1.\]

Hence the equation of the line is given by

\[\frac{y - \sqrt{8}}{x - (1 + \sqrt{8})} = -1\]

or

\[y - \sqrt{8} = -(x - (1 + \sqrt{8})).\]

4. (25 points) Carbon-16 is radioactive, and decays at a rate proportional to how much is present, with a half-life of about 5730 years. How
many kilograms will remain after a 1 kilogram sample has been allowed to decay for 1000 years?

Since we know the half-life $t_{\text{half}} = 5730\text{ yrs}$, we can get the rate constant $k$ for the decay $\frac{dy}{dt} = ky$ from the all-important solution equation

$$y(t) = y_0 e^{kt}$$

by noting that

$$\frac{1}{2} y_0 = y(t_{\text{half}}) = y_0 e^{k \cdot t_{\text{half}}}$$

$$\frac{1}{2} = e^{k \cdot 5730}$$

$$\ln\left(\frac{1}{2}\right) = 5730k$$

$$k = \frac{-\ln(2)}{5730}$$

Since we are also given $y_0 = 1\text{ kg}$, we now know

$$y(t) = (1\text{ kg}) \cdot e^{\frac{-\ln(2) \cdot t}{5730}}.$$ 

Hence at time $t = 1000$ years, one has

$$y(1000) = (1\text{ kg}) \cdot e^{\frac{-\ln(2) \cdot 1000}{5730}} = (1\text{ kg}) \cdot (e^{\frac{-\ln(2) \cdot 1000}{5730}}) = (1\text{ kg}) \cdot \left(\frac{1}{2}\right)^\frac{1000}{5730}.$$