Consider the graphs of $y = f(x)$ shown in Figure 1 for the functions

$$f(x) = 2x - x^3, \quad \frac{1}{x}, \quad \frac{2x^2 - 5x + 8}{x^2 + x + 1}, \quad e^x, \quad \ln(x), \quad \tan^{-1}(x).$$

How would you describe what happens to these functions $f(x)$ when $x$ gets large and positive, that is, as $x$ approaches $+\infty$? What about when $x$ gets large and negative, that is, as $x$ approaches $-\infty$?

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Figure 1. Various graphs of $y = f(x)$.

Behavior of functions at infinity: infinite limits and horizontal asymptotes

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We seek some language involving limits to describe this. Informally, one might say \( \lim_{x \to +\infty} f(x) = +\infty \) to mean that we can ensure that the values of \( f(x) \) are arbitrarily large and positive by choosing \( x \) sufficiently large and positive. Similarly, one might say informally \( \lim_{x \to -\infty} f(x) = L \) for some real number \( L \) to mean that we can ensure that the values of \( f(x) \) are arbitrarily close to \( L \) by choosing \( x \) sufficiently large and positive. One could suitably modify these descriptions to define informally when

\[
\lim_{x \to +\infty} f(x) = \begin{cases} 
+\infty & \text{as with } f(x) = e^x \text{ or } \ln(x) \\
-\infty & \text{as with } f(x) = 2x - x^3 \\
L & \text{as with } f(x) = \frac{1}{x} \text{ for } L = 0, \\
or f(x) = \frac{2x^2 - 5x + 8}{2x^2 + x + 1} \text{ for } L = 2, \\
or f(x) = \tan^{-1}(x) \text{ for } L = \frac{\pi}{2},
\end{cases}
\]

and

\[
\lim_{x \to -\infty} f(x) = \begin{cases} 
+\infty & \text{as with } f(x) = 2x - x^3 \\
-\infty & \text{as with } f(x) = \frac{1}{x} \text{ or } e^x \text{ for } L = 0, \\
or f(x) = \frac{2x^2 - 5x + 8}{2x^2 + x + 1} \text{ for } L = 2, \\
or f(x) = \tan^{-1}(x) \text{ for } L = -\frac{\pi}{2}.
\end{cases}
\]

Note that for some functions one might have no limit at all for \( f(x) \) as \( x \) approaches \( \pm\infty \), that is, there is no real number \( L \) for which \( \lim_{x \to \pm\infty} f(x) = L \), nor does \( \lim_{x \to \pm\infty} f(x) = +\infty \), nor does \( \lim_{x \to \pm\infty} f(x) = -\infty \). In this case, say that \( \lim_{x \to \pm\infty} \sin(x) \) does not exist.

**Example.** \( \lim_{x \to +\infty} \sin(x) \) does not exist. As \( x \) gets arbitrarily large and positive, the values of \( f(x) = \sin(x) \) do not get arbitrarily large and positive, nor arbitrarily large and negative, nor do they approach closer and closer to any real number \( L \). Rather the values of \( f(x) \) forever oscillate, staying bounded between \(-1\) and \(+1\).

As with definitions of the usual kinds of limits \( \lim_{x \to a} f(x) = L \), one can capture the intuition behind these informal definitions \( \lim_{x \to \pm\infty} f(x) \) with something formal.

**Definition.** Formally, define \( \lim_{x \to +\infty} f(x) = +\infty \) to mean that for every \( M > 0 \), there exists an \( N > 0 \) such that the inequality \( f(x) > M \) holds for all \( x > N \).

**Definition.** Similarly, define formally \( \lim_{x \to +\infty} f(x) = L \) for a real number \( L \) to mean that for every \( \epsilon > 0 \), there exists an \( N > 0 \) such that the inequality \(|f(x) - L| < \epsilon \) holds for all \( x > N \).

Similar modifications exist to define formally what is meant by the other variations \( \lim_{x \to \pm\infty} f(x) = \pm\infty \) or \( \lim_{x \to \pm\infty} f(x) = L \).

**Definition.** When either \( \lim_{x \to +\infty} f(x) = L \) or \( \lim_{x \to -\infty} f(x) = L \), one says that the horizontal line \( y = L \) is a horizontal asymptote for the graph \( y = f(x) \). One can also say that the curve \( y = f(x) \) approaches the line \( y = L \) asymptotically.

**Example.** Let’s check formally that \( \lim_{x \to +\infty} e^x = +\infty \). To do this, if our adversary names for us some \( M > 0 \), we must find an \( N \) such that \( e^x > M \) for all \( x > N \).
A little thought, foresight, or experience with such arguments\textsuperscript{2} might suggest trying $N = \ln(M)$. And indeed one can check that for $x > N = \ln(M)$ one has
\[ f(x) = e^x > e^N = e^{\ln(M)} = M \]
where that inequality in the middle is due to the fact that $f(x) = e^x$ is a monotonically increasing function of $x$.

The formal definitions can be used to prove limit laws similar to the ones we have seen for other limits: in situations where the limits of $f(x), g(x)$ which appear on the right side of these laws are real numbers (not $\pm \infty$), one has
\[
\lim_{x \to \pm \infty} (f(x) \pm g(x)) = \lim_{x \to \pm \infty} f(x) \pm \lim_{x \to \pm \infty} g(x)
\]
\[
\lim_{x \to \pm \infty} f(x)g(x) = \left( \lim_{x \to \pm \infty} f(x) \right) \cdot \left( \lim_{x \to \pm \infty} g(x) \right)
\]
\[
\lim_{x \to \pm \infty} \frac{f(x)}{g(x)} = \left( \lim_{x \to \pm \infty} f(x) \right) \div \left( \lim_{x \to \pm \infty} g(x) \right)
\]
assuming $\lim_{x \to \pm \infty} g(x) \neq 0$ in this last law.

Another such law says that when $f(x)$ is a continuous function for all values $x$ in the range of $g(x)$, and $\lim_{x \to \pm \infty} g(x) = L$ for some real number $L$, then $\lim_{x \to \pm \infty} f(g(x)) = f(L)$.

This does not exhaust all the possible such limit laws. Also, some of these limit laws still apply even when $f(x), g(x)$ do not have finite limits.

**Example.** If $\lim_{x \to +\infty} f(x) = L$ and $\lim_{x \to +\infty} g(x) = +\infty$, then
\[
\lim_{x \to +\infty} (f(x) + g(x)) = +\infty.
\]

We sometimes abbreviate this law informally by saying “$L + \infty = +\infty$”. Similarly, one has $\frac{0}{\infty} = 0$.

However, one has to be careful, as some cases where one would like to apply a limit law are indeterminate forms, like
\[
\frac{\pm \infty}{\pm \infty}, \quad \frac{0}{\infty}, \quad 0 \cdot (\pm \infty), \quad \infty \pm \infty.
\]

Sometimes in these cases, algebraic manipulation and/or L’Hôpital’s rule (a later calculus topic) comes to our aid.

**Example.** Starting from the fact (which one can justify from the formal definition) that integer powers $x^n$ of $x$ have
\[
\lim_{x \to \pm \infty} x^n = \begin{cases} 
\pm \infty & \text{if } n = 1, 2, 3, \ldots \\
1 & \text{if } n = 0 \\
0 & \text{if } n = -1, -2, -3, \ldots
\end{cases}
\]
it’s not hard to analyze the behavior at infinity for any rational function. Recall that a rational function is $h(x) = \frac{f(x)}{g(x)}$ where
\[
f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_r x^r
\]
\[
g(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_s x^s
\]
\textsuperscript{2}The sometimes subtle art of how to pick the $N$ correctly is not something we will emphasize in our version of Math 1271, as we won’t often ask students to prove a limit is correct via the formal definition!
are polynomials, say of degrees \( r \) and \( s \), so that \( a_r, b_s \neq 0 \).

If one tries to analyze \( \lim_{x \to \pm \infty} h(x) \) by immediately using the quotient rule for limits it often leads to the indeterminate form \( \infty/\infty \). However, a useful algebraic trick comes from realizing that if \( x^N \) is the highest power of \( x \) appearing anywhere in either \( f(x) \) or \( g(x) \) (so \( N \) is just maximum of the two degrees \( r \) and \( s \)), then these terms \( x^N \) wherever they occur will dominate the behavior of \( f(x) \) when \( x \to \pm \infty \).

And we can “scale them away” by multiplying by \( 1/x^N \), leaving an equivalent limit, for which the quotient limit law will now work.

For example,

\[
\lim_{x \to \pm \infty} \frac{2x^2 - 1}{x^2 + 1} = \lim_{x \to \pm \infty} \frac{2x^2 - 1}{x^2 + 1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x \to \pm \infty} \frac{2 - 1/x^2}{1 + 1/x^2} = \frac{2}{\pm \infty} = \pm \infty
\]

\[
\lim_{x \to \pm \infty} \frac{2x^2 - 1}{x^3 + 1} = \lim_{x \to \pm \infty} \frac{2x^2 - 1}{x^3 + 1} \cdot \frac{1/x^3}{1/x^3} = \lim_{x \to \pm \infty} \frac{2 - 1/x^3}{1 + 1/x^3} = \frac{0}{1} = 0
\]

The graphs of these three rational functions are shown in Figure 2.

Doing this analysis in general for the rational function \( \frac{f}{g} \) where \( f, g \) have degrees \( r, s \) and leading coefficients \( f_r, g_s \) shows the following:

\[
\lim_{x \to \pm \infty} f(x) = \begin{cases} 
0 & \text{if } r < s, \\
\frac{f_r}{g_s} & \text{if } r = s, \\
(-1)^{r-s} \text{sign}(g_s) \cdot (\pm \infty) & \text{if } r > s
\end{cases}
\]

where \( \text{sign}(x) = \frac{|x|}{x} \) give the sign \( \pm 1 \) of a nonzero number \( x \).

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3Alternatively, one can do this same trick but multiply by \( 1/x^N \), which will also provide an illuminating scaling, and still work with the limit laws.

4... which is not really worth memorizing; the trick of multiplying \( 1/x^N \) is more important.
Exercises.

(a) Say whether \( \lim_{x \to +\infty} f(x) \) is some real number \( L \), or \( +\infty \) or \( -\infty \) or nonexistent for each of the following functions \( f(x) \). Remember to give some justification for your answer.

(b) Do the same for \( \lim_{x \to -\infty} f(x) \).

(c) Then list any horizontal asymptotes for the graph \( y = f(x) \).

1. \( f(x) = \cos(x) \)
2. \( f(x) = x \cos(x) \)
3. \( f(x) = \frac{1}{x(2+\cos(x))} \)
4. \( f(x) = x^2 \sin(x) \)
5. \( f(x) = \frac{\sin(x)}{x} \)
6. \( f(x) = e^{-2x} \sin(x) \)
7. \( f(x) = e^{2x} \sin(x) \)
8. \( f(x) = \frac{x^{100}+x^3+x}{3x^{100}+x^3+x} \)
9. \( f(x) = \frac{x^{100}+x^3+x}{3x^{100}+x^3+x} \)
10. \( f(x) = \frac{x^{100}+x^3+x}{3x^{100}+x^3+x} \)
11. \( f(x) = \tan^{-1}(3x^3 + 4x - 100) \)
12. \( f(x) = \tan^{-1}(-3x^3 + 4x - 100) \)