Math 4707 Fall 2020 (online!)

Remember to record!!

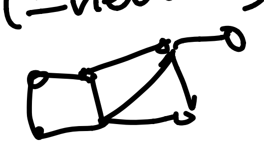
Gather.town meeting room
has password "Math 4707"

Proposed office hour slots
Tues – Thur 9:05 – 9:55am

CONTENT

$< \frac{1}{2}$ enumeration = counting
(e.g. computing probabilities,
analyzing runtime of
algorithms)

$> \frac{1}{2}$ graph theory
(= network)



Overlap classes:

5705 Enumeration

5707 Graph theory

5711 Combinatorial optimization

(ISysE 53...)

5248 } We'll skip this stuff on cryptology & Coding Theory in Chap. 6 & 15
5251 }

Chapter 1 Let's Count! [Enumeration]

Some basic counting principles

$\boxed{\text{Product principle}}$ – if one can choose an element of a set A by making r sequential choices, with d_1 choices for 1^{st} step

d_2 choices for 2^{nd}

$\vdots$

d_r choices for r^{th}

then $|A| = \#A = d_1 d_2 \cdots d_r$

EXAMPLES

① How many subsets of an n element set are there?

n	subsets of $\{1,2,\dots,n\}$	# of subsets
0	$\emptyset$	$1 = 2^0$
1	$\emptyset$, $\{1\}$	$2 = 2^1$
2	$\emptyset$, $\{1\}$, $\{2\}$, $\{1,2\}$	$4 = 2^2$
3	$\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$	$8 = 2^3$
4		$16 = 2^4$

Guess: 2^n

Pick the subset $S \subseteq \{1,2,\dots,n\}$ in n stages: at 1st stage, ask is 1 in S, yes/no? ($d_1 = 2$ possibilities)

at 2nd, ask is 2 in S? ($d_2 = 2$ possibilities)

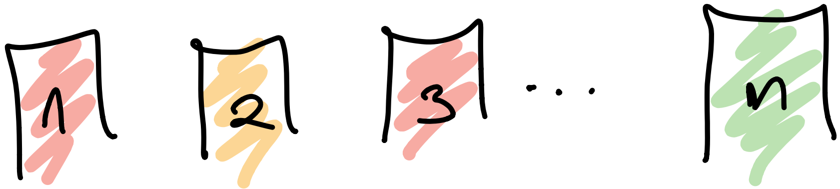
DEF'N $A := \{$all subsets S of $\{1,2,\dots,n\}\}$

⋮

at n^{th} stage, ask is n in S? ($d_n = 2$)

so $|A| = \# A = d_1 \cdot d_2 \cdots d_n = 2 \cdot 2 \cdots 2 = 2^n$

② How many ways to paint n posts labeled $1, 2, \ldots, n$ with k choices of colors?



$d_1 = k$ choices for 1st post | $d_2 = k$ choices for 2nd | . . . | $d_n = k$ for nth

$A = \{k\text{-colorings of } n \text{ posts } \{1, 2, \ldots, n\}\}$

$$|A| = \# A = d_1 \cdot d_2 \cdots d_n$$
$$= k \cdot k \cdots k = k^n$$

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③ How many ways to pick your 1st, 2nd, 3rd favorite ice cream flavors from the 31 flavors offered by Baskin-Robbins?

(3) How many ways to pick your 1st, 2nd, 3rd favorite ice cream flavors from the 31 flavors offered by Baskin-Robbins?

$$31 \cdot 30 \cdot 29$$

1st choice $d_1 = 31$

2nd flavor has $d_2 = 30$ choices (no repeats)

3rd flavor has $d_3 = 29$ choices (no repeats)

More generally, how many ways to choose an ordered list of k distinct elements from n choices?

($k=3$, $n=3$) above).

$$d_1 \cdot d_2 \cdot d_3 \cdots d_k$$

$$= n \cdot (n-1) \cdot (n-2) \cdots (n-(k-1))$$

$$= \frac{n!}{(n-k)!}$$

Think about this!

Recall $n! := n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$

(4) How many rearrangements are there of the letters in SNAKE?

$$\underbrace{5}_{d_1} \cdot \underbrace{4}_{d_2} \cdot \underbrace{3}_{d_3} \cdot \underbrace{2}_{d_4} \cdot \underbrace{1}_{d_5} \left(=: 5!\right)$$

KAENS

$\{S, N, A, K, E\}$

$n = 5$

$k = 5$

How many permutations of n objects?

$$n! := \underbrace{n}_{d_1}\underbrace{(n-1)}_{d_2}\underbrace{(n-2)}_{d_3} \cdots 3 \cdot \underbrace{2}_{d_{n-1}} \cdot \underbrace{1}_{d_n}$$

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(5) How many ways to pair off $2n$ campers $\{1, 2, \ldots, 2n-1, 2n\}$ as buddy pairs for swimming?

⑤ How many ways to pair off
$2n$ campers $\{1, 2, \ldots, 2n-1, 2n\}$
as buddy pairs for swimming?

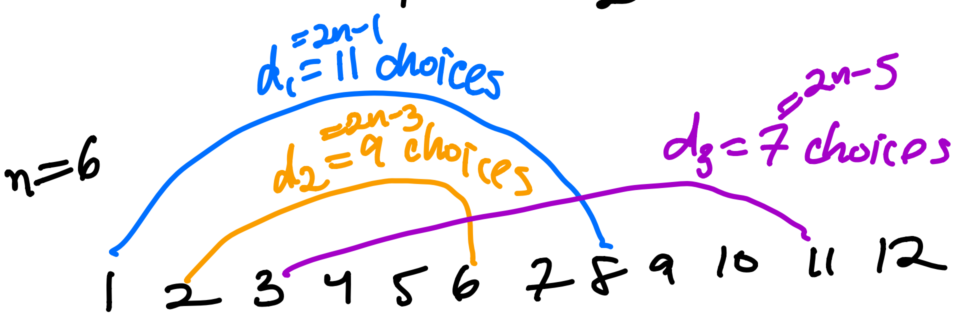
n	buddy pairings	# pairings
1	1 — 2	1
2	1 — 2 3 — 4 1 2 3 4 1 2 3 4	$3 = 3 \cdot 1$
3	1 2 3 4 5 6 1 2 3 4 5 6 1 2 3 4 5 6 1 2 3 4 5 6 ⋮	$15 = 5 \cdot 3 \cdot 1$
4		$105 = 7 \cdot 5 \cdot 3 \cdot 1$

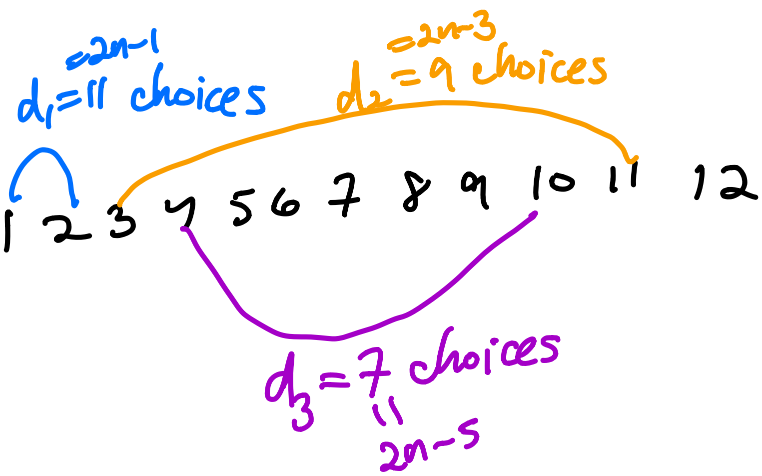
(5) How many ways to pair off $2n$ campers $\{1,2,\ldots,2n-1,2n\}$ as buddy pairs for swimming?

$$A = \{\text{all pairings of } \{1,2,\ldots,2n\}\}$$

$$\#A = \underbrace{(2n-1)}_{= d_1}\underbrace{(2n-3)}_{= d_2} \cdots \underbrace{7}_{= d_{n-2}} \cdot \underbrace{5}_{= d_{n-1}} \cdot \underbrace{3}_{= d_n} \cdot 1 = \frac{2n!}{n! \cdot 2^n}$$

Think about this!





SUM PRINCIPLE

If a set $A = \underbrace{A_1 \dot\cup A_2}$

disjoint union

$A = A_1 \cup A_2$

and $A_1 \cap A_2 = \phi$

then $|A| = |A_1| + |A_2|$

More generally, if

$$A = A_1 \dot\cup A_2 \dot\cup A_3 \dot\cup \dots \dot\cup A_r$$

then
$$|A| = |A_1| + |A_2| + \dots + |A_r|$$
$$= \sum_{i=1}^{r} |A_i|$$

EXAMPLE: How many rearrangements of SNAKE have S, N adjacent?

e.g.

AKSNE $\leftarrow$ there are 4! of these, rearranging A, K, SN, E

AKNSE $\leftarrow$ there are 4! of these, rearranging A, K, NS, E

EXAMPLE: How many rearrangements of SNAKE have S, N adjacent?

e.g.
AKSNE ← there are 4! of these, rearranging A, K, SN, E

AKNSE ← there are 4! of these, rearranging A, K, NS, E

$$A = A_1 \sqcup A_2$$

all such rearrangements

those with SN as a "superletter"

those with NS as a "superletter"

$$\Rightarrow |A| = |A_1| + |A_2|$$
$$= 4! + 4!$$
$$= 24 + 24 = 48$$

SOME NOTATION REVIEW

"For all" $\forall$

"There exists" $\exists$

$A \subset B$
and $A \neq B$

$\in$, $\subset$, $\subsetneq$, $\subseteq$, $\supset$, $\supseteq$,

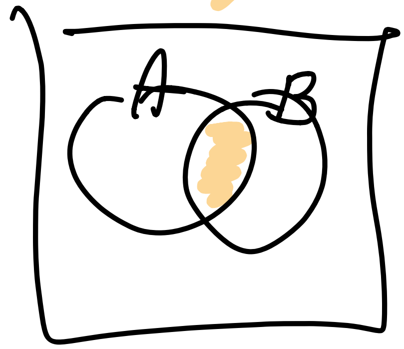
$x \in A$
x an element of A

$A \subseteq B$
A is a subset of B, (emphasizing maybe $A = B$)

$A \subset B$
A is a subset of B

$A \cup B$

$A \cap B$

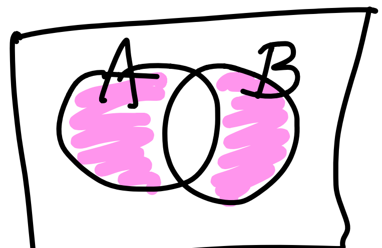


$A \setminus B$

$= A \cap B^c$

$= A - B$

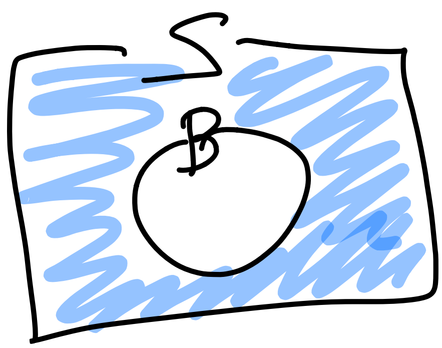
$= \{a \in A : a \notin B\}$



$$(A \setminus B) \sqcup (B \setminus A) =: A \triangle B$$

symmetric difference

$$B^c = S \setminus B = \text{complement of } B \text{ within the universe } S$$



Math 4707 Sept. 14, 2020

Wait until questions in lecture are written down before writing answers in the chat!

Wed Sept 23 lecture will be pre-recorded and posted, not synchronous.

Thur Sept. 24 <u>may</u> also be canceled!

We've been counting $|A| = \#A$ for various sets A ...

<u>2 principles:</u>

PRODUCT PRINCIPLE

$$|A| = d_1 d_2 \cdots d_r$$

SUM PRINCIPLE:

$$A = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_r$$

$$\underset{\text{implies}}{\Longrightarrow} \quad |A| = \sum_{i=1}^{r} |A_i|$$

DIFFERENCE PRINCIPLE:

Sometimes $A = B \setminus C$
$$= B - C$$
$$= \{ b \in B : b \notin C \}$$

for some $C \subseteq B$, so $|A| = |B| - |C|$

and sometimes its easier to compute $|B|$ and $|C|$ and <u>subtract</u>!

EXAMPLES:

① How many ways to paint n different posts $\{1, 2, \ldots, n\}$ with k colors so that at least 2 colors are used?

$$k^n - k$$

$$= \left|\{\text{all colorings}\}\right| - \Bigg|\overbrace{\left\{\begin{matrix}\text{colorings} \\ \text{with only} \\ \text{1 colors used}\end{matrix}\right\}}^{C}\Bigg|$$

$$= |B| - |C|$$

② How many subsets of $\{1,2,\dots,100\}$ contain at least one odd number?

$$2^{100} - 2^{50}$$

$$= \left|\{\text{all subsets}\}\right| - \Bigg|\underbrace{\left\{\begin{array}{l}\text{those with}\\ \text{no odd}\\ \text{numbers}\end{array}\right\}}_{\left\{\begin{array}{l}\text{those with only}\\ \text{even numbers}\\ 2,4,\dots,98,100\end{array}\right\}}\Bigg|$$

$$= |B| - |C|$$

SHEPHERD'S PRINCIPLE
(OVERCOUNTING)
(COUNTING IN TWO WAYS)

- sometimes it's easier to count a different set B than A, but you know every element of A corresponds to m different elements of B.

Then $|A| = \dfrac{|B|}{m}$

SHEPHERD'S PRINCIPLE
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- sometimes its easier to count a different set B than A, but you know every element of A corresponds to m different elements of B.

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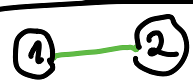
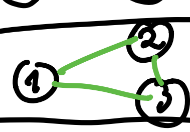
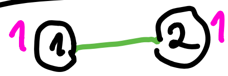
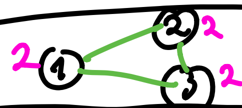
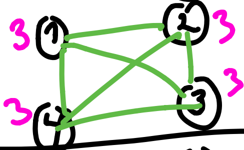
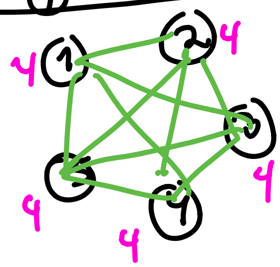
EXAMPLES:

2 How many handshakes occur when n people meet each other?

n	handshake = edges	# handshakes
1		0
2		1
3		3
4		
5		

② How many handshakes occur when n people meet each other?

n	handshake = edges	sum of degrees	# handshakes
1	① 0	0	$0 = \frac{1(1-1)}{2}$
2		2	$1 = 1 = \frac{2(2-1)}{2}$
3		6	$3 = 3 = \frac{3(3-1)}{2}$
4		12	$6 = 3 \cdot 2 = \frac{4(4-1)}{2}$
5		20	$10 = 5 \cdot 2 = \frac{5(5-1)}{2}$

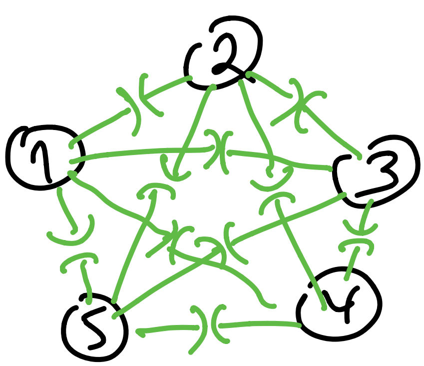
$$|A| \quad \frac{n(n-1)}{2} = \frac{|B|}{m}$$

$|A| = \{\text{handshakes}\}$

$B = \{(1^{st}\text{ person}, 2^{nd}\text{ person})$ in a handshake$\}$

$= \{$ half-edges in the pictures $\}$

$n=5$



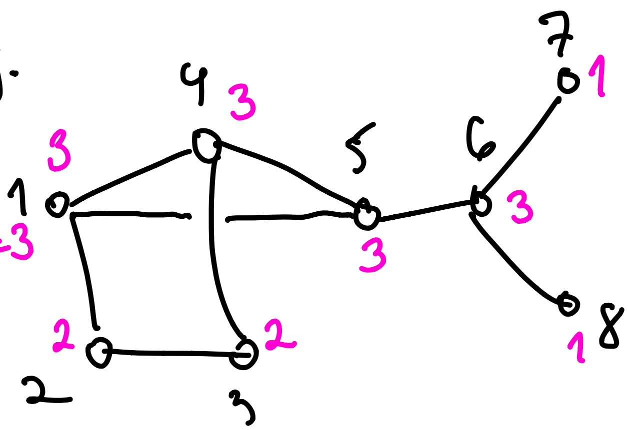
③ Let's generalize a bit to <u>graphs</u> and <u>vertex degrees</u> (§7.1)

<u>DEFINITION</u>: A (<u>simple</u>) <u>graph</u>

$$G = (\underbrace{V}_{\text{vertices}}, \underbrace{E}_{\text{edges (pairs of vertices)}})$$

is a set of vertices V and a set of pairs E of vertices

e.g.



$\deg_G(1) = 3$

$$G = (V, E)$$

$$V = \{1,2,3,4,5,6,7,8\}$$

$$E = \{\{1,2\}, \{2,3\}, \{3,5\}, 34, 15, 56, 67, 78\}$$

Each vertex $v \in V$ has a <u>degree</u> (<u>valence</u>)

$$\deg_G(v) := |\{w \in V : \{v,w\} \in E\}|$$

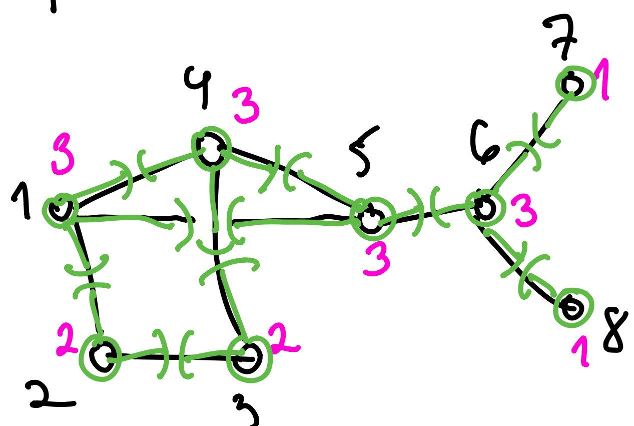
total of vertex degrees $= 1+1+2+2+3+3+3+3$
$= 18 = 2 \cdot 9, \; 9 = |E|$

(THM 7.1.2 on p.130)

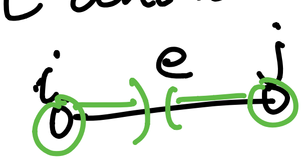
PROPOSITION: In any graph $G=(V,E)$

$$\sum_{v\in V} \deg_G(v) = 2\cdot |E|$$

proof: Count $\underbrace{\text{"half-edges"}}_{B}$ in 2 ways:



Every edge $e=\{i,j\}$ in E contributes to two half-edges



so $\underset{\{\text{half-edges}\}}{\underset{\shortparallel}{|B|}} = 2|E|$

Also every vertex $v\in V$ contributes $\deg_G(v)$ elements (half-edges) to B

Hence $|B| = \sum_{v\in V} \deg_G(v)$. So $\sum_{v\in V} \deg_G(v) = 2|E|$ ∎

COROLLARY: At a party with an odd number of guests, some guest will know an even number of other guests!

(i.e. if $G = (\underset{\substack{\shortparallel \\ \text{people}}}{V}, \underset{\substack{\shortparallel \\ \text{edges between people who know each other}}}{E})$ has $|V|$ being odd,

then some $v \in V$ has $\deg_G(v)$ even.)

proof: $\sum_{v \in V} \deg_G(v) = 2|E|$ is even.

If all $\deg_G(v)$ are odd, then $\sum_{v \in V} \deg_G(v)$ is a some of odd-ly many odd numbers, which is always odd (e.g. $1+7+5+5+7+7+7 \underset{\text{odd!}}{= 39}$)

That would be a contradiction ∎

In fact, this shows more strongly that...

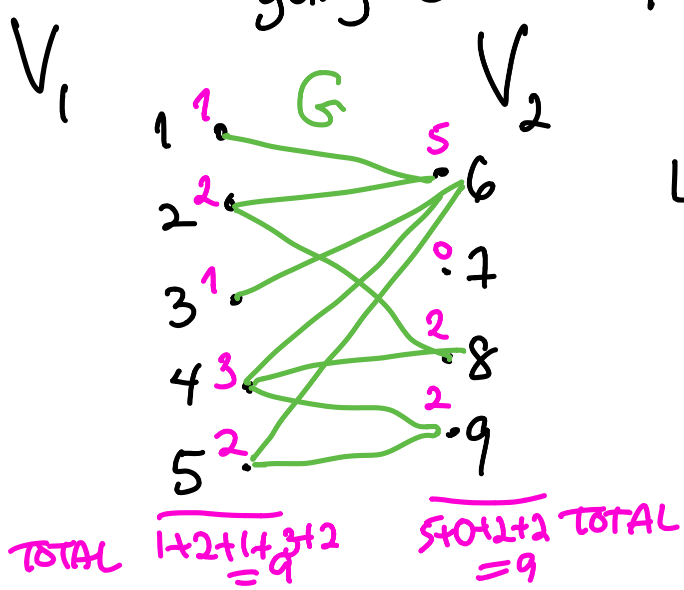
(THM 7.1.1) COROLLARY: In every graph, the number of odd degree vertices must be <u>even</u>.

(Think about it!)

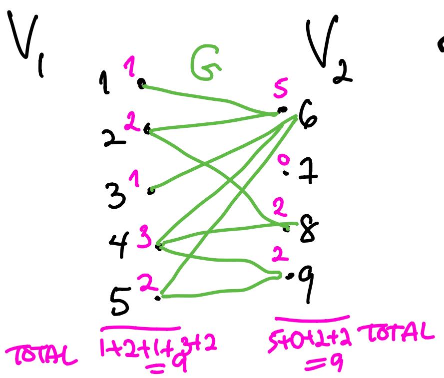
④ More modeling with graphs and degrees and counting in 2 ways...

A <u>bipartite</u> graph $G = (V, E)$, $V = V_1 \sqcup V_2$,

has every edge $\{i_1, i_2\}$ going between V_1 and V_2, meaning $i_1 \in V_1$ and $i_2 \in V_2$



Let's compare the sum of vertex degrees on the two sides V_1, V_2



PROPOSITION: In any bipartite graph

$$G = (V, E), \quad V = V_1 \sqcup V_2$$

one has
$$\sum_{v \in V_1} \deg_G(v_1) = \sum_{v \in V_2} \deg_G(v_2)$$

COROLLARY: In particular, their average degrees have this ratio:

$$\frac{\text{average degree in } V_1}{\text{average degree in } V_2} = \frac{\frac{1}{|V_1|}\cancel{\sum_{v \in V_1} \deg_G(v_1)}}{\frac{1}{|V_2|}\cancel{\sum_{v \in V_2} \deg_G(v_2)}} = \frac{|V_2|}{|V_1|}$$

SAME

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proof: $\displaystyle\sum_{v \in V_1} \deg_G(v_1) = |E| = \sum_{v \in V_2} \deg_G(v_2)$ ∎

ASSIGNMENT: Read "The Math, the Math, the Sex" linked on syllabus. Relate to COROLLARY.

MORE EXAMPLES OF SHEPHERD'S PRINCIPLE

(1) How many ways to pick 3 flavors of the 31 ice cream flavors to bring home (but don't care which is 1st or 2nd or 3rd)?

$$\frac{31 \cdot 30 \cdot 29}{3!} = \frac{|B|}{m}$$

$A = \{\text{choices as desired}\}$ $\quad B = \{\text{choices with 1st, 2nd, 3rd specified}\}$

$m = 3!$

$\binom{31}{3}$ = binomial coefficient

$\binom{n}{k} :=$ # of ways to pick a k-element subset of an n-element set

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

$$= \frac{n!}{k!\,(n-k)!}$$

$n = 31$ flavors

$k = 3$ ones brought home

② How many rearrangements are there of the letters in $BA_1N_1A_2N_2A_3$?

e.g. $\overbrace{AAA}^{3}\overbrace{NN}^{2}\overbrace{B}^{1}$

AAANBN

$\vdots$

$$\frac{6!}{3!\,2!\,1!} = \frac{|B|}{m}$$

$$\| $$

$$\binom{6}{3\ 2\ 1}$$

multinomial coefficient

B = rearrangements of $\{A_1, A_2, A_3, N_1, N_2, B_1\}$

$|B| = 6!$

More generally,

$$\binom{n}{k_1\ k_2 \dots k_\ell} = \text{\# ways to arrange}$$

n letters that have

k_1 copies of 1st letter

k_2 " " 2nd

$\vdots$

k_ℓ copies of the ℓ^{th} letter

$k_1 + k_2 + \dots + k_\ell = n$

$$= \frac{n!}{k_1!\,k_2! \cdots k_\ell!}$$

$$\binom{n}{k_1\ k_2 \cdots k_\ell} = \text{\# ways to arrange } n \text{ letters that have}$$

k_1 copies of 1st letter
k_2 —— " —— 2nd
⋮
k_ℓ copies of the ℓ^{th} letter

$k_1+k_2+\dots+k_\ell=n$

$$= \frac{n!}{k_1!\,k_2!\cdots k_\ell!}$$

Binomial coefficient

$$\binom{n}{k} = \text{\# } k\text{-element subsets of } n\text{-element set}$$

$$= \binom{n}{k\ \ n-k} = \frac{n!}{k!\,(n-k)!}$$

= # words with k A's
$n-k$ B's

e.g. $k=3$
$n=7$
$n-k=4$

1 2 3 4 5 6 7
B A B A B B A

GROUP WORK on poker hand probabilities

A standard deck has 52 cards

A 2 3 4 5 6 7 8 9 10 J Q K of ♣ ♦ ♥ ♠

A poker hand is a choice of 5 of the cards, as an unordered set, e.g.

{ 4♥, J♠, 6♠, 10♦, 2♣ }

① How many poker hands are there?

② What is the probability that it is a...

ROYAL FLUSH	e.g. {10♥, J♥, Q♥, K♥, A♥}
STRAIGHT FLUSH (not ROYAL)	e.g. {7♦, 8♦, 9♦, 10♦, J♦} (aces low or high)
STRAIGHT (not a straight flush)	e.g. {7♦, 8♠, 9♦, 10♣, J♣}
FLUSH (not a straight flush)	e.g. {5♦, 6♦, 9♦, Q♦, K♦}
4 OF A KIND	e.g. {5♦, 5♣, 5♥, 5♠, J♣}
FULL HOUSE	e.g. {5♦, 5♣, 5♥, J♦, J♠}
3 OF A KIND (not 4, not full house)	e.g. {5♦, 5♣, 5♥, J♦, 6♣}
2 PAIR	e.g. {5♦, 5♣, 10♦, 10♠, 3♥}
1 PAIR	e.g. {5♦, 5♣, 10♦, 8♠, K♥}