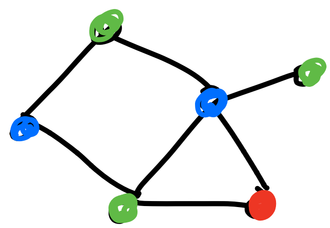
# Math 4707 Dec. 2, 2020



$G = (V, E)$ graph

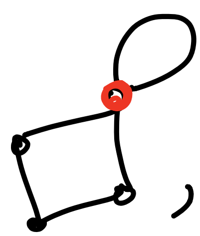
$\chi(G) = \min\{k : G$ has a proper vertex $k$-coloring$\}$

chromatic number of $G$

$\chi(G) = 3$

## A few easy observations about $\chi(G)$:

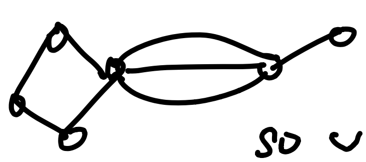
- $G$ has no proper colorings if it has any loops



,

so assume $G$ is loopless from now on.

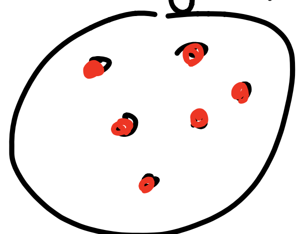
- Multiple/parallel edges don't affect properness of vertex colorings,



so we can assume when we want that there are no parallel edges in $G$.

• $\chi(G) \le 1$ i.e. $G$ is 1-colorable

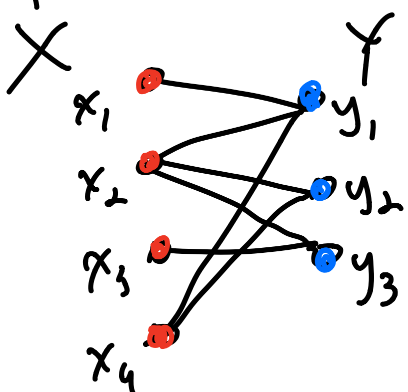
$\iff$ $G$ has no edges, i.e.

$G =$ 

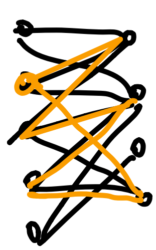
**PROPOSITION:** $\chi(G) \le 2 \overset{(a)}{\iff}$ $G$ is bipartite

$\iff$ all cycles in $G$ have even length

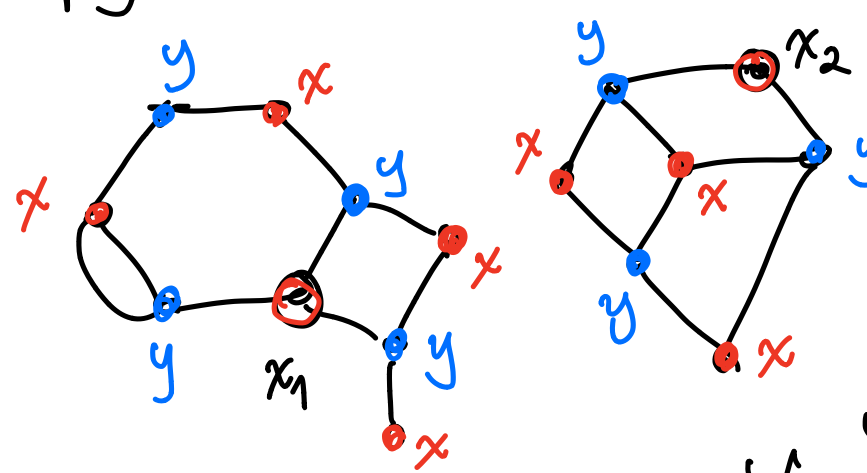
proof: $\chi(G) \le 2 \overset{(a)}{\iff} G = (\underset{\substack{\parallel \\ \{\text{vertices colored } 1\}}}{X} \sqcup \underset{\substack{\parallel \\ \{\text{vertices colored } 2\}}}{Y}, E)$




$G$ bipartite $\Longrightarrow$ all cycles in $G$ have even length



Why does $G$ having only even cycles imply $G$ is bipartite? (i.e. $\overset{(b)}{\Longleftarrow}$ ?)




Let's 2-color the vertices $V = \overset{\text{red}}{X} \sqcup \overset{\text{blue}}{Y}$ by first picking a root vertex $x_i$ in each connected component of $G$.

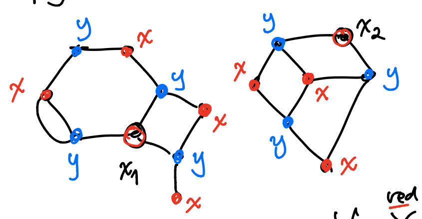
Then we claim that all other vertices $v \in V$ have the following coloring forced:

$$X := \{x \in V : x \text{ has an even length path to some root vertex } x_i\}$$

$$Y = \{y \in V : y \text{ has an odd length path to some root vertex } x_i\}$$

CLAIM: This partition $V = X \sqcup Y$ makes $G$ a bipartite graph.

Why does $G$ having only even cycles imply $G$ is bipartite? (i.e. $\overset{(b)}{\Longleftarrow}$ ?)

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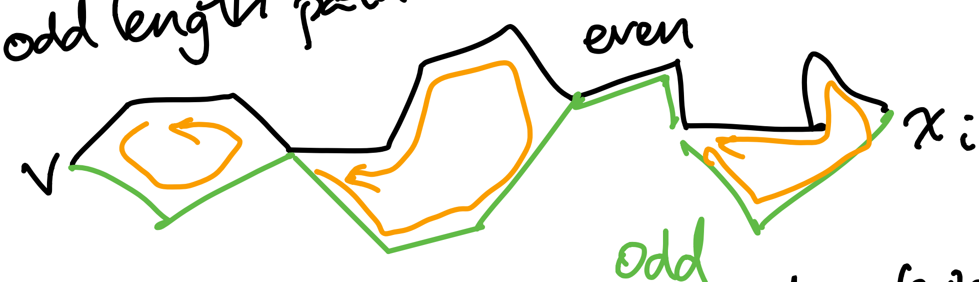
CLAIM: This partition $V = X \sqcup Y$ makes $G$ a bipartite graph.

## Must check:

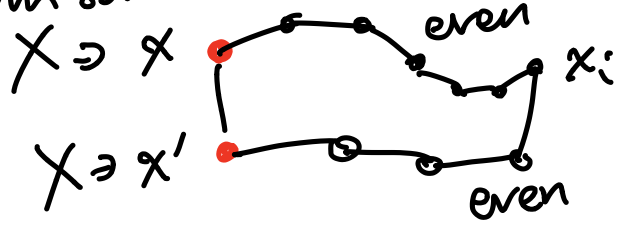
- $V = X \cup Y$
  since every vertex $v \in V$ has some path to some root vertex $x_i$ in its connected component.

- $X \cap Y = \emptyset$
  since if $v \in X \cap Y$ then $v$ has an even length path and an odd length path to some root vertex $x_i$




This creates at least one odd cycle in $G$.

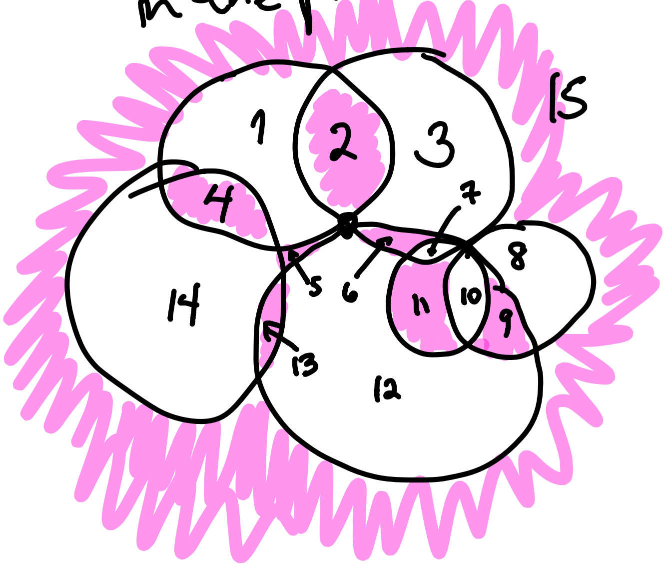
Lastly we must check there is no edge $x - x'$ from some $x \in X$ to some $x' \in X$, or $y - y'$ with $y, y' \in Y$



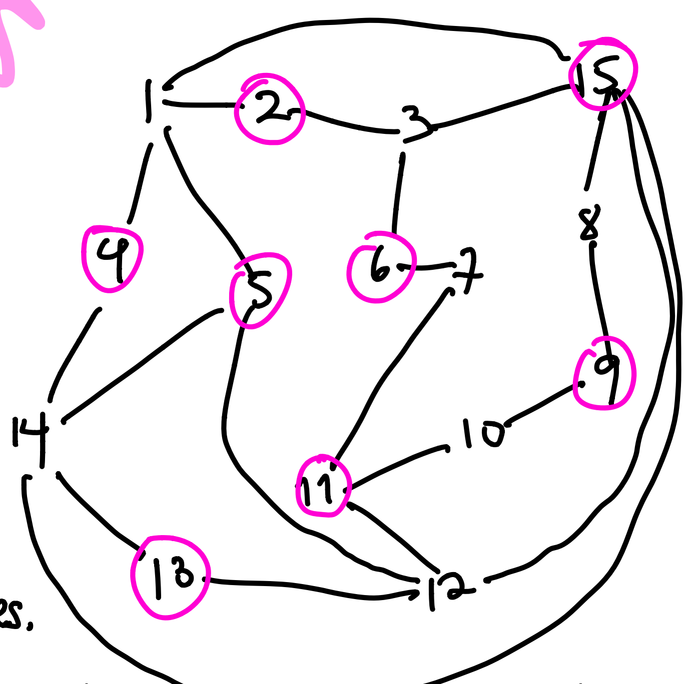

creates an odd cycle in $G$. ∎

EXAMPLE (THEOREM 13.1.1)

One can always properly 2-color the regions formed by an arrangement of circles in the plane:



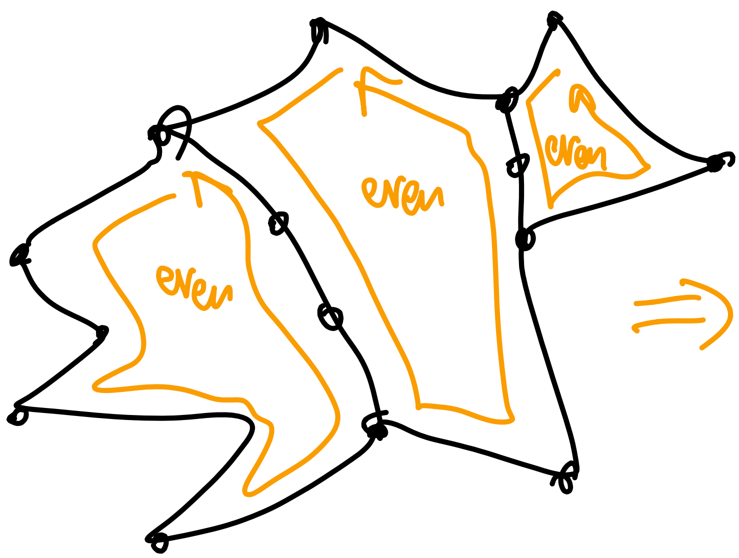

proof: Form the dual planar graph




We want to understand why this dual graph is bipartite ($\chi(G) \leq 2$), so enough to show it has only even cycles.

This holds because regions here → corresponding to circle intersection points in original, and if $m$ circles meet there, the region is a $2m$-cycle.

Hence the cycles bounding regions are all even, so any cycle




$\Longrightarrow$ big one is also even. $\boxtimes$

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## §13.3 Deciding colorability

$\chi(G) \le 2 \iff G$ bipartite

$\iff G$ contains no odd cycles

Q: $\chi(G) \le 3$? $\chi(G) \le 4$? ..

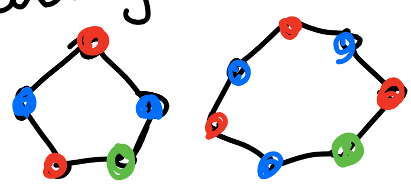
Not so simple to characterize, or compute algorithmically!

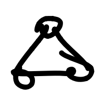
THEOREM (1970's) If one had a <u>fast</u>
Garey & Johnson
(polynomial-time) algorithm to decide
k-colorability (whether $\chi(G) \leq k$),
then there would also be fast algorithms
for TSP, Hamilton cycle problem, and
others ...

Certainly $\chi(G) \leq k \Rightarrow$ G contains no
complete graph $K_{k+1}$
as an edge subgraph

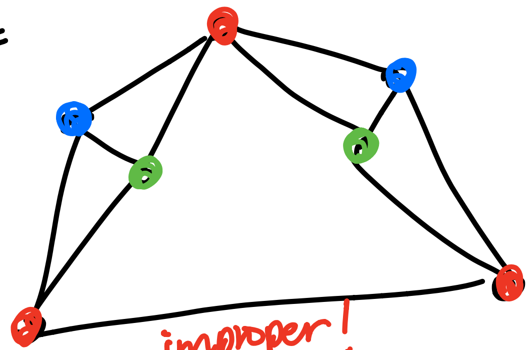
but the converse ($\Leftarrow$) is <u>false</u>,

e.g. odd cycles of size 5, 7, 9, 11, ...



contain no $K_3$ , but $\chi(G) = 3$.

EXAMPLE

$G =$  requires at least 4 colors, and $\chi(G) = 4$

improper!

$\chi(G) > 3$

but $G$ contains no $K_4$.

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Limiting vertex degrees in $G$ does limit $\chi(G)$.

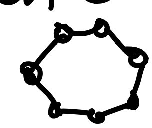
THEOREM (Brooks 1941) $G$ connected, loopless, no multiple edges

Let $\delta(G) := \max\{\deg_G(v) : v \in V\}$
max vertex degree

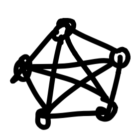
Then $\chi(G) \overset{(a)}{\leq} \delta(G) + 1$ (easy! we'll prove it)

AND not as easy! see Bondy & Murty §8.2

(b) If $\chi(G) = \delta(G) + 1$, then either

$G =$ odd cycle  so $\delta(G) = 2$, $\chi(G) = 3$

OR

$G = K_n$  so $\delta(G) = n-1$, $\chi(G) = n$

**THEOREM** (Brooks 1941) $G$ connected, loopless, no multiple edges

Let $\delta(G) := \max\{\deg_G(v) : v \in V\}$
max vertex degree

Then $\chi(G) \overset{(a)}{\leq} \delta(G)+1$

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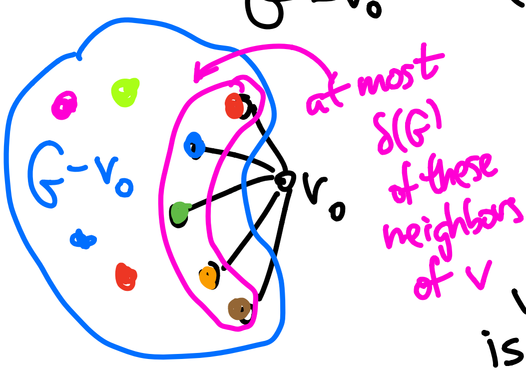
**proof of (a):** Let's use induction on $|V|=n$ and give an inductive algorithm to properly color $G$ with $\delta(G)+1$ colors.

**BASE CASE:** If $|V|=n \leq \delta(G)+1$, just color every vertex a different color!

**INDUCTIVE STEP:** If $G$ has $|V|=n$ vertices, pick any $v_0 \in V$ and consider

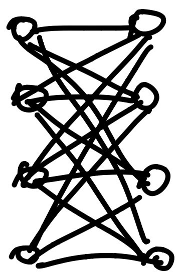
$$G - v_0 := \left(V - \{v_0\},\ E - \{\{v_0,v\} : v \in V\}\right)$$




Then $\delta(G-v_0) \leq \delta(G)$, and so by induction $G-v_0$ has a proper coloring with $\delta(G)+1$ colors. Since the neighbors of $v_0$ use $\leq \delta(G)$ colors, some color is left for $v_0$ to use. ■

REMARK: Note that Brooks's Thm. does not imply $\chi(G) \geq \delta(G)$, e.g.

$$\chi(K_{n,n}) = 2 \text{ but } \delta(K_{n,n}) = n$$

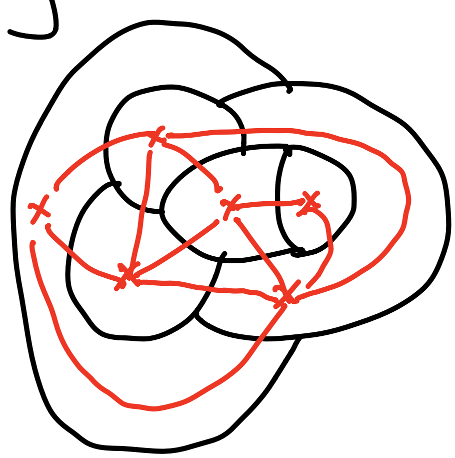
$K_{4,4} =$ 

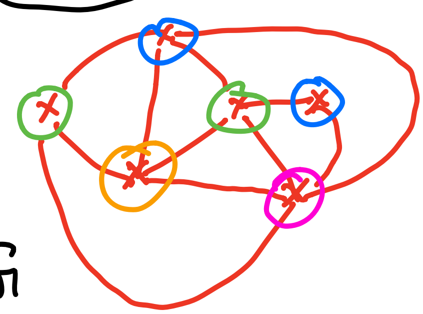
# Math 4707 Dec. 7, 2020

## §13.4 Coloring planar graphs and maps

CONJECTURE (Guthrie 1852) "The 4-color ~~conjecture~~ problem"

The regions of every planar map can be properly 4-colored





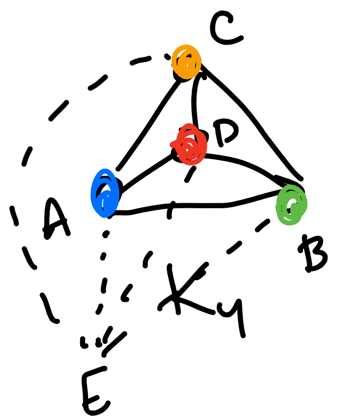
$\Updownarrow$

Every (loopless) planar graph $G$ has $\chi(G) \leq 4$

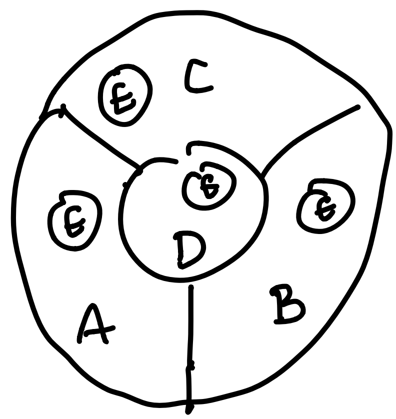
i.e. its vertices can be properly 4-colored.

REMARK: You have to insist the regions in the map are connected, simply-connected (no holes)

e.g.




dual map




is

$K_5$

$\chi(K_5)=5$

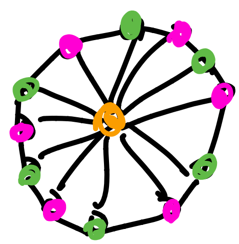
The history of the 4-color ~~problem~~ ~~conjecture~~ theorem is FASCINATING – read some in the book, Wikipedia

partly because of controversies that it raised...

How does planarity of $G$ bound $\chi(G)$ at all ??

It certainly doesn't bound $\deg_G(v)$,

e.g.



Let's see how Euler's formula and its consequences let us easily prove the 6-color theorem, and prove the 5-color theorem, then discuss the 4-color theorem.

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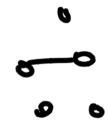
<u>6-color Theorem:</u> $G$ a loopless planar graph $\implies \chi(G) \leq 6$.

<u>proof:</u> Without loss of generally, let's assume $G$ has no parallel edges.

6-color Theorem: $G$ a loopless planar graph

$\Longrightarrow \chi(G) \le 6$.



proof: Without loss of generality, let's assume $G$ has no parallel edges.

Also, assume $G$ has at least 2 edges.

Hence $\boxed{e \le 3v-6}$ where $e = \#E$, $v = \#V$

This inequality implies $G$ has at least one vertex $v_0$ with $\deg_G(v_0) \le 5$,

otherwise

$$2e = \sum_{v \in V} \underbrace{\deg_G(v)}_{\ge 6} \ge 6v$$

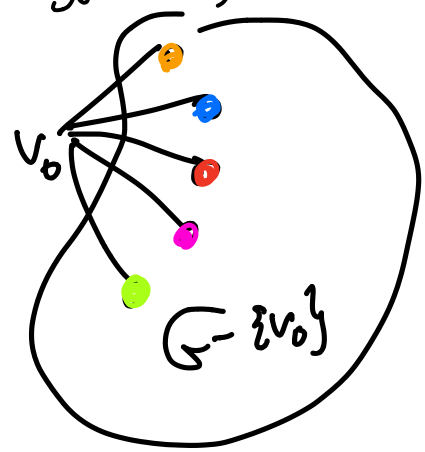
divide by 2

$$e \ge 3v$$

which contradicts $e \le 3v-6$.

We now use this to show $\chi(G) \le 6$ via induction on $\#V$: BASE CASE where $|V| \le 6$, one can color them with 6 different colors, and in the inductive step, find some $v_0 \in V$ with $\deg_G(v_0) \le 5$, inductively 6-color $G - \{v_0\}$

We now use this to show $\chi(G) \leq 6$ via induction on $\#V$: BASE CASE where $|V| \leq 6$, one can color them with 6 different colors, and in the inductive step, find some $v_0 \in V$ with $\deg_G(v_0) \leq 5$, inductively 6-color $G - \{v_0\}$




Since there are at most 5 colors used on the neighbors of $v_0$, a color is left for $v_0$ $\blacksquare$

Now let's use Kempe's (1879) idea to prove...

<u>5-color Theorem</u>: $G$ loopless planar $\Rightarrow \chi(G) \leq 5$.

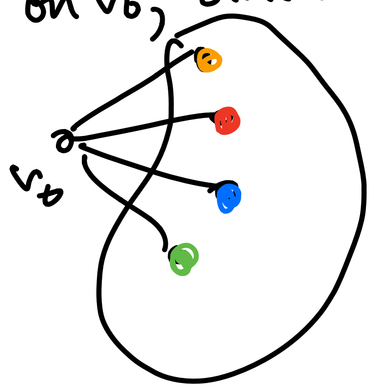
<u>proof</u>: Again, prove it by induction on $|V| = v$.

BASE CASE: If $|V| \leq 5$, color every vertex differently.

INDUCTIVE STEP: Again, use the existence of at least one vertex $v_0$ with $\deg_G(v_0) \leq 5$.

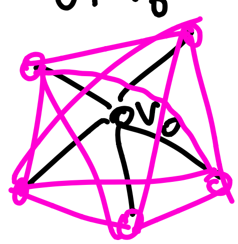
CASE 1: $\deg_G(v_0) \leq 4$

Do the Brooks argument: 5-color $G - \{v_0\}$ properly, and there is a color left over to use on $v_0$, since its neighbors use at most 4.




CASE 2: $\deg_G(v_0) = 5$.

~~CASE 2(a)~~: Every pair $\{u, v\}$ of neighbors of $v_0$ has an edge in $G$



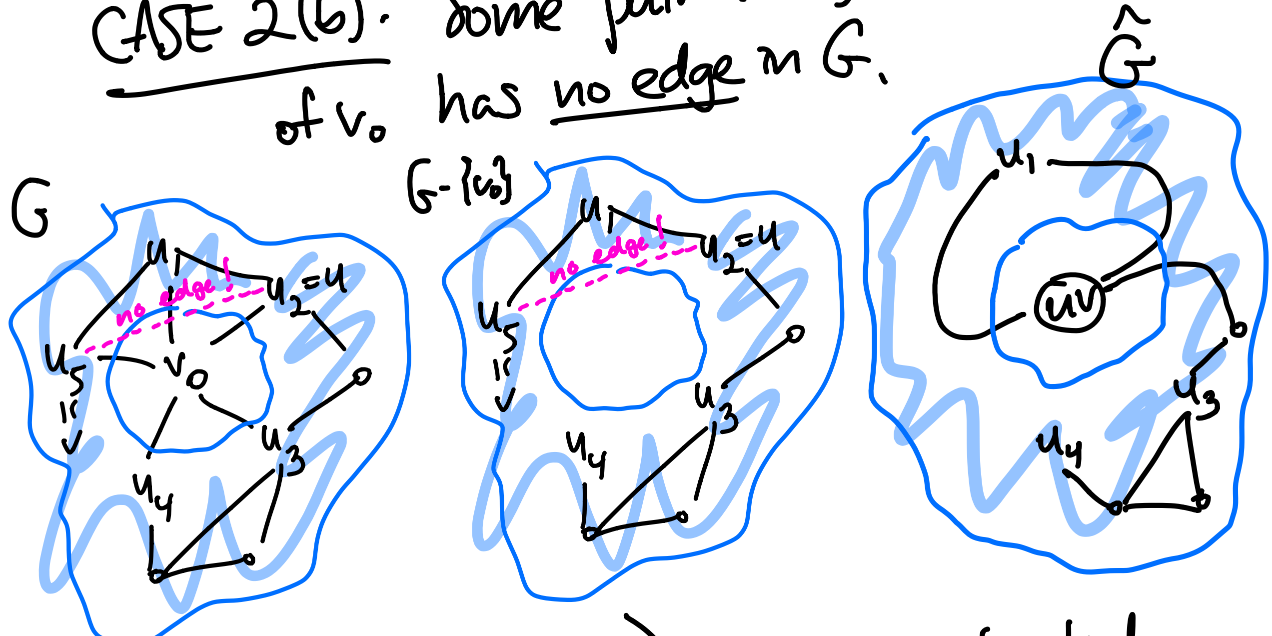

Impossible, since this would be a $K_5$ (nonplanar) inside $G$ (planar)!

CASE 1: $\deg_G(v_0) \leq 4$

CASE 2: $\deg_G(v_0) = 5$.

~~CASE 2(a)~~: Every pair $\{u,v\}$ of neighbors of $v_0$ has an edge in $G$

CASE 2(b): Some pair $\{u,v\}$ of neighbors of $v_0$ has no edge in $G$.

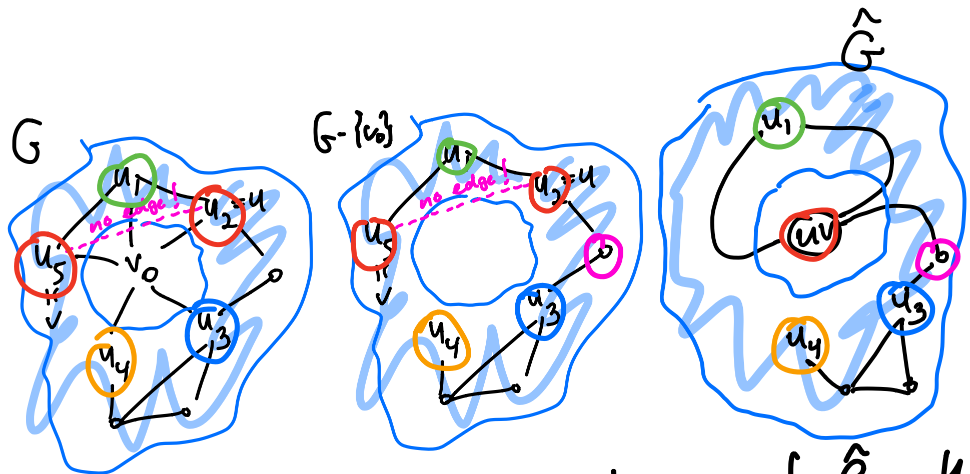



Consider $\hat{G} := \left(G - \{v_0\}\right)$ with $u, v$ contracted together,

which is still planar,

and loopless (because $\{u,v\}$ was not an edge of $G$).

Since $\hat{G}$ has fewer vertices, it has a proper 5-coloring, by induction.




Given any proper 5-coloring of $\tilde{G}$, that is the same thing as a proper 5-coloring of $G-\{v_0\}$ in which $u, v$ were assigned the same color. This lets us complete a proper 5-coloring for $G$, since the neighbors of $v_0$ use $\leq 4$ colors. $\blacksquare$

Eventually the 4-color theorem was proven in 1976 by Appel & Haken, but using a computer to check many cases (called "unavoidable configuration").

# The chromatic polynomial (not in book)

(Birkhoff & Lewis 1946)

In an attempt to prove the 4-color theorem, they defined...

<u>DEF'N</u>: $\chi(G, k)$ = # of proper vertex $k$-colorings of $G$

the <u>chromatic polynomial</u> of $G$

(so $\chi(G,k) \neq 0 \iff \chi(G) \le k$)

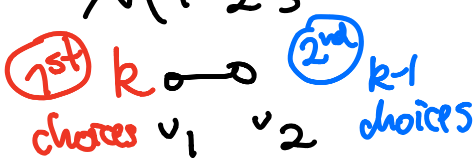
<u>EXAMPLES</u>:

1. Complete graphs $K_n$

$\chi(K_1, k) = k$

$K_1 = \circ$

$\chi(K_2, k) = k(k-1)$



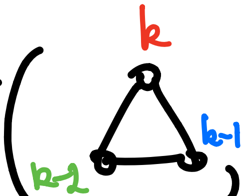
EXAMPLES:

① Complete graphs $K_n$

$\chi(K_1, k) = k$

$\chi(K_2, k) = k(k-1)$

1st: $k$ choices, $v_1$ — $v_2$, 2nd: $k-1$ choices

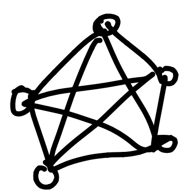
$\chi($  $, k) = k(k-1)(k-2)$



$K_3$

$$\chi(K_n, k) = k(k-1)(k-2)\cdots(k-(n-1))$$

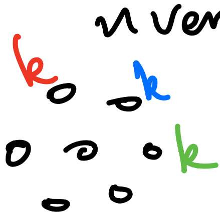
e.g. $\chi(K_5, k) = k(k-1)(k-2)(k-3)(k-4)$

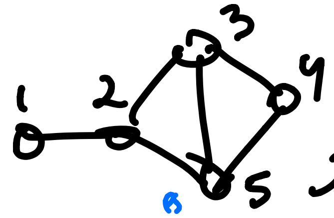


evaluate at $k = 1, 2, 3, 4$ → $0$

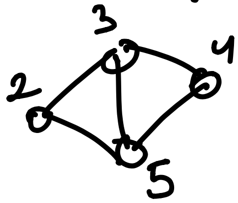
evaluate it at $k \geq n$ → $n!\binom{k}{n} = \dfrac{k!}{(k-n)!}$

② $G$ has no edges, $n$ vertices, has $\chi(G, k) = k^n$

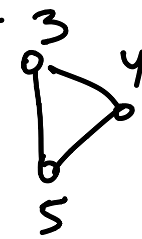


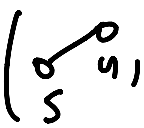
③ $\chi\Big($  $, k\Big) = ?$

$G$

$= \underbrace{(k-1)}_{\text{\# of choices for coloring vertex 1, given any proper } k\text{-coloring of } G-\{1\}} \cdot \chi\Big($  $, k\Big)$

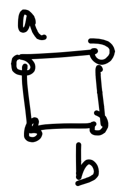
$G-\{1\}$

$= (k-1)\underbrace{(k-2)}_{\text{\# of choices for coloring vertex 2 given any coloring of } G-\{1,2\}} \chi\Big($  $, k\Big)$

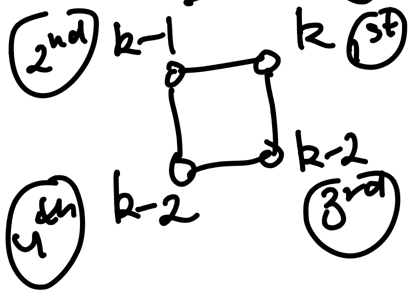
$= (k-1)(k-2)(k-2)\,\chi\Big($  $, k\Big)$

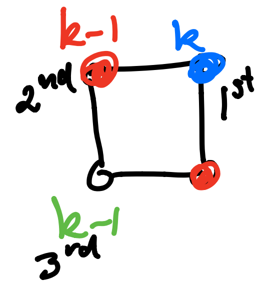
$= (k-1)(k-2)(k-2)(k-1)\cdot k$

$= k(k-1)^2(k-2)^2$

④ $\chi\Big($ 

$, k\Big) = \#\left\{\text{proper } k\text{-colorings with } a, b \text{ colored differently}\right\} + \#\left\{\text{those with } a, b \text{ colored same}\right\}$




$$= k(k-1)(k-2)^2 + k(k-1)(k-1)$$

$$\vdots$$

$$= k^4 - 4k^3 + 6k^2 - 3k$$

a polynomial in $k$.