The chromatic polynomial (not in book)

(Birkhoff & Lewis 1946)

In an attempt to prove the 4-color theorem, they defined...

DEF'N: $\chi(G, k) =$ # of proper vertex k-colorings of G

the chromatic polynomial of G

(so $\chi(G,k) \neq 0 \iff \chi(G) \leq k$)

In particular, if one could show $\chi(G,4) \neq 0$ for planar G this would prove the 4-color theorem.

EXAMPLES:

① Complete graphs K_n

$$\chi(K_1, k) = k$$

$$\chi(K_2, k) = k(k-1)$$

1st: k choices, v_1 v_2, 2nd: $k-1$ choices

EXAMPLES:

① Complete graphs K_n

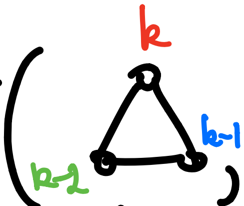
$\chi(K_1, k) = k$

$K_1 = \circ$

$\chi(K_2, k) = k(k-1)$

1st: k choices  2nd: $k-1$ choices

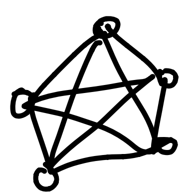
v_1 v_2

$\chi($  $, k) = k(k-1)(k-2)$

K_3

$$\chi(K_n, k) = k(k-1)(k-2)\cdots(k-(n-1))$$

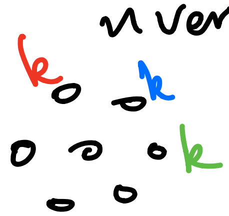
e.g. $\chi(K_5, k) = k(k-1)(k-2)(k-3)(k-4)$

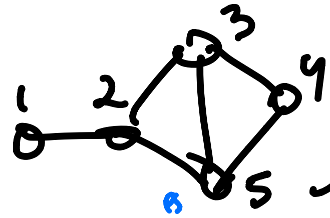


evaluate at $k = 1, 2, 3, 4$ → 0

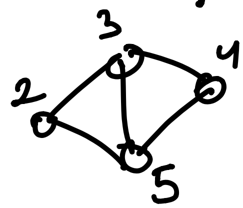
evaluate it at $k \geq n$ → $n! \binom{k}{n} = \frac{k!}{(k-n)!}$

② G has no edges, n vertices, has $\chi(G, k) = k^n$



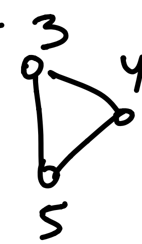
③ $\chi\Big($  $, k\Big) = ?$

G

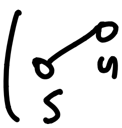
$= (k-1)\cdot\chi\Big($  $, k\Big)$

$G - \{1\}$

# of choices for coloring vertex 1, given any proper k-coloring of $G - \{1\}$

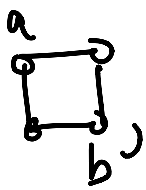
$= (k-1)(k-2)\,\chi\Big($  $, k\Big)$

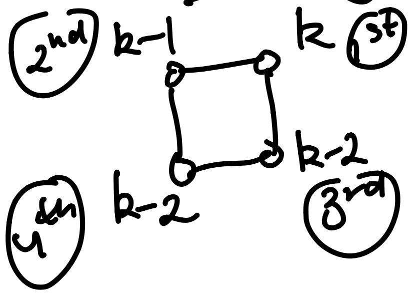
# of choices for coloring vertex 2 given any coloring of $G - \{1,2\}$

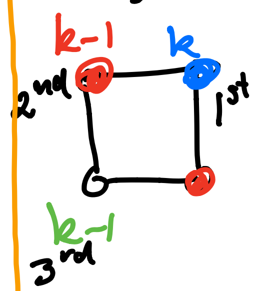
$= (k-1)(k-2)(k-2)\,\chi\Big($  $, k\Big)$

$= (k-1)(k-2)(k-2)(k-1)\cdot k$

$= k(k-1)^2(k-2)^2$

4 $\chi\Big($  $, k\Big) = \#\left\{\begin{array}{c}\text{proper } k\text{-colorings with}\\ a, b \text{ colored}\\ \text{differently}\end{array}\right\} + \#\left\{\begin{array}{c}\text{those}\\ \text{with}\\ a, b\\ \text{colored}\\ \text{same}\end{array}\right\}.$





$$= k(k-1)(k-2)^2 + k(k-1)(k-1)$$

$$\vdots$$

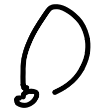
$$= k^4 - 4k^3 + 6k^2 - 3k$$

a polynomial in k.

Math 4707 Dec. 9, 2020

PROPOSITION (Birkhoff & Lewis 1946)

Let $G=(V,E)$, $n=\#V$



EASY! ① $\chi(G,k)=0$ if G has any loops

② If e is a non-loop edge in G, then

$$\chi(G,k) = \chi(\underbrace{G\setminus e}_{\text{deletion of } e},k) - \chi(\underbrace{G/e}_{\text{contraction on } e},k)$$

③ If G has no loops, $\chi(G,k)$ is a polynomial in k, thought of as a variable, of the following form:

$$\chi(G,k) = k^n - (\#E)k^{n-1} + a_{n-2}k^{n-2} - a_{n-3}k^{n-3} + \dots \pm a_c k^c$$

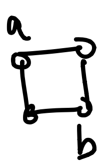
where $c :=$ # of connected components in G

and $a_i > 0$ for $i = c, c+1, \dots, a_{n-2}$

EXAMPLE

$n=4$

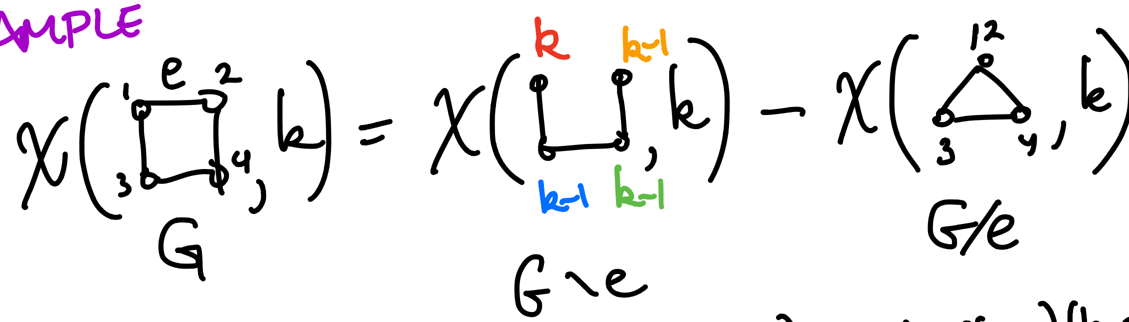
$\#E=4$

$\chi($ 

$, k) = k^4 - 4k^3 + 6k^2 - 3k^1$

$c=1$ connected component

EXAMPLE



$$= k(k-1)(k-1)(k-1) - k(k-1)(k-2)$$
$$= k(k-1)^3 - k(k-1)(k-2)$$
$$= k(k-1)\left[(k-1)^2 - (k-2)\right]$$
$$= k(k-1)\left[k^2 - 3k + 3\right]$$
$$= k^4 - 4k^3 + 6k^2 - 3k \quad \checkmark$$

---

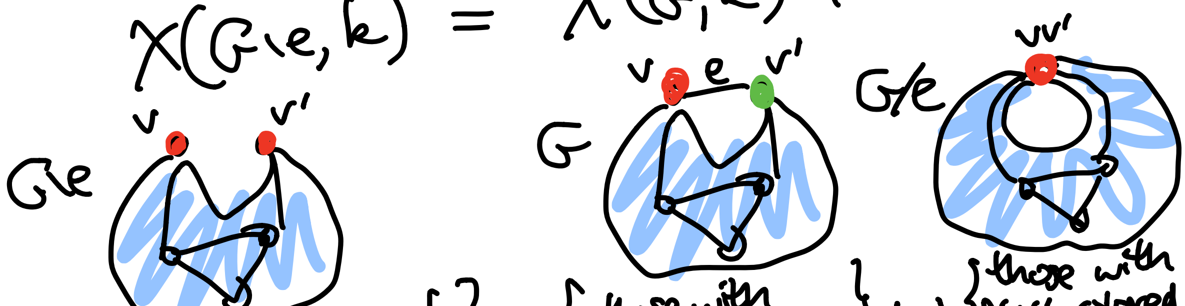
proof of:

(2) If $e = \{v, v'\}$ is a non-loop edge in G, then

$$\chi(G,k) = \chi(\underbrace{G\setminus e}_{\text{deletion of } e}, k) - \chi(\underbrace{G/e}_{\text{contraction on } e}, k)$$

It's equivalent to show:

$$\chi(G\setminus e, k) = \chi(G,k) + \chi(G/e, k)$$



$$\{\text{proper } k\text{-colorings of } G\setminus e\} = \{\text{those with } v, v' \text{ colored \underline{differently}}\} \sqcup \{\text{those with } v, v' \text{ colored \underline{same}}\}$$

$$\chi(G,k) = \chi(\underbrace{G \setminus e}_{\text{deletion of } e}, k) - \chi(\underbrace{G/e}_{\text{contraction on } e}, k)$$

If G has no loops,

③ $\chi(G,k)$ is a polynomial in k, thought of as a variable, of the following form:

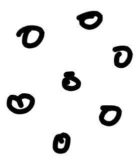
$$\chi(G,k) = k^n - (\#E)k^{n-1} + a_{n-2}k^{n-2} - a_{n-3}k^{n-3} + \dots \pm a_c k^c$$

where $c :=$ # of connected components in G

and $a_i > 0$ for $i = c, c+1, \dots, a_{n-2}$

proof of ③: Induction on $\#E$.

BASE CASE: $\#E = 0$



G

n vertices

$$\chi(G,k) = k^n = k^n - 0 \cdot k^{n-1} + 0 \cdot k^{n-2} - \dots \pm$$

fits the pattern in ③.

$c = n$

$\#E = 0$

INDUCTIVE STEP: Assume G has some nonloop edge e

$$\chi(G,k) = \chi(G \setminus e, k) - \chi(G/e, k)$$

use induction on $\#E$

$$= k^n - (\#E - 1)k^{n-1} + a'_{n-2}k^{n-2} - \dots \pm a'_{c'}k^{c'} - (k^{n-1} - a''_{n-2}k^{n-2} + \dots \pm a''_{c''}k^{c''})$$

where $a'_i, a''_j > 0$

where $\boxed{\underbrace{c''}_{c(G/e)} = \underbrace{c}_{c(G)} \le \underbrace{c'}_{c(G \setminus e)}}$

$$= k^n - (\#E)k^{n-1} + a_{n-2}k^{n-2} - a_{n-3}k^{n-3} + \dots \pm a_c k^c \quad \blacksquare$$

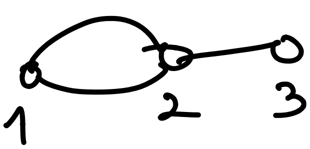
Acyclic orientations and the (-1)-color theorem

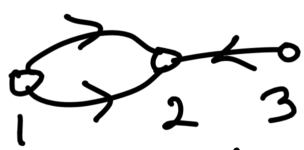
(not in book)

DEF'N: If $G=(V,E)$ is undirected, then an orientation ω of G is a choice of orientation for each edge $e=\{v,v'\}$ $\overset{v}{\circ}\overset{e}{—}\overset{v'}{\circ}$ either as $\overset{v}{\circ}\rightarrow\overset{v'}{\circ}$ or $\overset{v}{\circ}\leftarrow\overset{v'}{\circ}$ to make a directed graph $D=(V,A)$ from G.

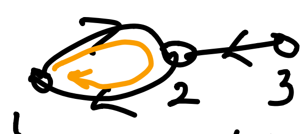
It's an acyclic orientation if it has no directed cycles.

e.g.

$G=$ 



an acyclic orientation



an orientation, but not acyclic

Let $\Omega(G) := \{$all acyclic orientations ω of $G\}$

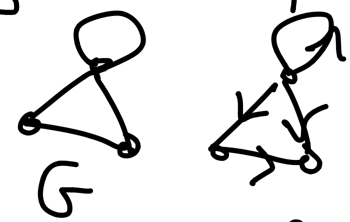
Q: What $\#\Omega(G)$?

EXAMPLES:

① $\Omega(G) = 0 \iff G$ has a <u>loop</u>.

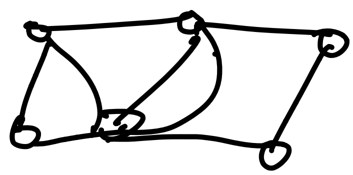
$\Longleftarrow$ G has a loop

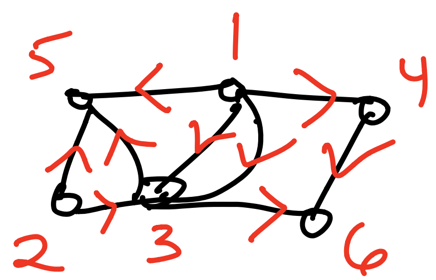
EASY

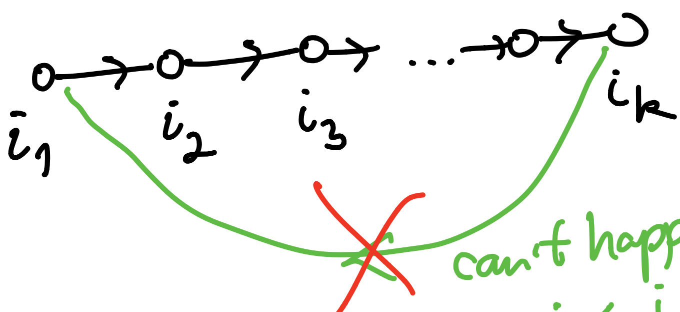


To see why a loopless G has some orientation, number the vertices $1, 2, \ldots, n$ and orient edges i — j with $i \lneq j$

as $i \to j$

$G =$ 





can't happen

since $i_1 \lneq i_2 \lneq \cdots \lneq i_k$

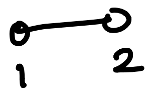
so $i_k \not\lneq i_1$

② Complete graphs K_n

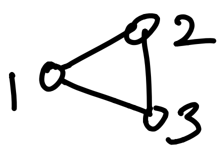
$\#\Omega(K_1) = 1 = 1!$

$\Omega(K_2) = \{ \underset{1}{\circ} \to \underset{2}{\circ} \, , \, \underset{1}{\circ} \leftarrow \underset{2}{\circ} \}$

$\#\Omega(K_2) = 2 = 2!$



$\#\Omega(K_3) = 2^3 - 2 = 6 = 3!$

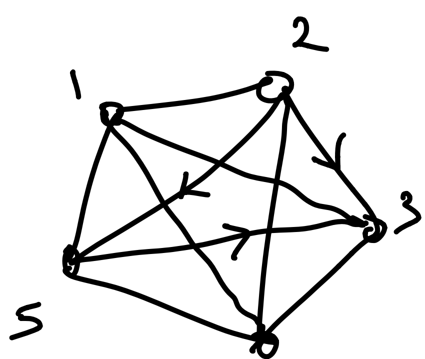


$=$ 

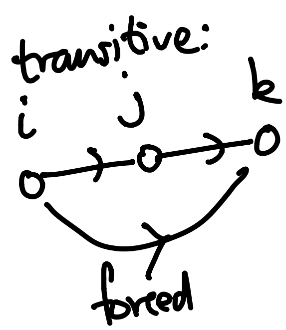
$\#\Omega(K_n) = \#$ linear orderings of $V = \{1,2,3,\dots,n\}$

$= n!$

not obvious!

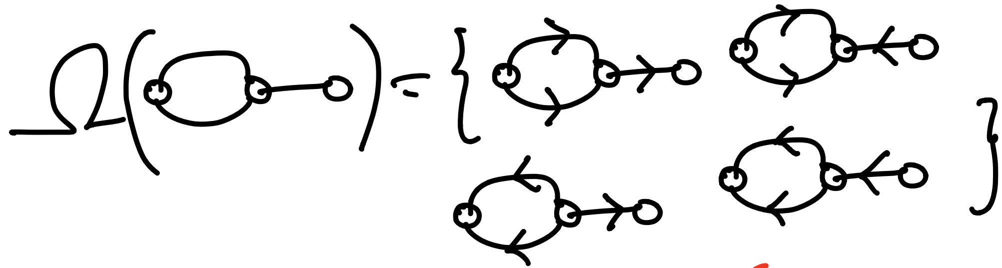


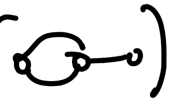
transitive:

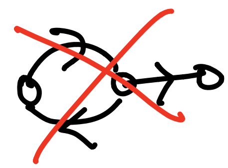


EXERCISE:

Show $\exists$ some i with $i \to j$ for all j, and induct to get a linear ordering on $\{1,2,\dots,n\}$

③ 

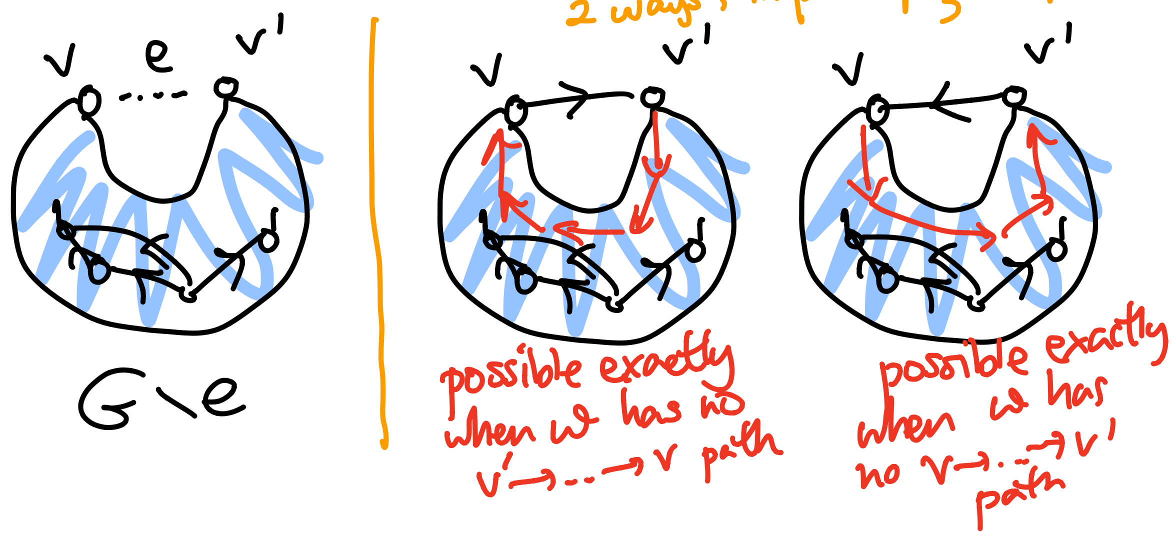
$\#\Omega($  $) = 4$



PROPOSITION: Assume e is a non-loop of G.
Then $\#\Omega(G) = \#\Omega(G \setminus e) + \#\Omega(G/e)$

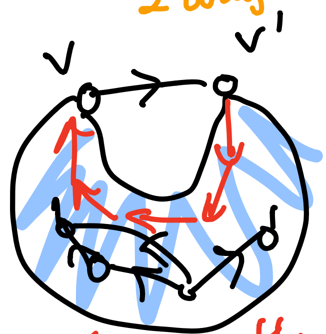
proof: Consider any acyclic orientation ω of $G \setminus e$.
How many ways can I extend ω by choosing a direction on e to get an acyclic orientation of G.

2 ways, in principle are possible:



$G \setminus e$

2 ways, in principle are possible:



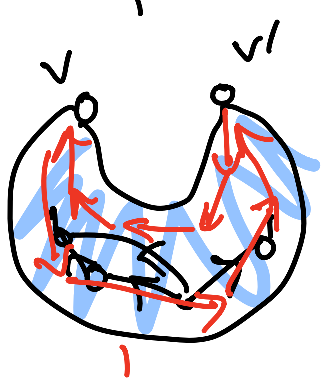
possible exactly when ω has no $v' \to \cdots \to v$ path

possible exactly when ω has no $v \to \cdots \to v'$ path

Sometimes, ω can only be directed one way, either $v \to v'$ or $v' \to v$

Can one ever have 0 ways?

<u>No</u>, because one would need both obstructions present in ω on $G \setminus e$



Now, calculate with:

$\alpha_1 = \#$ acyc. orientations of $G \setminus e$ that extend 1 way only

$\alpha_2 = \#$ acyc. orientations of $G \setminus e$ that extend both ways.

$= \#$ acyc. orientations of G/e

THINK ABOUT IT!

Now, calculate with:

$\alpha_1 = \#$ acyc. orientations of $G \setminus e$ that extend 1 way only

$\alpha_2 = \#$ acyc. orientations of $G \setminus e$ that extend both ways.

$= \#$ acyc. orientations of G/e

THINK ABOUT IT!

THINK ABOUT IT!

$$\#\Omega(G) = \alpha_1 + 2\alpha_2$$

$$\#\Omega(G\setminus e) = \alpha_1 + \alpha_2$$

$$\#\Omega(G/e) = \alpha_2$$

$$\Longrightarrow \#\Omega(G) = \#\Omega(G\setminus e) + \#\Omega(G/e)$$

COROLLARY (Stanley's "(-1)-color theorem")

$$\#\Omega(G) = (-1)^{\#V}\chi(G,-1)$$

proof: Induct on $\#E$. BASE CASE $\#E=0$

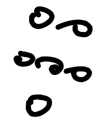
$\#\Omega(G)=1$

$\chi(G,k)=k^n$

$(-1)^{\#V}\chi(G,-1) = (-1)^n \cdot (-1)^n = 1$

COROLLARY ("Stanley's (−1)-color theorem")

$$\#\Omega(G) = (-1)^{\#V}\chi(G,-1)$$

proof: Induct on $\#E$. BASE CASE $\#E=0$ 

$\#\Omega(G)=1$

$\chi(G,k)=k^n$

$(-1)^{\#V}\chi(G,-1) = (-1)^n\cdot(-1)^n = 1$

INDUCTIVE STEP: Pick a nonloop edge e,

$$\#\Omega(G) = \#\Omega(G\setminus e) + \#\Omega(G/e)$$

$$= (-1)^{\#V}\chi(G\setminus e,-1) + (-1)^{\#V-1}\chi(G/e,-1)$$

by induction

$$= (-1)^{\#V}\left[\chi(G\setminus e,-1) - \chi(G/e,-1)\right]$$

$$= (-1)^{\#V}\chi(G,-1) \quad \blacksquare$$

EXAMPLES:

① $\chi(K_n,k) = k(k-1)(k-2)\cdots(k-(n-1))$

$(-1)^{\#V}\chi(K_n,-1) = (-1)^n(-1)(-2)(-3)\cdots(-n) = n! = \#\Omega(K_n)$

② $\chi($ 

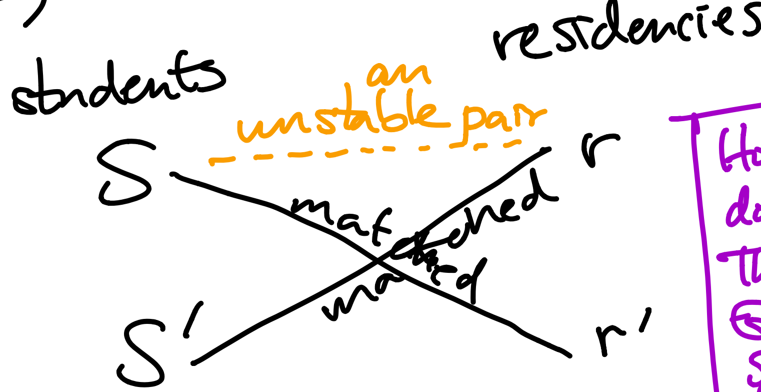
$, k) = k(k-1)^2$ $\quad (-1)^3\chi($  $,-1)$

$= (-1)^3(-1)(-2)^2 = 4$

Math 4707 Dec. 14, 2020

Stable matchings

Example: The National Residency Matching Program (NRMP) matches graduating medical students to residency program slots, each with lists of preferences for each other, so that things are stable in the sense that one avoids a matched pair (S, r)



but S prefers r to r'
and r prefers S to S'

How do they do this? They used Gale-Shapley algorithm (1962).
since 1952!

REMARK: Also used in NYC school system for 8th graders and high school slots.

[In 2012, Alvin Roth & Lloyd Shapley got Econ Nobel Prize for their work on stable matchings.]

Let's make some assumptions, then fix them later.

Assume n students and n residency slots.

Assume each linearly orders all n of the otherside.

EXAMPLE: Cosmetic pulmonology has only 4 slots, 4 students (A, B, C, D)

U. Minn. has two slots m_1, m_2
Rochester-Mayo has 1 slot r
Harvard has 1 slot h
with these preferences...

EXAMPLE: Cosmetic pulmonology has only 4 slots, 4 students (A, B, C, D)

U. Minn. has two slots m_1, m_2
Rochester-Mayo has 1 slot r
Harvard has 1 slot h
with these preferences...

preferences

same since both U. Minn.

A: m_1, m_2, r, h	m_1: A, B, C, D, Ω
B: m_1, m_2, h, r	m_2: A, B, C, D, Ω
C: m_1, m_2, r, h	r: B, A, D, C, Ω
D: r, m_1, m_2, h	h: B, C, A, D, Ω

fake horrible student Ω = Dr. Jekyll

The algorithm starts with "conditional matchings"

$\Omega - m_1$
$\Omega - m_2$
$\Omega - r$
$\Omega - h$

[all matched to Ω]

In a cycle of the algorithm, a proposer S starts out being unmatched and proposes to the highest residency r left on their list.

If r was matched to S' that they prefer to S

then they reject S, who crosses r off their list, and continues as proposer.

If r was matched to S' that they like worse than S, then they reject S', who crosses r off their list, r conditionally matches with S, and S' becomes the new proposer.

The cycle ends when Ω is rejected (and doesn't cross off anyone,) doesn't become proposer.

The number k of non-Jekyll-matched slots increases by 1.

Algorithm stops in n cycles, when $k=n$.

A: m_1, m_2, r, h	m_1: A, B, C, D, Ω
B: $\cancel{m_1}, m_2, h, r$	m_2: A, B, C, D, Ω
C: m_1, m_2, r, h	r: B, A, D, C, Ω
D: r, m_1, m_2, h	h: B, C, A, D, Ω

Cycle 1: $S = A$ proposes to m_1, rejecting Ω

proposer

Cycle over, with

$A - m_1$

$\Omega - m_2$

$\Omega - r$

$\Omega - h$

Cycle 2: $S = B$ proposes to m_1,

who reject B, as m_1 prefers A.

B crosses m_1 off their list

$S = B$ proposes to m_2, who says conditional yes, rejecting Ω.

Cycle over, with

$A - m_1$

$B - m_2$

$\Omega - r$

$\Omega - h$

$A: m_1, m_2, r, h$ | $m_1: A, B, C, D, \Omega$
$B: \cancel{m_1}, m_2, h, r$ | $m_2: A, B, C, D, \Omega$
$C: \cancel{m_1}, \cancel{m_2}, \cancel{r}, h$ | $r: B, A, D, C, \Omega$
$D: r, m_1, m_2, h$ | $h: B, C, A, D, \Omega$

Cycle 3: $S = C$ proposes to m_1, rejected
$S = C$ proposes to m_2, rejected
$S = C$ proposes to r, conditionally accepted & rejecting Ω

Cycle over, with
$A - m_1$
$B - m_2$
$C - r$
$\Omega - h$

Cycle 4: $S = D$ proposes to r, who conditionally accepts them, rejecting C.
$S = C$ proposes to h, who accepts, rejecting Ω.

Cycle over with
$A - m_1$
$B - m_2$
$D - r$
$C - h$

Does it work?

<u>PROP</u>: The algorithm never gets stuck, no student runs out of their list, and it finishes in n^2+n steps (proposals).

<u>proof</u>: Notice that only Ω can be matched to multiple slots. So if a student S has crossed every slot off their list, then each of those slots rejected S and is matched to some real student and there are only $n-1$ other real students.

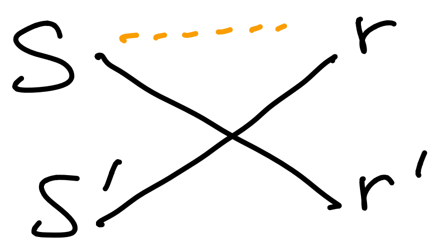
There are at most n^2+n proposals since n of them end up with Ω being rejected, and all of the other proposals cause a slot to be crossed off someone's list. There are $\leq n \cdot n$ positions to cross off. $\blacksquare$

PROP: The matching produced is stable.

proof: KEY OBSERVATION:

- A residency slot's match only improves over time.
- A student's match only gets worse over time.

Now suppose we ended up with an unstable pair: (S, r)



S prefers r to r'

r prefers S to S'

S must never have proposed to r during the algorithm since the KEY OBSERVATION says r only saw proposals from students it liked less than S'. Since S matched with r', it must have crossed r off its list at some point. So S did propose to r. contradiction $\boxtimes$

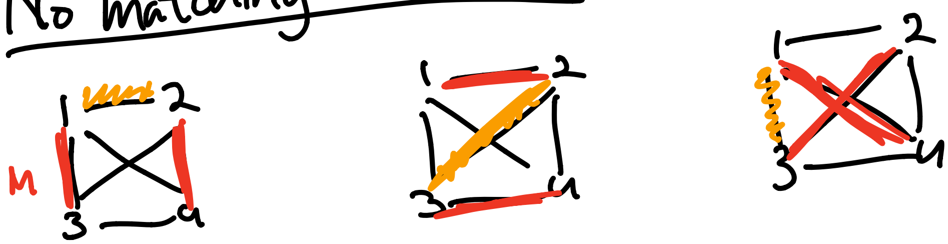
REMARK: Gale & Shapley (1962) pointed out that nonbipartite graphs & preferences ("the roommate problem") can have no stable matchings

EXAMPLE:

1, 2, 3, 4 people

1 likes 2 best
2 likes 3 best
3 likes 1 best
and everybody hates 4, likes them worst.

No matching is stable:



unstable pair

The Gale-Shapley matching result has an interesting characterization:

THEOREM:

Let S, r be matched in Gale-Shapley

proposers ... rejectors/acceptors

Then in any other stable matching M,

(i) if S is matched to some r' in M, then S prefers r to r' (or r is at least as good as r')

(ii) if r is matched to S' in M, then r prefers S' to S (or S' is at least as good as S)

In other words, Gale-Shapley gives, among all stable matchings,

- the worst outcome for all residencies r
- the best outcome for all students S.