Math 4707 Dec. 16, 2020 (Last day!!)

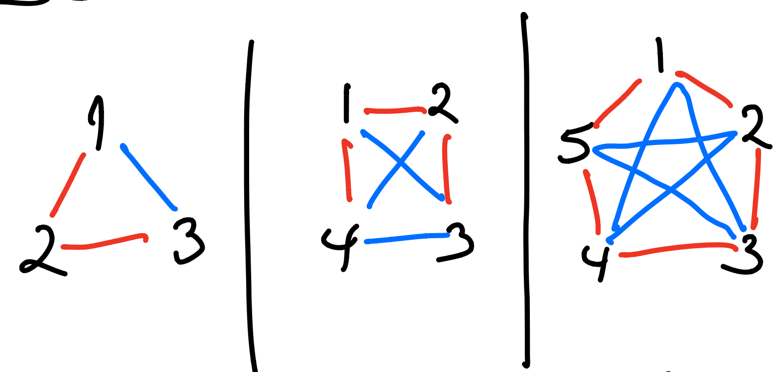
Ramsey numbers

(see Wikipedia entry on "Ramsey's Theorem")

Q: How many people need to walk into a room before we can be sure that there will either be
3 mutual acquaintances
or 3 mutual strangers ?

≤ 5 will not suffice:

e.g.

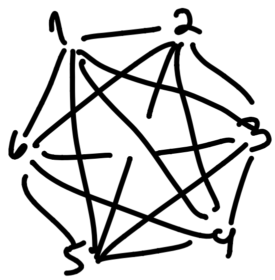


acquaintances (red)

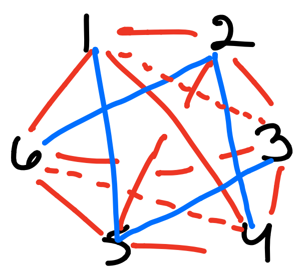
strangers (blue)

no monochromatic triangles! K_3
(all red or all blue)

CLAIM: 6 is big enough, that is,
if we color all the edges of K_6
either red or blue,



then there will always be
a monochromatic triangle K_3



proof of CLAIM:

Look at person #1, and either
they know ≥ 3 others
or they don't know ≥ 3 others,
since otherwise $\leq \underbrace{2+2}_{4 < 5}$ others

CASE 1: Person #1 knows ≥ 3 others.

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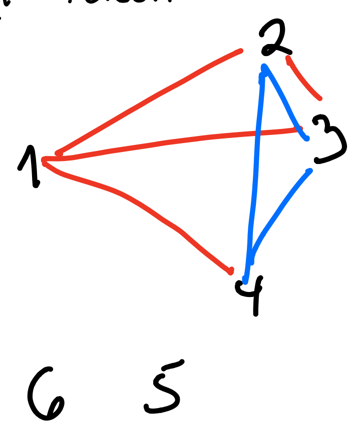

Subcase 1a: Some pair among {2,3,4} knows each other
This pair with 1 makes a red K_3

Subcase 1b: No pair among {2,3,4} knows each other.
Then they make a blue K_3

CASE 2: Person #1 does not know ≥ 3 others.
Same argument, swapping red & blue.

DEF'N: The (2-color) Ramsey number $R(k,\ell)$ for $k,\ell \geq 2$ is the smallest number n such that every red-blue 2-coloring of the edges of K_n leads to either a red K_k or a blue K_ℓ inside it.

We just showed $R(3,3) = 6$

complete red K_3 or complete blue K_3

THEOREM (Ramsey 1930, Erdös-Szekeres 1960)

$R(k,\ell)$ exists $\forall\, k,\ell \geq 2$, and satisfies

$$R(k,2) = k$$
$$R(2,\ell) = \ell$$
$$R(k,\ell) = R(\ell,k)$$

and $R(k,\ell) \leq \binom{k+\ell-2}{k-1} = \binom{k+\ell-2}{\ell-1}$

THEOREM (Ramsey 1930, Erdös-Szekeres 1960)

$R(k,l)$ exists $\forall\, k,l \geq 2$, and satisfies

$$R(k,2) = k = \binom{k+2-2}{k-1} = \binom{k}{k-1}$$

$$R(2,l) = l = \binom{2+l-2}{l-1} = \binom{l}{l-1}$$

✓ $R(k,l) = R(l,k)$

and $R(k,l) \leq \binom{k+l-2}{k-1} = \binom{k+l-2}{l-1}$

EXAMPLE We showed $R(3,3) = 6$

$$\leq \binom{3+3-2}{3-1} = \binom{4}{2} = 6$$

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proof of THM: $R(k,l) = R(l,k)$

comes from swapping the roles of red, blue.

$R(k,2) = k$ says in a red-blue coloring of edges of K_k, either there is a blue edge ($=$ blue K_2) or the whole thing is a red K_k.

By symmetry, $R(2,l) = l$.

To show $R(k,\ell) \overset{(*)}{\le} \binom{k+\ell-2}{k-1}$

we'll actually show

$$\boxed{R(k,\ell) \le R(k-1,\ell) + R(k,\ell-1)}$$

To see this implies the above bound, use induction on $k+\ell$

BASE CASE $k=\ell=2$ we checked $(*)$ holds.

INDUCTIVE STEP

$$
\begin{aligned}
R(k,\ell) &\boxed{\le} R(k-1,\ell) + R(k,\ell-1) \\
&\overset{\text{by induction}}{\le} \binom{k-1+\ell-1}{k-2} + \binom{k+\ell-1-1}{k-1} \\
&= \binom{k+\ell-2}{k-2} + \binom{k+\ell-2}{k-1} \\
&\overset{\text{Pascal's recurrence}}{=} \binom{k+\ell-1}{k-1}
\end{aligned}
$$

Why does

$$\boxed{R(k,\ell) \le R(k-1,\ell) + R(k,\ell-1)}$$

hold?

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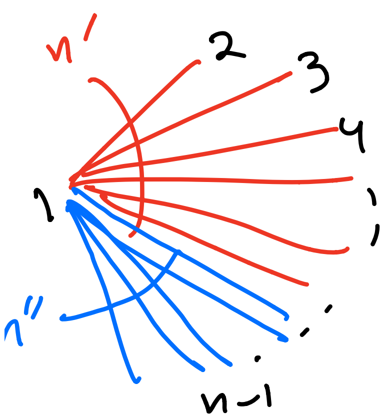
hold?

Assume $n \ge R(k-1,\ell) + R(k,\ell-1)$, and look in our red-blue coloring of edges of K_n at the numbers

$n' = $ # red edges emanating from vertex 1

$n'' = $ # blue edges emanating from vertex 1



Since $n' + n'' = n-1$
$\ge R(k-1,\ell) + R(k,\ell-1) - 1$,

either

$n' \ge R(k-1,\ell)$ CASE 1

or

$n'' \ge R(k,\ell-1)$ CASE 2

Assuming CASE 1, 1 has red edges to at least $R(k-1,\ell)$ vertices $\{v_2, v_3, \dots, v_{R(k-1,\ell)}\} = N$. Either N contains a red K_{k-1} (which combines with 1 to give a red K_k, or N contains a blue K_ℓ.

CASE 2 is symmetric in red & blue ∎

EXERCISE: $R(3,4) \le \binom{3+4-2}{3-1} = \binom{5}{2} = 10$

$R(3,4) = 9$

What are the actual values of $R(k,\ell)$?

Very few are known –
see the table on Wikipedia
page on Ramsey's Theorem.

We don't know a formula for $R(3,\ell)$!
nor a formula for $R(k,k)$!

We don't even know their asymptotics!

For $R(k,k)$ we get an upper bound

$$R(k,k) \le \binom{k+k-2}{k-1} = \binom{2(k-1)}{k-1} \underset{\sim}{\approx} \frac{1}{2\sqrt{\pi(k-1)}} 2^{2k-2}$$

asymptotically in k via Stirling's approx

$$\le 4^k$$

Accompanying the asymptotic upper bound $R(k,k) \leq 4^k$,

one can show

THM (P. Erdös 1947) $R(k,k) \geq (\sqrt{2})^k$ asymptotically in k

so $(\sqrt{2})^k \leq R(k,k) \leq 4^k$

but these bounds have not been (significantly) tightened since 1947!

Erdös's proof was the first example of what is called the "Probabilistic method" in combinatorics.

He showed that if $n < (\sqrt{2})^k$ (or some function roughly like this)

then a randomly chosen red-blue coloring of edges of K_n has expected number of monochromatic K_k's lying between 0 & 1, but <1. So some coloring has 0.