Math 4707 Mon Sept. 21

REMINDER: Wed Sept. 23 class is asynchronous (recorded) and (probably) no Thur Sept. 24 office hour.

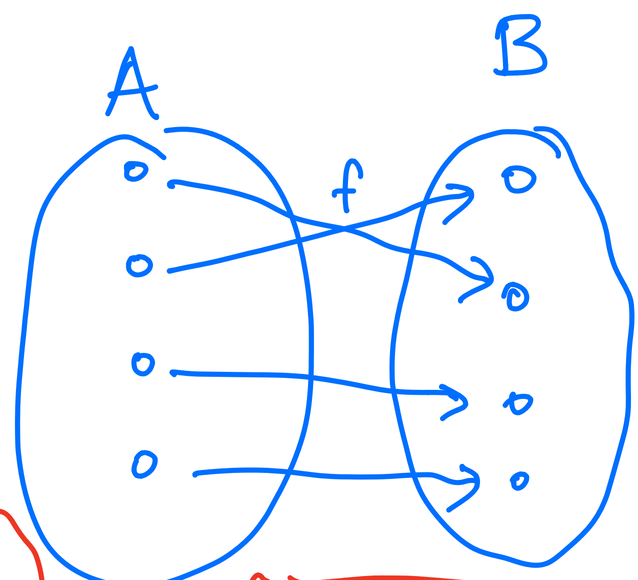
BIJECTION PRINCIPLE –

Suppose two sets A, B have a **bijection** $A \xrightarrow{f} B$

$$a \stackrel{f}{\longmapsto} f(a) = b$$

i.e. both

1–1, one-to-one, injective,	and	onto, surjective,



then $\#A = \#B$, and sometimes B is easier to count than A.

$$\#A = |A|, \quad \#B = |B|$$

EXAMPLES

① $\underbrace{\left\{ \substack{\text{all} \\ \text{subsets of } \{1,2,\dots,n\}} \right\}}_{A:=}$ Recall had $|A| = 2^n$

but A also has a nice bijection to $B = \{0,1,2,\dots,2^n-1\}$

(so $\#B = 2^n$
$\quad\quad\;\; \| $
$\quad\quad\; \#A$)

given by writing down the characteristic vector of length n consisting of 0's, 1's for the subset, and then reading it as a binary number:

e.g. $n=6$

$\{1,2,4,6\} \rightsquigarrow \overset{1\;2\;3\;4\;5\;6}{(1,1,0,1,0,1)}$

a subset $\quad$ characteristic vector

$$\rightsquigarrow \overset{5\,4\,3\,2\,1\,0}{110101}_{\text{two}}$$

$$= 1\cdot 2^5 + 1\cdot 2^4 + 0\cdot 2^3 + 1\cdot 2^2 + 0\cdot 2^1 + 1\cdot 2^0$$

$$= 32+16+4+1 = 53 \in \{0,1,2,\dots,2^6-1\} = B$$

This map $f: A \to B$ is a bijection since we can write down its inverse bijection f^{-1}:

given $m \in \{0,1,2,\dots,2^n-1\} = B$ write it uniquely in binary, and read the digits as the characteristic vector of a subset in A

e.g. $n=6$

$$m = 38 \in \{0,1,\dots,\underset{\substack{\parallel \\ 2^6-1}}{63}\} \rightsquigarrow 38 = 32+4+2$$

$$= 2^5 + 2^2 + 2^1$$

$$= 1\cdot 2^5 + 0\cdot 2^4 + 0\cdot 2^3 + 1\cdot 2^2 + 1\cdot 2^1 + 0\cdot 2^0$$

$$= \overset{1\,2\,3\,4\,5\,6}{100110}_{\text{two}}$$

$$\rightsquigarrow \{1,4,5\} \in A$$

$$\subset \{1,2,3,4,5,6\}$$

e.g. $n=3$

	123	
$\emptyset$	000_{two}	$= 0$
$\{1\}$	100_{two}	$= 4$
$\{2\}$	010	$= 2$
$\{3\}$	001	$= 1$
$\{1,2\}$	110	$= 6$
$\{1,3\}$	101	$= 5$
$\{2,3\}$	011	$= 3$
$\{1,2,3\}$	111_{two}	$= 7$

all of $\{0,1,\dots,\underset{\substack{\parallel \\ 2^3-1}}{7}\}$

② Which are there more of,
even-sized subsets
or
odd-sized subsets of $\{1,2,\ldots,n\}$?

n	even-sized	odd-sized
1	ϕ (1)	$\{1\}$ (1)
2	ϕ, $\{1,2\}$ (2)	$\{1\}$, $\{2\}$ (2)
3	ϕ, $\{1,2\}$ $\{1,3\}$ $\{2,3\}$ (4)	$\{1\}$ $\{2\}$ $\{3\}$ $\{1,2,3\}$ (4)
4	ϕ $\{1,2\}$ 13 14 23 24 34 1234 (8)	$\{1\}$ $\{2\}$ $\{3\}$ $\{4\}$ $\{123\}$ 124 134 234 (8)

GUESS: There are as many of each.

Q: Can we find a bijection ?

$$A \xrightarrow{f} B$$

$A = \{\text{even-sized subsets of } \{1,2,\ldots,n\}\}$

$B = \{\text{odd-sized subsets of } \{1,2,\ldots,n\}\}$

Q: Can we find a bijection ?

$$A \xrightarrow{f} B$$

$A = \{\text{even-sized subsets of } \{1,2,\dots,n\}\}$, $B = \{\text{odd-sized subsets of } \{1,2,\dots,n\}\}$

If n is odd, then set complementation

$$S \longmapsto \underbrace{\{1,2,\dots,n\} \setminus S}_{\text{then this has size } n-k}$$

if $\#S = k$

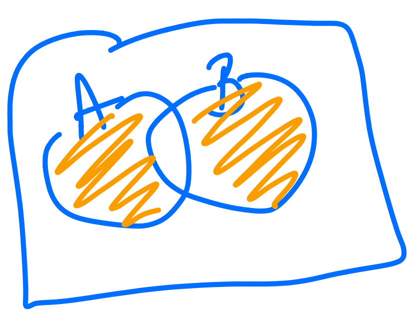
Here is such a bijection:

$$A \xrightarrow{f} B$$

$A = \{\text{even-sized subsets of } \{1,2,\dots,n\}\}$, $B = \{\text{odd-sized}\}$

$$S \overset{f}{\longmapsto} f(S) = \begin{cases} S - \{1\} & \text{if } 1 \in S \\ S \cup \{1\} & \text{if } 1 \notin S \end{cases}$$

$$= S \triangle \{1\}$$



symmetric difference

$A \triangle B = (A \setminus B) \cup (B \setminus A)$

Q: What is f^{-1}?

Same as f.

③ Prove $\binom{n}{k} = \binom{n}{n-k}$ two ways:

(a) directly from the formula with factorials

(b) via a bijection

$$\left\{\begin{matrix}k\text{-element}\\ \text{subsets of } \{1,2,\dots,n\}\end{matrix}\right\} \xrightarrow{f} \left\{\begin{matrix}n-k \text{ element}\\ \text{subsets of } \{1,2,\dots,n\}\end{matrix}\right\}$$

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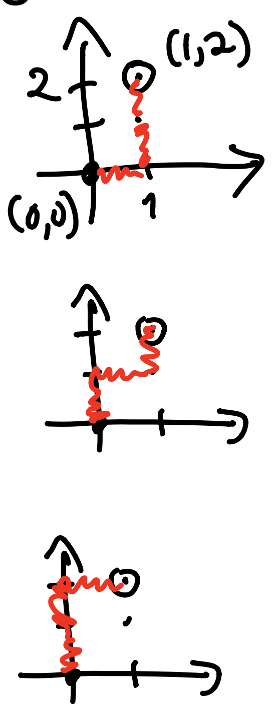
④ "Block-walking"

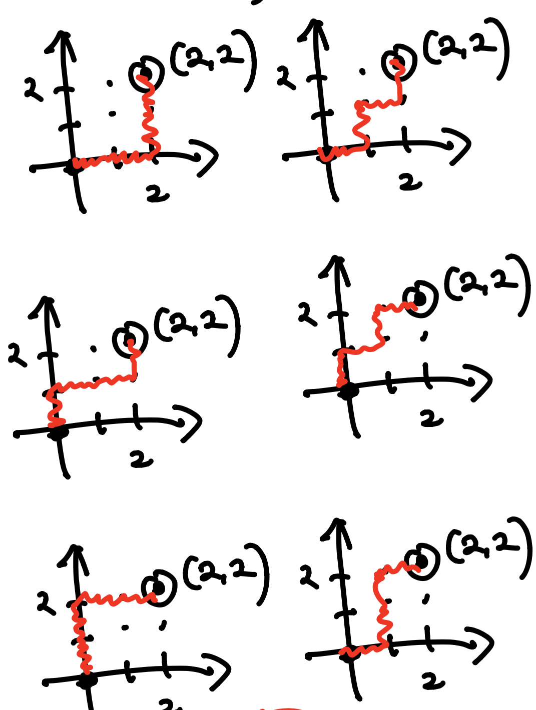
How many ways are there to walk from $(0,0)$ to (m,n) taking unit steps north or east at each step?

e.g. $(m,n) = (1,2)$



③ ways

$(m,n) = (2,2)$

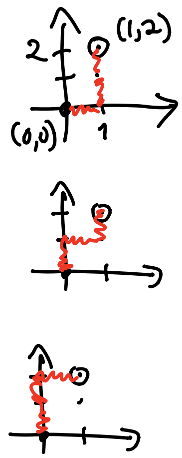


⑥ ways

4 "Block-walking"

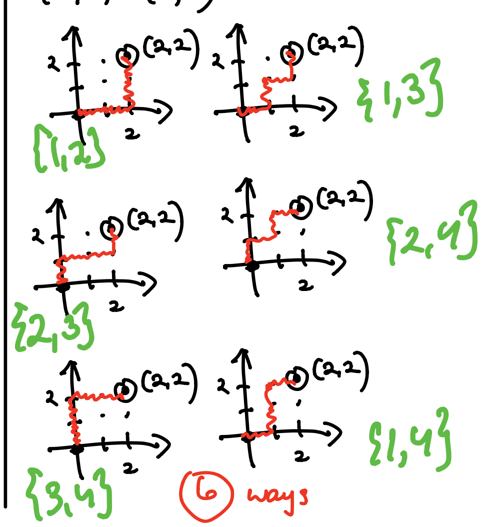
How many ways are there to walk from $(0,0)$ to (m,n) taking unit steps north or east at each step?

e.g. $(m,n)=(1,2)$



3 ways

$(m,n)=(2,2)$



6 ways

$\binom{4}{2}=6$

ANSWER: $\binom{m+n}{m} = \binom{m+n}{n} = \frac{(m+n)!}{m!\,n!}$

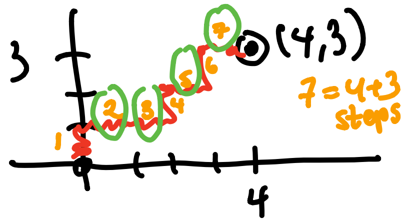
$\parallel$

\# m-element subsets of $\{1,2,\dots,m+n\}$

There's a bijection

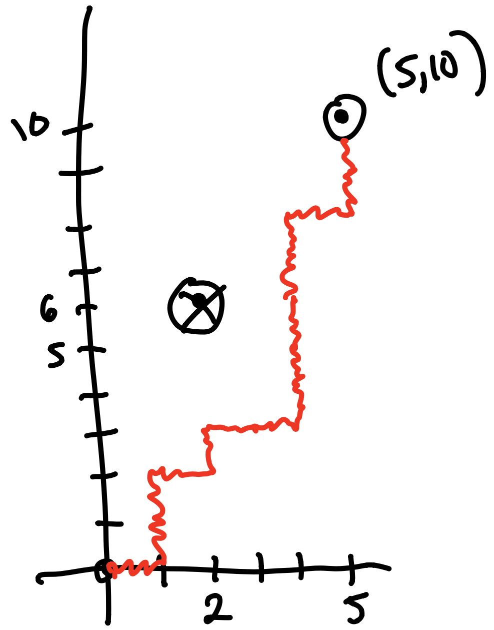
$$\left\{\begin{array}{l}\text{walks N, E}\\ \text{from } (0,0) \text{ to } (m,n)\end{array}\right\} \xrightarrow{f} \left\{\begin{array}{l}m \text{ element}\\ \text{subsets of}\\ \{1,2,\dots,m+n\}\end{array}\right\}$$

walk $W \xmapsto{f}$ subset of E steps



$\longmapsto \{2,3,5,7\}$

⑤ How many ways to walk from (0,0) to (5,10) that avoid the point (2,6)?



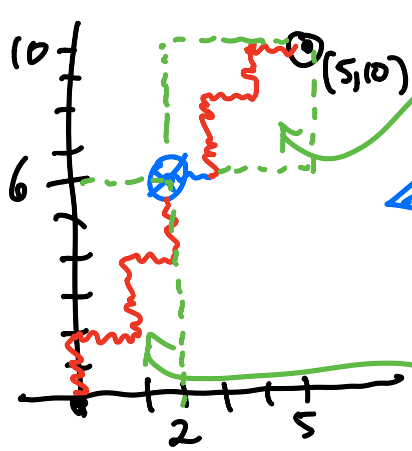
Use <u>difference principle</u>:

$A = \{\text{such walks}\}$

$= \underbrace{B}_{\text{all walks N, E from (0,0) to (5,10)}} - \underbrace{C}_{\text{bad walks, i.e. going through (2,6)}}$

$$|A| = |B| - |C|$$

$$= \binom{5+10}{5} - \binom{?}{} = \binom{5+10}{5} - \binom{2+6}{2}\binom{3+4}{3}$$

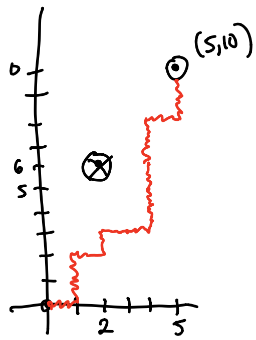


$\binom{\overset{5-2}{3}+\overset{10-6}{4}}{3}$ ways to go (2,6) to (5,10)

a bad walk

$\binom{2+6}{2}$ ways to go (0,0) to (2,6)

⑤ How many ways to walk from $(0,0)$ to $(5,10)$ that avoid the point $(2,6)$?



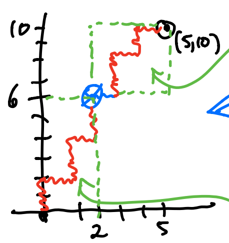
Use difference principle:

$A = \{\text{such walks}\}$

$$= \underbrace{B}_{\left\{\substack{\text{all walks} \\ N,E \text{ from} \\ (0,0) \text{ to } (5,10)}\right\}} - \underbrace{C}_{\left\{\substack{\text{bad walks,} \\ \text{i.e. going} \\ \text{through} \\ (2,6)}\right\}}$$

$$|A| = |B| - |C|$$

$$= \binom{5+10}{5} - (?) = \binom{5+10}{5} - \binom{2+6}{2}\binom{3+4}{3}$$



$\binom{3+4}{3}$ ways to go $(2,6)$ to $(5,10)$, with $3 = 5-2$, $4 = 10-6$

a bad walk

$\binom{2+6}{2}$ ways to go $(0,0)$ to $(2,6)$

What if we're walking $(0,0)$ to (m,n) avoiding going through (a,b)?

$0 \le a \le m$

$0 \le b \le n$

$$\binom{m+n}{m} - \binom{a+b}{a}\binom{(m-a)+(n-b)}{m-a}$$

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⑥ How many ways to choose your baker's dozen (13) of bagels from 5 flavors at Bruegger's?

{asiago, garlic, onion, plain, sesame}

There's a bijection

$A = \{\text{choices of 13 bagels from 5 flavors}\}$

$$\downarrow f$$

$B = \{\text{choices of 4 dividing lines between 13 symbols } \times\}$

⑥ How many ways to choose your baker's dozen (13) of bagels from 5 flavors at Bruegger's ?

$\{$asiago, garlic, onion, plain, sesame$\}$

There's a bijection

$$\begin{array}{l} A = \{\text{choices of 13 bagels from 5 flavors}\} \\ \quad \downarrow f \\ B = \{\text{choices of 4 dividing lines between 13 symbols } x\} \end{array}$$

3 asiago
0 garlic
1 onion
4 plain
5 sesame

$$\mapsto \left[\begin{array}{ccccccccc} xxx & | & & | & x & | & xxxx & | & xxxxx \\ \text{asiago} & & \text{garlic} & & \text{onion} & & \text{plain} & & \text{sesame} \end{array}\right]$$

$$= \underbrace{xxx||x|xxxx|xxxxx}_{\text{17 symbols total, } x\text{'s or } |\text{'s}}$$

$$17 = 13\ x\text{'s} + 4\ |\text{'s}$$

$$\binom{17}{4} \text{ choices}$$

$$\binom{17}{4} = \binom{13+4}{4} = \binom{13+4}{13}$$

PROPOSITION: The number of ways to pick a k-element multiset (= a set with multiplicities) from a set with n elements is

$k = 13$ above
13 x's

$n=5$ flavors $\rightsquigarrow$ $n-1$ dividers

$$\binom{k+n-1}{k} = \binom{k+n-1}{n-1}$$

proof: The above bijection ∎

Math 4707 Wed Sept. 23 (asynchronous lecture)

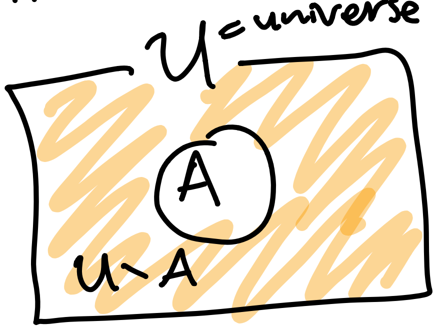
Our last basic counting principle:

PRINCIPLE OF INCLUSION-EXCLUSION (P.I.E.)

= How to deal with overlapping sets in counting problems without going crazy!

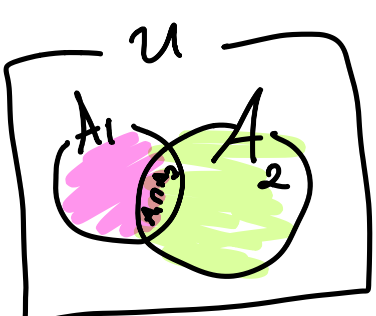
It generalizes

DIFFERENCE PRINCIPLE restated:



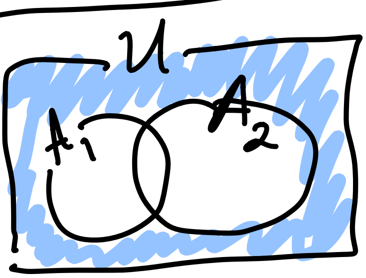
$$|\mathcal{U} \setminus A| = |\mathcal{U}| - |A|$$

What about 2 overlapping sets A_1, A_2?



What is $|A_1 \cup A_2|$?

$$= |A_1| + |A_2| - |A_1 \cap A_2|$$



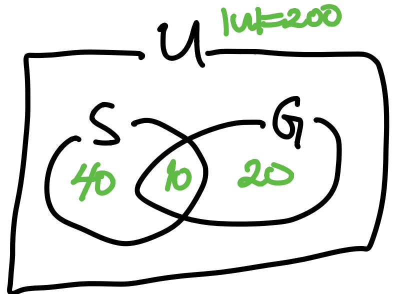
What is $|\mathcal{U} \setminus A_1 \cup A_2|$?

$$= |\mathcal{U}| - |A_1 \cup A_2|$$

$$= |\mathcal{U}| - |A_1| - |A_2| + |A_1 \cap A_2|$$

Either formula is P.I.E. for 2 sets.

EXAMPLE using P.I.E. for 2 sets



e.g $|U| = 200$ students in a school

$|S| = 50$ take Spanish

$|G| = 30$ take German

$|S \cap G| = 10$ take both

How many students take neither?

$$|U - S \cup G|$$

$$= |U| - |S| - |G| + |S \cap G|$$

$$= 200 - 50 - 30 + 10$$

$$= 130 \checkmark$$

$200 - 70$
$= 130$

How many take at least some language?

$40 + 10 + 20 = 70$

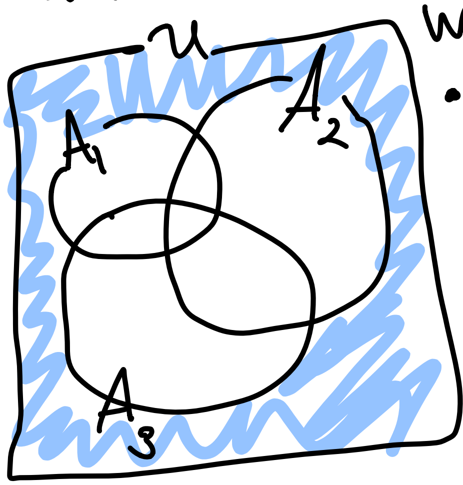
$$|S \cup G| = |S| + |G| - |S \cap G|$$

$$= 50 + 30 - 10$$

$$= 80 - 10$$

$$= 70 \checkmark$$

What about 3 overlapping sets?



What is

- $$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

P.I.E. for 3 sets

- $$|U - A_1 \cup A_2 \cup A_3| = |U| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|$$

What about for n overlapping sets?

PROPOSITION: For $A_1, A_2, \ldots, A_n \subset \mathcal{U}$ finite sets

P.I.E for n sets

$$|\mathcal{U} \setminus A_1 \cup A_2 \cup \ldots \cup A_n| = |\mathcal{U}| - |A_1| - |A_2| - \ldots - |A_n| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| + \ldots + |A_{n-1} \cap A_n| - |A_1 \cap A_2 \cap A_3| - |A_1 \cap A_2 \cap A_4| - \ldots + \ldots \pm |A_1 \cap A_2 \cap \ldots \cap A_n|$$

$$= \left|\mathcal{U} \setminus \bigcup_{i=1}^{n} A_i\right|$$

$$= \overbrace{|\mathcal{U}|}^{k=0} - \overbrace{\sum_{i=1}^{n} |A_i|}^{k=1} + \overbrace{\sum_{1 \le i < j \le n} |A_i \cap A_j|}^{k=2} - \overbrace{\sum_{1 \le i_1 < i_2 < i_3 \le n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}|}^{k=3} + \ldots + \overbrace{(-1)^n |A_1 \cap A_2 \cap \ldots \cap A_n|}^{k=n}$$

$$= \sum_{k=0}^{n} (-1)^k \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|$$

proof: (We'll skip the details)

Use induction on n = the # of overlapping sets A_i

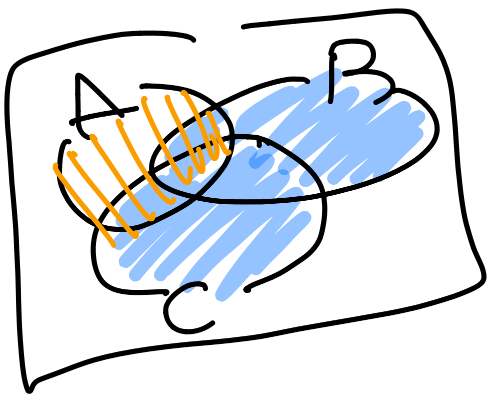
We did base case $n=1$

and even $n=2$.

In the inductive step, you can use

distributivity of $\cap$ over $\cup$:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$



$\blacksquare$

PRINCIPLE OF INCLUSION-EXCLUSION

$n=1$: $|U \setminus A| = |U| - |A|$

$n=2$: $|U \setminus (A_1 \cup A_2)| = |U| - |A_1| - |A_2| + |A_1 \cap A_2|$

general n: $|U \setminus A_1 \cup A_2 \cup \ldots \cup A_n| = |U \setminus \bigcup_{i=1}^{n} A_i|$

$$= \sum_{k=0}^{n} (-1)^k \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|$$

$k=0$ term is $|U|$ itself

EXAMPLES:

① How many rearrangements of BAMBINI avoid consecutive double letters, that is avoiding BB and also II?

$U = \{$all rearrangements of BAMBINI$\}$

$A_1 = \{$those with consecutive BB$\}$

$A_2 = \{$those with consecutive II$\}$

We want $|U \setminus \underbrace{A_1 \cup A_2}_{\text{those with either BB or II or both.}}|$

EXAMPLES:

① How many rearrangements of BAMBINI avoid consecutive double letters, that is avoiding BB and also II?

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n=2
P.I.E. worked!

$$|U \setminus A_1 \cup A_2| = |U| - |A_1| - |A_2| + |A_1 \cap A_2|$$

$$= \binom{7}{2,2,1,1,1} - \binom{6}{1,2,1,1,1} - \binom{6}{2,1,1,1,1}$$

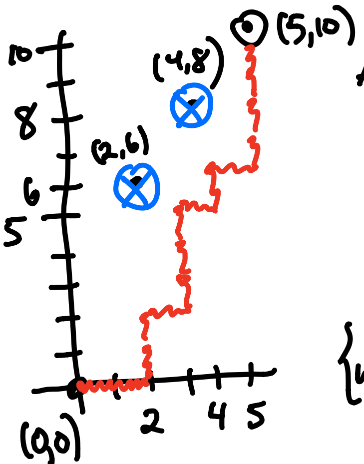
$\underbrace{BB}_{2}\underbrace{II}_{2}\underbrace{A}_{1}\underbrace{M}_{1}\underbrace{N}_{1}$ $\quad$ $\underbrace{BB}_{1}\underbrace{II}_{2}\underbrace{A}_{1}\underbrace{M}_{1}\underbrace{N}_{1}$ $\quad$ $\underbrace{BB}_{2}\underbrace{II}_{1}\underbrace{A}_{1}\underbrace{M}_{1}\underbrace{N}_{1}$

$$+ \binom{5}{1,1,1,1,1}$$

$\underbrace{BB}_{1}\underbrace{II}_{1}\underbrace{A}_{1}\underbrace{M}_{1}\underbrace{N}_{1}$

$$= \frac{7!}{2!2!} - \frac{6!}{2!} - \frac{6!}{2!} + 5!$$

② How many walks from $(0,0)$ to $(5,10)$
(taking unit N, E steps)
avoid both $(2,6)$ and $(4,8)$?



ANSWER =

$$|U - A_1 \cup A_2| =$$

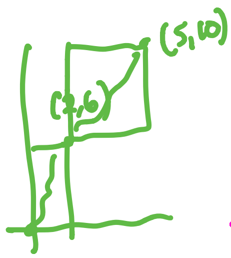
P.I.E for $n=2$!

$$|U| - |A_1| - |A_2| + \underbrace{|A_1 \cap A_2|}$$

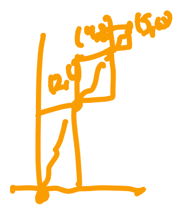
{all walks} {walks through (2,6)} {walks through (4,8)}

{walks through (2,6) and (4,8)}

$$= \binom{5+10}{5} - \binom{2+6}{2}\binom{3+4}{3} - \binom{4+8}{4}\binom{1+2}{1}$$



$$+ \binom{2+6}{2}\binom{2+2}{2}\binom{1+2}{1}$$



(3) The "hatcheck" or "derangement" or "Secret Santa" problem

n people give their hats to the hatcheck person, who later gives them back randomly. What is the probability that nobody gets their own hat back?

Let's get a formula, and as a function of n, figure out how it behaves for n large. Approaches 0? Approaches 1?

$U = \{$all arrangements of n hats (distributions)$\}$

$|U| = n!$

h_5	h_4	h_1	...	h_3
1	2	3	...	n

Let $A_1, A_2, \ldots, A_n$ be defined by $A_i := \{$arrangements where person i gets their own hat$\}$

$U = \{$all arrangements of n hats (distributions)$\}$

$|U| = n!$

$$\begin{array}{ccccc} H_5 & H_7 & H_1 & \cdots & H_3 \\ 1 & 2 & 3 & \cdots & n \end{array}$$

Let $A_1, A_2, \ldots, A_n$ be defined by $A_i = \{$arrangements where person i ~~gets their~~ own hat$\}$

We want # derangements

$$= |U \setminus (A_1 \cup \ldots \cup A_n)|$$

$$\underset{\text{P.I.E.}}{=} |U| - \sum_{i=1}^{n} |A_i| + \sum_{1 \le i < j \le n} |A_i \cap A_j| - \ldots$$

for n sets

$$= \sum_{k=0}^{n} (-1)^k \sum_{1 \le i_1 < \ldots < i_k < n} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|$$

$|U| = n!$

$|A_1| = (n-1)!$ $\quad |A_i| = (n-1)!$ for all i $\quad |A_i \cap A_j| = (n-2)!$

$$\begin{array}{ccccc} H_1 & H_7 & H_3 & & H_5 \\ 1 & 2 & 3 & \cdots & n \end{array}$$

$|A_1 \cap A_2| = (n-2)!$

$$\begin{array}{cccccc} H_1 & H_2 & H_5 & H_7 & & H_8 \\ 1 & 2 & 3 & 4 & \cdots & n \end{array}$$

derangements

1 $|U \setminus (A_1 \cup \dots \cup A_n)|$

$$= \sum_{k=0}^{n} (-1)^k \sum_{1 \le i_1 < \dots < i_k < n} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}|$$

with $|U| = n!$

$|A_i| = (n-1)!$

$|A_i \cap A_j| = (n-2)!$

$|A_{i_1} \cap \dots \cap A_{i_k}| = (n-k)!$

$$= \sum_{k=0}^{n} (-1)^k \sum_{1 \le i_1 < \dots < i_k \le n} (n-k)!$$

$$= \sum_{k=0}^{n} (-1)^k (n-k)! \binom{n}{k}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$$

$$\# \text{ derangements} = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$$

so probability that we get a derangement is

$$\frac{\#\text{derangements}}{|U|} = \sum_{k=0} \frac{(-1)^k}{k!}$$

$$= \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}$$

$$= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}$$

$$\Big\downarrow \lim_{n \to \infty}$$

$$= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots$$

$$= \left[e^x\right]_{x=-1} = e^{-1} = \frac{1}{e} \approx \frac{1}{3}$$