Math 4707 Sept. 28, 2020

Some asymptotics / size estimation
(book §1.4, 2.2)

Digits

EXAMPLE: How many digits (base ten) will

$$\left|\left\{\begin{matrix}\text{subsets of}\\ \{1,2,\ldots,n\}\end{matrix}\right\}\right| = 2^n$$

have for large n?

How many digits base ten does some positive integer x have?

x has 3 digits $\iff$ (if and only if) $\overset{10^2}{\overset{\shortparallel}{100}} \le x < \overset{10^3}{\overset{\shortparallel}{1000}}$

x has k digits $\iff$ $10^{k-1} \le x < 10^k$

How many digits base ten does some positive integer x have?

x has 3 digits $\Longleftrightarrow$ $\overset{10^2}{\overset{\shortparallel}{100}} \le x < \overset{10^3}{\overset{\shortparallel}{1000}}$

if and only if

x has k digits $\Longleftrightarrow$ $10^{k-1} \le x < 10^k$

$f(x) = \log_{10}(x)$ is strictly increasing

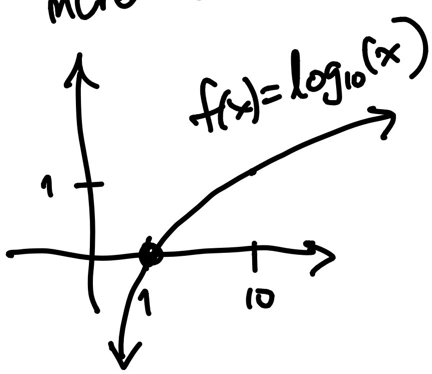
$$\Longleftrightarrow \underbrace{\log_{10}(10^{k-1})}_{=\,k-1} \le \log_{10}(x) < \underbrace{\log_{10}(10^k)}_{=\,k}$$

$$\Longleftrightarrow k-1 \le \log_{10}(x) < k$$

$$\Longleftrightarrow k \le 1+\log_{10}(x) < k+1$$

$$\Longleftrightarrow k = \lfloor 1+\log_{10}(x) \rfloor$$

$$\Longleftrightarrow k = 1 + \lfloor \log_{10}(x) \rfloor$$



$\lfloor x \rfloor$ = "floor of x"
= greatest integer $\le x$
= "x rounded down"

$\lceil x \rceil$ = "ceiling of x"
= "x rounded up"

x has k digits $\iff$ $k = 1 + \lfloor \log_{10}(x) \rfloor$

so 2^n has $1 + \lfloor \log_{10}(2^n) \rfloor$ digits base ten

$$= 1 + \lfloor n \log_{10}(2) \rfloor$$

"approximately equal" $\approx$ $1 + n(0.3)$ digits

Some comparisons

Let $[n] := \{1, 2, \ldots, n\}$, and let's compare as n gets large the size of...

$\{\text{subsets of } [n]\}$ $\quad 2^n$

$\{\text{5-element subsets of } [n]\}$ $\quad \binom{n}{5}$

If n is odd, $\binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}$ (why?)

$\{(\text{half-sized}) \ \frac{n}{2}\text{-element subsets of } n\}$ $\quad \binom{n}{n/2}$

$\{\text{permutations of } [n]\}$ $\quad n!$

PROPOSITION: For n large,

$$\binom{n}{5} \overset{(1)}{\leq} \binom{n}{n/2} \overset{(2)}{\leq} 2^n \overset{(3)}{\leq} n!$$

5-element subsets; n/2-element subsets; subsets; permutation

proof: This should be clear enough.

To prove (1), look at binomial coefficients $\binom{n}{k}$ with n fixed

e.g. n=6

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

$$1 \leq 6 \leq 15 \leq 20 \geq 15 \geq 6 \geq 1$$

You'd guess $\binom{n}{k} \leq \binom{n}{k+1}$ if $k \leq \frac{n-1}{2}$

$$\binom{5}{0} \leq \binom{5}{1} \leq \binom{5}{2} = \binom{5}{3} \geq \binom{5}{4} \geq \binom{5}{5}$$

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

So let's check this:

To prove: $\binom{n}{k} \le \binom{n}{k+1}$ if $k \le \frac{n-1}{2}$

$$\iff \frac{n!}{k!\,(n-k)!} \le \frac{n!}{(k+1)!\,(n-k-1)!}$$

divide by $\frac{n!}{k!(n-k-1)!}$

$$\iff \frac{1}{n-k} \le \frac{1}{k+1}$$

mult. by $(k+1)(n-k)$

$$\iff k+1 \le n-k$$

$$\iff 2k \le n-1$$

$$\iff k \le \frac{n-1}{2}$$

To prove $2^n \le n!$, it's false for $n \le 3$

n	2^n		$n!$
1	2	>	1
2	4	>	2
3	8	>	6
4	16	<	24
5	32	<	120
6	64	<	720
	⋮		⋮

so prove its true for $n \ge 4$, by induction on n.

BASE CASE: $n=4$ check $\underset{16}{\overset{\shortparallel}{2^4}} < \underset{24}{\overset{\shortparallel}{4!}}$ ✓

INDUCTIVE STEP: Given $2^n \leq n!$
$(2^n < n!)$

then $2^{n+1} = 2 \cdot 2^n \underset{\text{by inductive hypothesis}}{\leq} 2 \cdot n! \leq (n+1) \cdot n! = (n+1)!$

since $2 \leq n+1$ (as $n \geq 4$)

We've shown for n large
$$\binom{n}{5} \leq \binom{n}{n/2} \leq 2^n \leq n!$$

but really how quickly do they grow as a function of n?

$$\binom{n}{5} < \binom{n}{n/2} = \frac{n!}{\left(\frac{n}{2}!\right)^2} \underset{\text{via Stirling's approximation}}{\overset{\text{below}}{\approx}} \frac{\sqrt{2/\pi}}{\sqrt{n}} 2^n$$

HW problem:
$$\frac{n^5}{5^5} \leq \binom{n}{5} \leq \frac{n^5}{5!}$$
$$\frac{n^5}{5^5} = \frac{n^5}{3125}, \qquad \frac{n^5}{5!} = \frac{n^5}{120}$$

, so $\binom{n}{5}$ grows like $C \cdot n^5$ for some constant C

In fact, it grows more like $\frac{n^5}{120}$ since it's $\frac{n(n-1)(n-2)(n-3)(n-4)}{120}$

$$\binom{n}{5} < \binom{n}{n/2} < 2^n < n!$$

$\binom{n}{5} \approx \frac{n^5}{120}$

in fact

$$\lim_{n\to\infty} \frac{\binom{n}{5}}{n^5/120} = 1$$

$$= \lim_{n\to\infty} \frac{\frac{n(n-1)(n-2)(n-3)(n-4)}{120}}{n^5/120}$$

$\binom{n}{n/2} = \frac{n!}{((n/2)!)^2} \underset{\text{Stirling}}{\approx} \sqrt{\frac{2}{\pi}} \frac{2^n}{\sqrt{n}}$

$n! \approx$ Stirlings approximation $\sqrt{2\pi \cdot n}\left(\frac{n}{e}\right)^n$

We need the approximation for $n!$

<u>THEOREM (2.2.1)</u> (Stirling's formula)

For n large, $n! \approx \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n}$

$$\lim_{n\to\infty} \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} = 1$$

COROLLARY:

$$\binom{n}{n/2} \underset{\text{Stirling}}{\approx} \sqrt{\frac{2}{\pi}} \frac{2^n}{\sqrt{n}}$$

proof:

$$\binom{n}{n/2} = \frac{n!}{\left(\frac{n}{2}!\right)^2} \overset{\text{Stirling}}{\approx} \frac{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}}{\left[\left(\frac{n/2}{e}\right)^{n/2} \sqrt{2\pi \cdot \frac{n}{2}}\right]^2}$$

$$= \frac{\frac{n^n}{e^n} \sqrt{2\pi n}}{\left(\frac{n}{2e}\right)^n \pi n}$$

$$= 2^n \cdot \frac{\sqrt{2\pi n}}{\pi n}$$

$$= \frac{2^n}{\sqrt{n}} \sqrt{\frac{2}{\pi}} \quad \blacksquare$$

$$\binom{n}{5} < \binom{n}{n/2} < 2^n < n!$$

$$\approx \frac{n^5}{120} \qquad \approx \sqrt{2/\pi}\, \frac{2^n}{\sqrt{n}} \qquad 2^n \qquad \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

Chap 3 Binomial, multinomial coefficients & identities

PROPOSITION:

$$\underbrace{(x+y)}_{x+y \text{ is a binomial}}{}^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

and more generally

$$\underbrace{(x_1+x_2+\dots+x_\ell)}_{\text{a multinomial}}{}^n = \sum_{\substack{(k_1,k_2,\dots,k_\ell)\in \mathbb{Z}_{\geq 0} \\ k_1+k_2+\dots+k_\ell=n}} \binom{n}{k_1\ k_2\ \cdots k_\ell} x_1^{k_1} x_2^{k_2} \cdots x_\ell^{k_\ell}$$

$\mathbb{Z}_{\geq 0}$: the nonnegative integers

proof: Why is $(x+y)^5 = \dots + \binom{5}{2} x^2 y^3 + \dots$?

$n=5$, $k=2$

$$= \overset{①}{(x+y)}\overset{②}{(x+y)}\overset{③}{(x+y)}\overset{④}{(x+y)}\overset{⑤}{(x+y)}$$

pick 2 parentheses to take x not y;
pick y in the rest

To prove

$$\underbrace{(x_1+x_2+\dots+x_\ell)}_{\text{a multinomial}}{}^n = \sum_{\substack{(k_1,k_2,\dots,k_\ell)\in\underbrace{\mathbb{Z}_{\geq 0}}_{\text{the nonnegative integers}} \\ k_1+k_2+\dots+k_\ell=n}} \binom{n}{k_1\ k_2\ \cdots k_\ell} x_1^{k_1} x_2^{k_2} \cdots x_\ell^{k_\ell},$$

why is

$$(x_1+x_2+x_3)^n = \sum_{\substack{(k_1,k_2,k_3)\in\mathbb{Z}_{\geq 0} \\ k_1+k_2+k_3=n}} \binom{n}{k_1\ k_2\ k_3} x_1^{k_1} x_2^{k_2} x_3^{k_3} \ ?$$

$$\|$$

$$\overset{1}{(x_1+x_2+x_3)}\overset{2}{(x_1+x_2+x_3)}\cdots\cdots\overset{n}{(x_1+x_2+x_3)}$$

choose k_1 parentheses to pick x_1

choose k_2 parentheses to pick x_2

choose k_3 parentheses to pick x_3

Same as picking a word of length n with k_1 x_1's, k_2 x_2's, k_3 x_3's, i.e. $\binom{n}{k_1 k_2 k_3}$ ways

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PROPOSITION:

$$(x+y)^n = \sum_{k=0}^{n} \underbrace{\binom{n}{k}}_{\text{binomial coefficient}} x^k y^{n-k}$$

More generally,

$$(x_1+x_2+\dots+x_\ell)^n = \sum_{\substack{(k_1,k_2,\dots,k_\ell)\in\mathbb{Z}_{\geq 0} \\ k_1+k_2+\dots+k_\ell=n}} \binom{n}{k_1\, k_2 \dots k_\ell} x_1^{k_1} x_2^{k_2} \cdots x_\ell^{k_\ell}$$

Some consequences ...

Set $y=1$ to get

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Set $y=1$ to get

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

set $x=1$

$$2^n = \sum_{k=0}^{n} \underbrace{\binom{n}{k}}$$

all subsets of $[n]:=\{1,2,\dots,n\}$

classifying them by cardinality k

$x=-1$

$$(-1+1)^n = \sum_{k=0}^{n} \binom{n}{k}(-1)^k$$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + (-1)^n\binom{n}{n}$$

$$\underbrace{\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots}_{\left|\left\{\text{even-sized subsets}\right\}\right|} = \underbrace{\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots}_{\left|\left\{\text{odd-sized subsets}\right\}\right|}$$

proved earlier by bijection

Let's take d/dx to get

$$n(x+1)^{n-1} = \sum_{k=\cancel{0}\,1}^{n} \binom{n}{k} k x^{k-1}$$

and then set $x=1$, to get a new identity:

$$n2^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k$$

Pascal's recurrence & triangle

PROPOSITION:

$$\begin{cases} \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} & \text{for } 1 \le k \le n \\ \binom{n}{0} = 1 \\ \binom{n}{n} = 1 \end{cases}$$

$$\begin{array}{ccccccccc} & & & & \binom{0}{0} & & & & \\ & & & \binom{1}{0} & & \binom{1}{1} & & & \\ & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\ & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\ \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\ \iddots & & & & \vdots & & & & \ddots \end{array} \qquad \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ \iddots & & & & & & & & \ddots \end{array}$$

proof: Why is $\underbrace{\binom{n}{k}}_{\substack{k\text{-element} \\ \text{subsets of} \\ [n]}} = \underbrace{\binom{n-1}{k}}_{\substack{\text{those} \\ \text{avoiding} \\ n}} + \underbrace{\binom{n-1}{k-1}}_{\substack{\text{those} \\ \text{containing} \\ n}}$ $\leftrightarrow$ $(k-1)$-element subsets of $[n-1]$ ■

follows from SUM PRINCIPLE.

More identities with combinatorial proofs

① $$\sum_{k=0}^{n} \underbrace{\binom{n}{k}^2}_{\binom{n}{k}\cdot\binom{n}{k}} = \binom{2n}{n}$$

$\binom{n}{k}\cdot\binom{n}{k}$: # walks from (0,0) to (k,n-k) then (k,n-k) to (n,n)

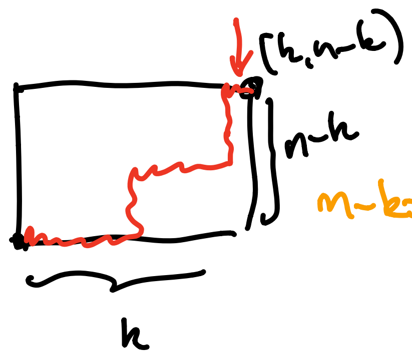
$\binom{2n}{n}$ = #block-walks from (0,0) to (n,n) taking N, E steps

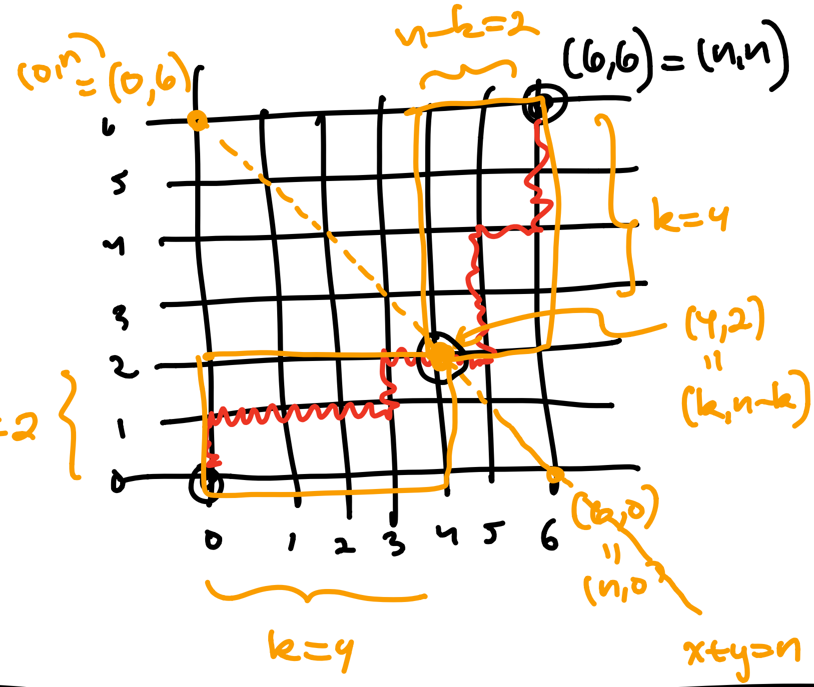
n=3

$\binom{3}{0}^2+\binom{3}{1}^2+\binom{3}{2}^2+\binom{3}{3}^2=$

$1^2+3^2+3^2+1^2=$

$20=\binom{6}{3}$





② $$k\binom{n}{k} = n\underbrace{\binom{n-1}{k-1}}$$

k: 2nd pick the chair

$\binom{n}{k}$: 1st pick k-person committee

n: pick chair first

$\binom{n-1}{k-1}$: pick k-1 other committee members

can be proven by counting in 2 ways the number of choices of a k-person committee and their chair from n people.

The birthday paradox

What's the chance that in a room of $40 = n$ people there are at least two people with the same birthday out of 365 days?

How large does n need to be in order to make the chance $\geq 50\%$?

Easiest to calculate

$$\text{PROB}\left(\begin{array}{c}\text{no two same}\\ \text{birthdays}\\ \text{among the 40 people}\end{array}\right) = \frac{\#\text{ choices of } \underline{\text{distinct}} \text{ birthdays for 40 people}}{\#\text{choices of birthdays for 40 people}}$$

1st person picks — 2nd person picks a different one

$$= \frac{\overset{\text{1st person picks}}{365} \cdot \overset{\text{2nd person picks a different one}}{364} \cdot 363 \cdots (365-39)}{\underbrace{365 \cdot 365 \cdot 365 \cdots 365}_{40 \text{ terms}}}$$

$$= \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365-39}{365}$$

$$= \left(1 - \frac{0}{365}\right)\left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right) \cdots \left(1 - \frac{39}{365}\right)$$

$$\text{PROB}\left(\begin{array}{l}\text{no two same} \\ \text{birthdays} \\ \text{among the 40 people}\end{array}\right)$$

$$= \left(1-\frac{0}{365}\right)\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\cdots\left(1-\underbrace{\frac{39}{365}}_{}\right)$$

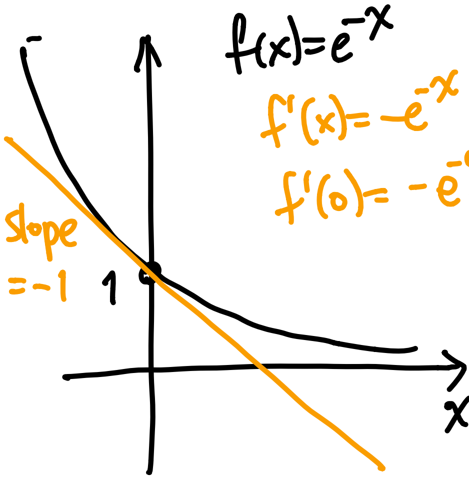
a small positive number $x = \frac{39}{365}$

Q: How to approximate $1-x$ for x small positive?

CLAIM: $1-x \approx e^{-x}$ for x small positive

$1-x \leq e^{-x}$ for x small positive

follows from concavity upward via $f''(x) = \cdots$



$f(x) = e^{-x}$

$f'(x) = -e^{-x}$

$f'(0) = -e^{-0} = -1$

$\ell(x) = 1-x$

or

$$f(x) = e^{-x} = \boxed{\overset{1-x}{1 - \frac{x^1}{1!}}} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

alternating series for $0<x<1$

$$\overset{\text{CLAIM}}{\Longrightarrow} \text{PROB}\left(\begin{array}{l}\text{all distinct} \\ \text{birthdays}\end{array}\right) = \left(1-\frac{0}{365}\right)\left(1-\frac{1}{365}\right)\cdots\left(1-\frac{39}{365}\right)$$

$$\leq e^{-\frac{0}{365}} \cdot e^{-\frac{1}{365}} \cdots e^{-\frac{39}{365}}$$

$$= e^{\frac{-(0+1+2+\cdots+39)}{365}}$$

$$\text{PROB}\begin{pmatrix}\text{all distinct}\\ \text{birthdays}\end{pmatrix} = \left(1-\frac{0}{365}\right)\left(1-\frac{1}{365}\right)\cdots\left(1-\frac{39}{365}\right)$$

$$\le e^{-\frac{0}{365}} \cdot e^{-\frac{1}{365}} \cdots e^{-\frac{39}{365}}$$

$$= e^{\frac{-(0+1+2+\cdots+39)}{365}}$$

$$= e^{\frac{-\frac{39\cdot 40}{2}}{365}} = e^{\frac{-\binom{40}{2}}{365}} \approx 0.12$$

EXERCISE:

$$0+1+2+\cdots+(n-1) = \frac{n(n-1)}{2} = \binom{n}{2} = \frac{n^2-n}{2}$$

so PROB of a birthday coincidence for 40 $\ge 1-0.12 = 0.88$

In general, given n people

$$\text{PROB}\begin{pmatrix}\text{no two birthdays}\\ \text{same}\end{pmatrix} \le e^{\frac{-\binom{n}{2}}{365}}$$

To get $\ge 50\%$ of birthday <u>coincidence</u>,

need $e^{-\binom{n}{2}/365} \le \frac{1}{2}$

take $\log_e(-) = \ln(-)$

$$-\binom{n}{2}\Big/365 \le \ln\left(\frac{1}{2}\right)$$

$$-\binom{n}{2} \le 365\ln\left(\frac{1}{2}\right) = -365\ln(2)$$

$$-\binom{n}{2} \le 365 \ln\left(\frac{1}{2}\right) = -365 \ln(2)$$

$$\binom{n}{2} \ge 365 \ln(2)$$

$$\frac{n^2 - n}{2} \ge 365 \ln(2)$$

$\Downarrow$ approximately

$$\frac{n^2}{2} \ge 365 \ln(2)$$

$$n^2 \ge 2 \cdot 365 \ln(2)$$

$\sqrt{\cdot}$

$$n \ge \sqrt{2 \cdot \ln(2) \cdot 365} \approx 22.5$$

In general, replacing 365 by N, one needs n = # trials of sampling from N possibilities to have $n \ge \sqrt{2 \ln(2) \cdot N} \approx \sqrt{N}$ before expecting a repeat among the samples.

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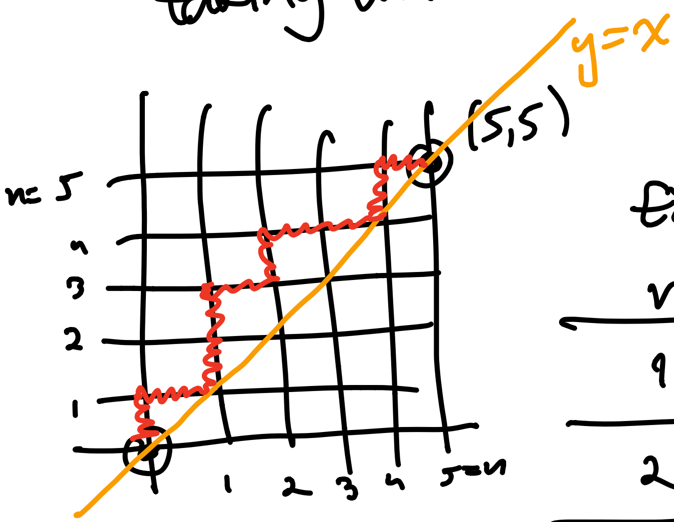
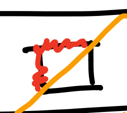
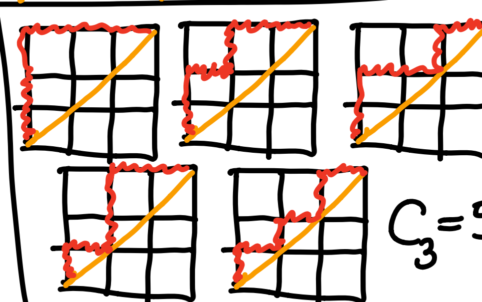
Catalan numbers - not in the book, but see handout by D. White on syllabus.

How many block walks from $(0,0)$ to (n,n) taking unit steps N or E stay weakly above the line $y=x$?



EXAMPLE

n	$\left\|\left\{\begin{matrix}\text{such}\\ \text{walks}\end{matrix}\right\}\right\| =: C_n$
1	 $C_1 = 1$
2	 $C_2 = 2$
3	 $C_3 = 5$
4	$14 = C_4$
5	$42 = C_5$

THEOREM

The number of block walks from $(0,0)$ to (n,n) taking units steps N or E stay weakly above the line $y=x$,

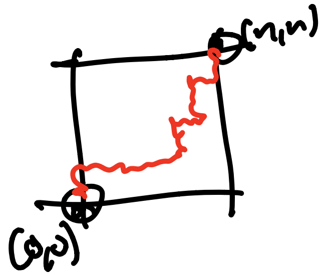
called the n^{th} Catalan number

is $$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n}$$
$$= \frac{1}{n}\binom{2n}{n-1} = \frac{(2n)!}{n!\,(n+1)!}$$
$$= \binom{2n}{n} - \binom{2n}{n-1}$$

COROLLARY

What's the probability that a walk from $(0,0)$ to (n,n) taking N, E steps stays weakly above $y=x$?

ANSWER: $$\frac{C_n}{\#\text{ of walks total}} = \frac{\frac{1}{n+1}\binom{2n}{n}}{\binom{2n}{n}} = \frac{1}{n+1}$$



<u>proof</u>: Let's prove the formula $C_n = \binom{2n}{n} - \binom{2n}{n-1}$ using a combination of difference + bijection principle:

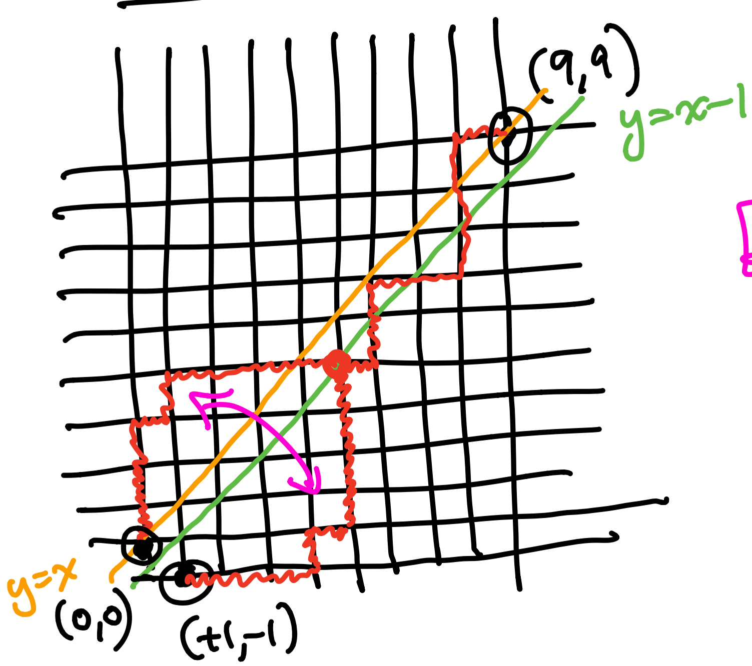
$$\#\left\{\begin{array}{l}\text{walks}\\ \text{weakly}\\ \text{above } y=x\end{array}\right\} = \underbrace{\#\left\{\begin{array}{l}\text{walks}\\ \text{in total}\\ \text{from}\\ (0,0) \text{ to } (n,n)\end{array}\right\}}_{\binom{2n}{n} \text{ of these}} - \underbrace{\#\left\{\begin{array}{l}\text{bad ones,}\\ \text{i.e. crossing}\\ \text{through}\\ y=x\end{array}\right\}}$$

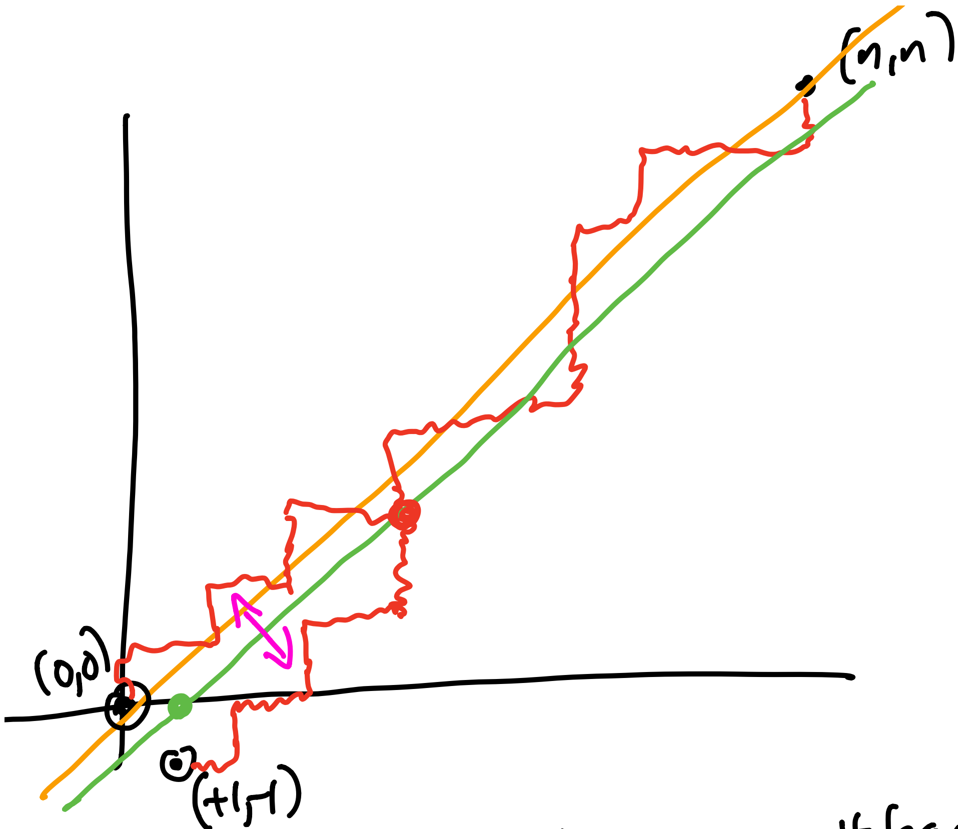
biject with

$$\#\left\{\begin{array}{l}\text{all walks from}\\ (+1,-1) \text{ to}\\ (n,n)\\ \text{taking N, E}\\ \text{steps}\end{array}\right\}$$

$\boxed{\text{reflect}}$ the initial part of the red path (up to the 1st contact with $y=x-1$) across $y=x-1$

Picture:





$$C_n = \#\text{good paths} = \#\text{all paths } (0,0) \text{ to } (n,n) - \underbrace{\#\text{bad paths } (0,0) \text{ to } (n,n)}_{\underbrace{\#\text{all paths } (+1,-1) \text{ to } (n,n)}_{\substack{n-1 \text{ horizontal steps} \\ n+1 \text{ vertical steps}}}}$$

$$= \binom{2n}{n} - \binom{n-1+n+1}{n-1}$$

$$= \binom{2n}{n} - \binom{2n}{n-1}$$

$$= \frac{(2n)!}{n!\,n!} - \frac{(2n)!}{(n-1)!\,(n+1)!}$$

$$= \frac{(2n)!}{(n-1)!\,n!}\left[\frac{1}{n} - \frac{1}{n+1}\right] = \frac{(2n)!}{(n-1)!\,n!}\left[\frac{1}{n(n+1)}\right] = \frac{(2n)!}{n!\,(n+1)!} \quad \blacksquare$$

Chapter 4 Fibonacci numbers

Sometimes a sequence of numbers comes to you by a recurrence and initial conditions, as in ...

Fibonacci's recurrence

$$F_{n+1} = F_n + F_{n-1} \quad \text{for } n \geq 1$$

with initial condition $F_0 = 0$, $F_1 = 1$

$$F_2 = F_1 + F_0 = 1 + 0 = 1$$

leading to a tabulation

n	F_n	even/odd
0	0	e
1	1	o
2	1	o
3	2	e
4	3	o
5	5	o
6	8	e
7	13	o

Note: If $5 \mid m$ then $5 \mid F_m$

divides evenly into

n	F_n	even/odd
8	21	o
9	34	e
10	55	o
11	89	o
12	144	e
13	233	o
14	377	o
15	610	e
⋮	⋮	

Guess..

PROPOSITION: F_{3m} is even

F_{3m+1}, F_{3m+2} are odd

proof: Induct on m.

BASE CASE: $m=0$

$F_{3\cdot 0} = F_0 = 0$ even ✓

$F_{3 \cdot 0+1} = F_1 = 1$ odd ✓

$F_{3\cdot 0+2} = F_2 = 1$ odd ✓

INDUCTIVE STEP: Assume it for $m-1, m-2, \ldots$ and prove it for m:

$$F_{3m} = F_{3m-1} + F_{3m-2}$$

$$= \underbrace{F_{3(m-1)+2}}_{\text{odd by induction}} + \underbrace{F_{3(m-1)+1}}_{\text{odd by induction}}$$

$= \text{even}$ ✓

Check $F_{3m+1} = \underbrace{F_{3m}}_{\substack{\text{even}\\ \text{by the}\\ \text{case}\\ \text{just}\\ \text{done!}}} + \underbrace{F_{3m-1}}_{\underbrace{F_{3(m-1)+2}}_{\substack{\text{odd}\\ \text{by induction}}}}$

$= \text{odd}$ ✓

F_{3m+2} case is similar $\blacksquare$

We also guessed...

PROPOSITION: F_{5m} is divisible by 5.

proof: Induct on m. BASE CASE $m=0$

$F_{5\cdot 0} = F_0 = 0$ ✓

INDUCTIVE STEP: Assume it for smaller m

$$F_{5m} = F_{5m-1} + F_{5m-2}$$

$$= F_{5m-2} + F_{5m-3} + F_{5m-3} + F_{5m-4}$$

$$= F_{5m-2} + 2F_{5m-3} + F_{5m-4}$$

$$= 3F_{5m-3} + 2F_{5m-4}$$

PROPOSITION: F_{5m} is divisible by 5.

proof: Induct on m. **BASE CASE** $m=0$

$$F_{5\cdot 0} = F_0 = 0 \checkmark$$

INDUCTIVE STEP: Assume it for smaller m

$$F_{5m} = F_{5m-1} + F_{5m-2}$$

$$= F_{5m-2} + F_{5m-3} + F_{5m-3} + F_{5m-4}$$

$$= F_{5m-2} + 2F_{5m-3} + F_{5m-4}$$

$$= 3F_{5m-3} + 2F_{5m-4}$$

$$= 3\left(F_{5m-4} + F_{5m-5}\right) + 2F_{5m-4}$$

$$= \underbrace{5F_{5m-4}}_{\text{divisible by 5}} + \underbrace{3\underbrace{F_{5m-5}}_{5(m-1)}}_{\text{divisible by 5 by induction}}$$

$$= \text{divisible by } 5 \qquad \blacksquare$$

n	F_n	even/odd
0	0	e
1	1	o
2	1	o
3	2	e
4	3	o
5	5	o
6	8	e
7	13	o

Note: If $5 \mid m$ then $5 \mid F_m$

divides evenly into

n	F_n	even/odd
8	21	o
9	34	e
10	55	o
11	89	o
12	144	e
13	233	o
14	377	o
15	610	e
⋮	⋮	

$\underset{F_5}{\overset{5}{\shortparallel}}$ divides F_{5m}

$\underset{F_3}{\overset{2}{\shortparallel}}$ divides F_{3m}

$\underset{F_7}{\overset{13}{\shortparallel}}$ divides $F_{2\cdot 7} = F_{14} = 377 = 13 \cdot 29$

<u>GUESS</u>: F_k divides F_{km} for $k, m \geq 0$.

THEOREM: More generally $F_k \mid F_{km}$.

proof: **LEMMA**: $\forall a, b \geq 0$

$$F_{a+b+1} = F_{a+1} F_{b+1} + F_a F_b$$

How would LEMMA prove the THEOREM?

By induction on m,

with BASE CASE $m=1$: $F_k \mid F_k$ ✓

and INDUCTIVE STEP:

Write $n = km = a + b + 1$

$$= k(m-1) + (k-1) + 1$$

$$F_n = F_{km} = F_{a+b+1} \overset{\text{use LEMMA}}{=} F_{a+1} F_{b+1} + F_a F_b$$

$$= \underbrace{F_{k(m-1)+1} F_k}_{\text{divisible by } F_k} + \underbrace{F_{k(m-1)}}_{\text{divisible by } F_k} F_{k-1}$$

$= \text{divisible by } F_k$