Math 4707 Oct. 7, 2020

Note: There are <u>two</u> resources in PDF on generating functions on the syllabus:
one by G. Musiker (intro)
S. Hopkins (exercises)

Fibonacci numbers

<u>recurrence</u> $\{ F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$

<u>initial conditions</u> $\begin{cases} F_0 = 0 \\ F_1 = 1 \end{cases}$

n	F_n
0	0
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
9	34
10	55

Noticed that

$\underset{F_3}{\overset{2}{\shortparallel}} \mid F_{3m}$ is even

$\underset{F_5}{\overset{5}{\shortparallel}} \mid F_{5m}$ is divisible by 5

<u>Guessed</u>: $F_k \mid F_{km} \quad \forall \begin{matrix} m \geq 1 \\ k \geq 1 \end{matrix}$

THEOREM: $F_k \mid F_{km}$ for $k,m \geq 1$

proof: We deduce it from

LEMMA: $F_{a+b+1} = F_{a+1}F_{b+1} + F_a F_b$ for $a,b \geq 0$

via choosing
$$a := k(m-1)$$
$$b := k-1$$

then Lemma says

$$\begin{aligned}
F_{a+b+1} &= F_{a+1}F_{b+1} + F_a F_b \\
&= \underbrace{F_{k(m-1)+1}\,F_k}_{\text{divisible by } F_k} + \underbrace{F_{k(m-1)}}_{\substack{\text{divisible by } F_k \\ \text{using induction on } m}} F_{k-1}
\end{aligned}$$

$$F_{a+b+1} = F_{k(m-1)+(k-1)+1} = F_{km}$$

$\blacksquare$

It remains to prove the LEMMA.

LEMMA: $F_{a+b+1} = F_{a+1}F_{b+1} + F_a F_b$ for $a, b \geq 0$

Let's check the instances

a=0: $F_{b+1} = F_1 \cdot F_{b+1} + F_0 F_b$

$= F_{b+1}$ says nothing!

a=1: $F_{b+2} = F_2 F_{b+1} + F_1 F_b$

$F_{b+2} = F_{b+1} + F_b$ is the original recurrence

a=2: $F_{b+3} = F_3 F_{b+1} + F_2 F_b$

$F_{b+3} = 2F_{b+1} + F_b$ seems new

$$F_{b+3} = F_{b+2} + F_{b+1} = F_{b+1} + F_b + F_{b+1}$$

✓

proof of LEMMA: Let's use induction on a, so fix a and prove the assertion $\forall b \geq 0$

BASE CASE: $a=0$ We checked $\forall b$ it says $F_{b+1} = F_{b+1}$ ✓

INDUCTIVE STEP: Assuming we've proven $F_{a+b+1} = F_{a+1}F_{b+1} + F_a F_b$ $\forall$ smaller values of a, and all b.

and now prove it:

$$F_{a+b+1} = F_{\underbrace{(a-1)}_{\text{smaller value of } a} + \underbrace{(b+1)}_{} + 1}$$

don't care how big this is, when a is smaller.

inductive hypothesis

$$= F_a F_{b+2} + F_{a-1} F_{b+1}$$

$$= F_a (F_{b+1} + F_b) + F_{a-1} F_{b+1}$$

$$= (F_a + F_{a-1}) F_{b+1} + F_a F_b$$

$$= F_{a+1} F_{b+1} + F_a F_b \qquad \blacksquare$$

Q: How quickly do F_n grow as $n \to \infty$?

linear?

quadratic?

geometric? $a, ar, ar^2, ar^3, \ldots$?

$\frac{F_{n+1}}{F_n}$ $\frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \ldots$

§4.3 An exact (!) formula for F_n

We'll get this formula by considering a power series

$$f(x) = F_0 \cdot x^0 + F_1 \cdot x^1 + F_2 \cdot x^2 + F_3 \cdot x^3 + \dots$$

$$= \sum_{n=0}^{\infty} F_n x^n$$

$$= 0 \cdot x^0 + 1 \cdot x^1 + 1 \cdot x^2 + 2 \cdot x^3 + 3 \cdot x^4 + 5x^5 + 8x^6 + \dots$$

called the <u>generating function</u> for the sequence $\{F_n\}_{n=0,1,2,\dots}$

---

e.g. $a_n = 2^n = \#$ subsets of $\{1,2,\dots,n\}$

has generating function

$$a(x) = 2^0 \cdot x^0 + 2^1 \cdot x^1 + 2^2 \cdot x^2 + 2^3 \cdot x^3 + \dots$$

$$= 1 + 2x + 4x^2 + 8x^3 + \dots$$

$$= \frac{1}{1-2x}$$

$a + ar + ar^2 + ar^3 + \dots$

$= \dfrac{a}{1-r}$

e.g. Fix n, and consider

$$b_k = \binom{n}{k}$$

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
⋮

Then

$$b(x) = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$$

$$= (x+1)^n$$

$x=1$: $2^n = \sum \binom{n}{k}$

$x=-1$: $0 = \sum (-1)^k \binom{n}{k}$

d/dx, then $x=1$: $n2^{n-1} = \sum k\binom{n}{k}$

THEOREM: Fibonacci #'s $\{F_n\}$ have

(a) $$f(x) = \sum_{n=0}^{\infty} F_n x^n = 0\cdot x^0 + 1\cdot x^1 + 1\cdot x^2 + 2x^3 + 5x^4 + \ldots$$

$$= \frac{x}{1-x-x^2}$$

(b) ... and from these we will deduce

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \beta^n\right) \text{ where}$$

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots = \text{golden ratio} > 1$$

$$\beta = \frac{1-\sqrt{5}}{2} \approx -0.618\ldots, \quad |\beta| < 1$$

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(c) ...and hence $F_n \approx \frac{1}{\sqrt{5}} \phi^n$ as $n \to \infty$

and $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi$ = golden ratio.

In fact, $F_n = \left[\frac{1}{\sqrt{5}} \phi^n\right]$ $\forall n \geq 0$

rounding or nearest-integer function

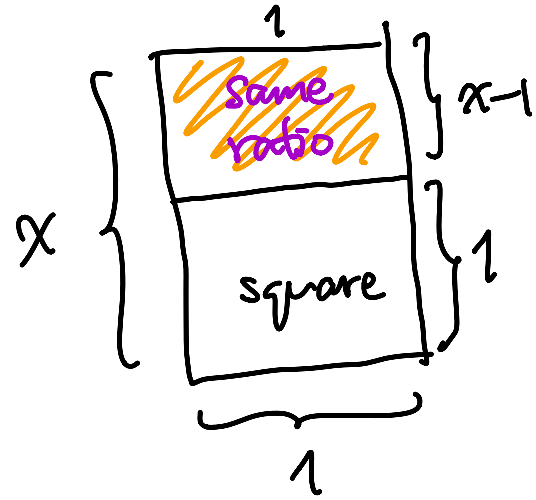
$$= \left\lfloor \frac{1}{\sqrt{5}} \phi^n + \frac{1}{2} \right\rfloor$$

ASIDE

$$\phi = \frac{1+\sqrt{5}}{2} = \text{golden ratio}$$

$$\beta = \frac{1-\sqrt{5}}{2}$$

supposedly most pleasing ratio x



$$\frac{x}{1} = \frac{1}{x-1}$$

$$x(x-1) = 1$$

$$x^2 - x = 1$$

$$x^2 - x - 1 = 0$$

$$x = \frac{+1 \pm \sqrt{(-1)^2 - 4\cdot 1(-1)}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2} = \{\phi, \beta\}$$

THEOREM: Fibonacci #'s $\{F_n\}$ have

(a) $f(x) = \sum_{n=0}^{\infty} F_n x^n = 0\cdot x^0 + 1\cdot x^1 + 1\cdot x^2 + 2x^3 + 5x^4 + \dots$

$$= \frac{x}{1-x-x^2}$$

proof:

PLAN:
recurrence relation
$\Big\}$ mult. by x^n, sum on n
set functional equation for $f(x)$
$\Big\}$
solve the functional equation for $f(x)$
$\Big\}$
extract the coefficient of x^n in $f(x)$

$$F_{n+1} = F_n + F_{n-1} \quad \text{for } n \geq 1$$

$F_0 = 0$

$F_1 = 1$

so $F_{n+1} x^n = F_n x^n + F_{n-1} x^n$ for $n \geq 1$

$$\sum_{n=1}^{\infty} F_{n+1} x^n = \sum_{n=1}^{\infty} F_n x^n + \sum_{n=1}^{\infty} F_{n-1} x^n$$

$$\frac{1}{x} \sum_{n=1}^{\infty} F_{n+1} x^{n+1} = f(x) - F_0 x^0 + x \sum_{n=1}^{\infty} F_{n-1} x^{n-1}$$

$$\frac{1}{x}\left(f(x) - F_0 x^0 - F_1 x^1\right) = f(x) - F_0 x^0 + x f(x)$$

$$\frac{1}{x}\left(f(x) - x\right) = f(x) + x f(x)$$

$$f(x) - x = x f(x) + x^2 f(x)$$

$$-x = (x^2 + x - 1) f(x)$$

$$f(x) = \frac{-x}{x^2 + x - 1} = \frac{x}{1-x-x^2}$$

(6) $f(x) = \dfrac{x}{1-x-x^2}$

from these we will deduce

$F_n = \dfrac{1}{\sqrt{5}}\left(\phi^n - \beta^n\right)$ where

$\phi = \dfrac{1+\sqrt{5}}{2} \approx 1.618\ldots$ = golden ratio > 1

$\beta = \dfrac{1-\sqrt{5}}{2} \approx -0.618\ldots$, $|\beta| < 1$

<u>proof</u>: Let's rewrite $f(x)$ using

$$1-x-x^2 = (1-\phi x)(1-\beta x)$$

check:

$$= 1-(\phi+\beta)x + \phi\beta x^2$$
$$= 1-\left(\frac{1+\sqrt{5}}{2}+\frac{1-\sqrt{5}}{2}\right)x + \frac{1+\sqrt{5}}{2}\cdot\frac{1-\sqrt{5}}{2}x^2$$
$$= 1-x-x^2 \checkmark$$

One can do the partial fraction algorithm to write

$$f(x) = \frac{x}{1-x-x^2} = \frac{x}{(1-\phi x)(1-\beta x)} = \frac{\frac{1}{\sqrt{5}}}{1-\phi x} + \frac{\frac{-1}{\sqrt{5}}}{1-\beta x}$$

use geometric series $\frac{1}{1-r} = 1+r+r^2+\ldots$

$$= \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\phi^n x^n - \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\beta^n x^n$$

so

$$f(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\left(\phi^n - \beta^n\right)x^n$$

i.e. $F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \beta^n\right)$.

(c) $F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \beta^n\right)$ $\qquad \phi = \frac{1+\sqrt{5}}{2} \approx 1.618$, $\quad \beta = \frac{1-\sqrt{5}}{2} \approx -0.618$

... and hence $F_n \approx \frac{1}{\sqrt{5}}\phi^n$ as $n \to \infty$

and $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi =$ golden ratio.

In fact, $F_n = \left[\frac{1}{\sqrt{5}}\phi^n\right]$ $\forall n \geq 0$

rounding or nearest-integer function

$$= \left\lfloor \frac{1}{\sqrt{5}}\phi^n + \frac{1}{2} \right\rfloor$$

Not much to say in proving this, except

$$F_n = \frac{1}{\sqrt{5}}\phi^n + \left(\text{error term } = -\frac{1}{\sqrt{5}}\beta^n \xrightarrow[n\to\infty]{} 0\right)$$

$\blacksquare$

Math 4707 Oct. 12, 2020

Stirling numbers of the second kind

$S(n,k) :=$ # ways to partition $\{1,2,\ldots,n\}$ into k unlabelled nonempty blocks

$n \geq k \geq 1$

e.g. $n=4$, $k=2$

$S(4,2) = 7 = \#\{$ 123-4, 124-3, 134-2, 234-1, 12-34, 13-24, 14-23 $\}$

$S(3,1) = 1 = \#\{123\}$

$S(3,2) = 3 = \#\{$12-3, 13-2, 23-1$\}$

$S(3,3) = 1 = \#\{$1-2-3$\}$

n \ k	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

#2 (a) $S(n,1) = 1 = S(n,n)$

only all of $\{1,2,\ldots,n\}$ in a single block

only 1-2-3-...-n each in a singleton block

(b) $S(n,n-1) = \binom{n}{2}$

pick the unique pair $\{i,j\}$ that forms the only non-singleton block

e.g. $n=9$, $n-1=8$ $\quad 1-2-58-3-4-6-7-9$

(c) $S(n,2) = \dfrac{2^n - 2}{2}$ $\quad \left(= 2^{n-1} - 1\right)$

correct for overcounting by 2 since $S-S^c$ is same partition as S^c-S

numerator picks a subset $S \neq \emptyset, \{1,2,\ldots,n\}$ to be the 1st block, and its complement is the 2nd block S^c

(e) $S(5,3) = \binom{5}{3} + \frac{1}{2}\binom{5}{2,2,1}$

counts ijk-l-m

counts ij-kl-m $(= kl-ij-m)$

#3. $$S(n,k) = \underbrace{S(n-1,k-1)}_{\substack{\text{n goes into a} \\ \text{singleton block,} \\ \{1,2,\dots,n-1\}\text{ go into} \\ \text{k-1 blocks}}} + \underbrace{k}_{\substack{\text{n} \\ \text{chooses} \\ \text{which of the} \\ \text{k blocks to be in}}}\underbrace{S(n-1,k)}_{\substack{\{1,2,\dots,n-1\} \\ \text{partitioned} \\ \text{into k} \\ \text{blocks}}}$$

#4. (a) $$f_1(x) = S(1,1)x^1 + S(2,1)x^2 + S(3,1)x^3 + \dots$$
$$= x^1 + x^2 + x^3 + \dots$$
$$= \frac{x}{1-x}$$

(b) $$f_2(x) = S(2,2)x^2 + S(3,2)x^3 + S(4,2)x^4 + \dots$$
$$= (2^{2-1}-1)x^2 + (2^{3-1}-1)x^3 + (2^{4-1}-1)x^4 + \dots$$
$$= (2x^2 + 4x^3 + 8x^4 + \dots) - (x^2 + x^3 + x^4 + \dots) = \frac{2x^2}{1-2x} - \frac{x^2}{1-x}$$
$$= \frac{1}{(1-x)(1-2x)}\left[(1-x)2x^2 - x^2(1-2x)\right]$$
$$= \frac{x^2}{(1-x)(1-2x)}\left[2(1-x) - (1-2x)\right] = \frac{x^2}{(1-x)(1-2x)} = \frac{x}{1-x} \cdot \frac{x}{1-2x}$$

#5. $S(n,k) = S(n-1,k-1) + kS(n-1,k)$ for $n \geq k \geq 1$

$$\sum_{n \geq k} S(n,k)x^n = \sum_{n \geq k} S(n-1,k-1)x^n + \sum_{n \geq k} kS(n-1,k)x^n$$

$$f_k(x) = x f_{k-1}(x) + x \sum_{n \geq k} kS(n-1,k)x^{n-1}$$

$$f_k(x) = x f_{k-1}(x) + kx f_{k-1}(x)$$

$$(1-kx)f_k(x) = x f_{k-1}(x)$$

$$f_k(x) = \frac{x}{1-kx}$$

$$= \frac{x}{1-kx} \cdot \frac{x}{1-(k-1)x} f_{k-2}(x)$$

$$= \frac{x}{1-kx} \cdot \frac{x}{1-(k-1)x} \cdots \frac{x}{1-2x} \frac{x}{1-x}$$

$$= \frac{x^k}{(1-kx)(1-(k-1)x)\cdots(1-2x)(1-x)}$$

e.g.

$$f_4(x) = S(4,4)x^4 + S(5,4)x^5 + S(6,4)x^6 + \dots$$

$$= \sum_{n\geq 4} S(n,4)x^n$$

$$= \frac{x^4}{(1-x)(1-2x)(1-3x)(1-4x)}$$

partial fractions

$$= \frac{A}{1-x} + \frac{B}{1-2x} + \frac{C}{1-3x} + \frac{D}{1-4x}$$

$$= A\sum_{n\geq 0} x^n + B\sum_{n\geq 0} 2^n x^n + C\sum_{n\geq 0} 3^n x^n + D\sum_{n\geq 0} 4^n x^n$$

$$= \sum_{n\geq 0} \underbrace{\left(A + 2^n B + 3^n C + 4^n D\right)}_{= S(n,4)} x^n$$

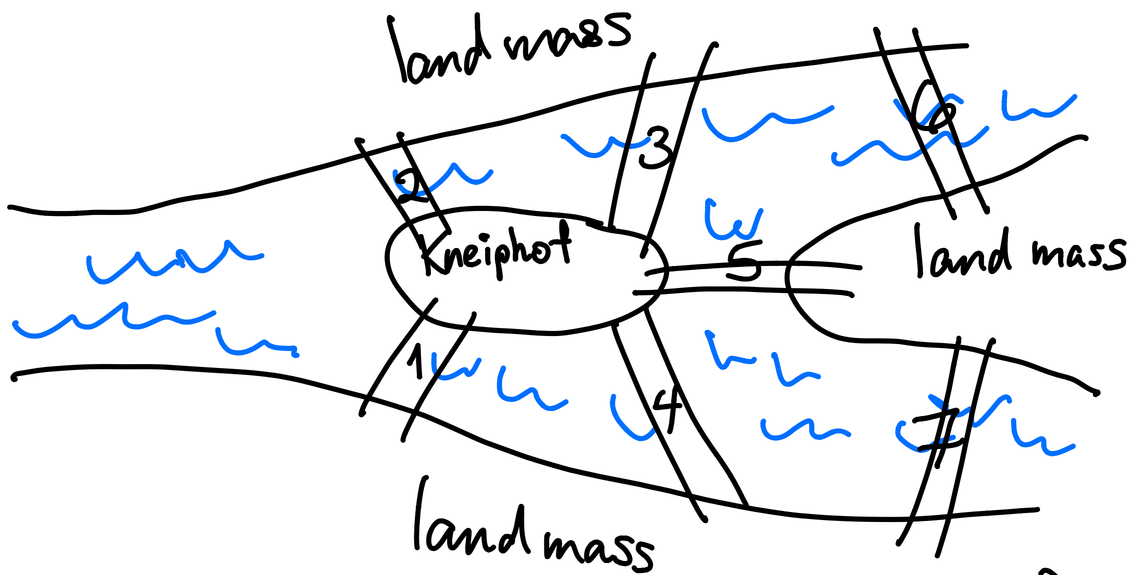
has a formula with 4 terms (see Wikipedia page)

MATH 4707 Oct. 14, 20

GRAPH THEORY

Euler walks and Euler tours (§7.3)

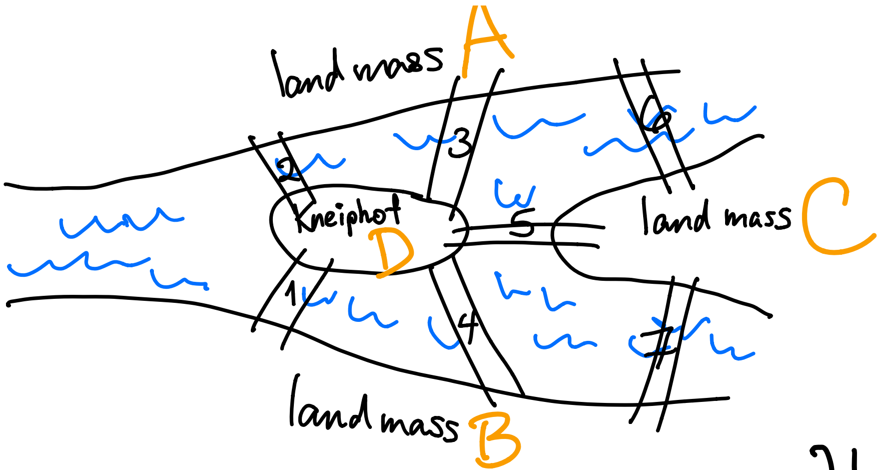
EXAMPLE: The 7 bridges of Königsberg



Q: Can one find a tour (= a closed walk) that goes over each bridge exactly once?

same starting and ending landmass

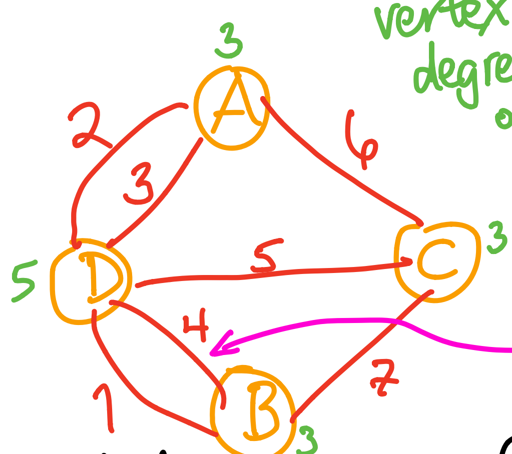
Can one find a walk (possibly different start/end masses) using each bridge exactly once?



L. Euler proved both are impossible in 1736, and came up with a graph theory abstraction that solved all such problems!

Abstraction: A graph $G = (V, E)$, where V (vertices) $= \{\text{land masses}\} = \{A, B, C, D\}$ and E (edges) $= \{\text{bridges}\}$.

some (all!) vertex degrees odd

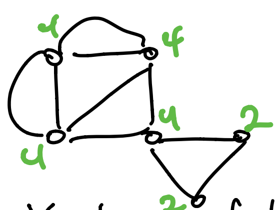


note that we're allowing multiple edges with same endpoints! (not simple graphs)

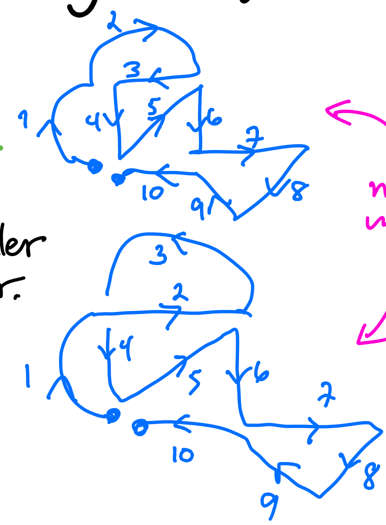
DEFINITION: An Euler tour in G is a closed walk from vertex-to-vertex along edges, using each edge exactly once. An Euler walk/path is same but not necessarily closed.

Q: Which graphs have Euler tours, Euler walks?
Are they unique in any sense?

all even $\deg_G(v)$



Yes, has an Euler tour.



not unique

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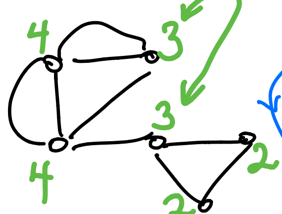
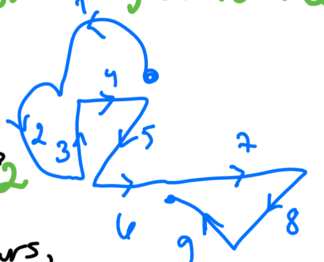
odd degree vertices





No Euler tours, but has Euler walks.

THEOREM (Euler 1736) $G = (V, E)$ a graph with no isolated vertices has an Euler tour

$\Longleftrightarrow$ (a) G is connected i.e. every pair $v, v' \in V$ has some path v ——— v' between them

AND

(b) every vertex $v \in V$ has $\deg_G(v)$ even

valence or degree of v in G

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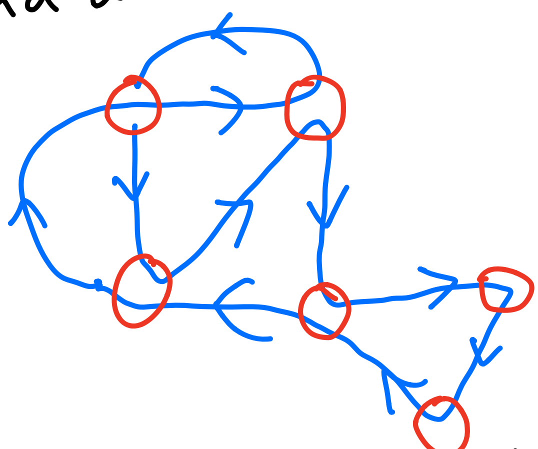
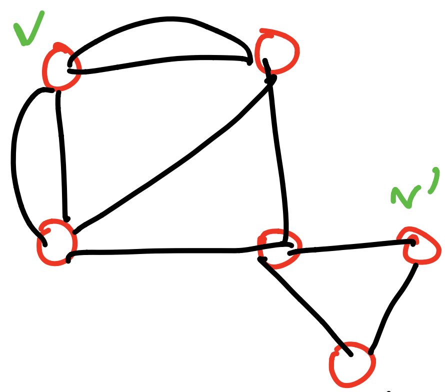
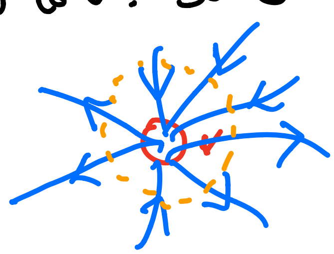
valence or degree of v in G

proof: ($\Longrightarrow$): Close off the Euler tour, and direct it with arrows:





Given any $v,v' \in V$, pick edges incident to them e,e', and following the infinite tour from e to e' gives a path from v to v'. So G is connected.



Also, fixing $v \in V$, the tour pairs off edges incident to v as it enters v and exits v along those edges.

THEOREM (Euler 1736) $G=(V,E)$ a graph with no isolated vertices has an Euler tour

$\iff$ (a) G is connected i.e. every pair $v, v' \in V$ has some path v — — — v' between them

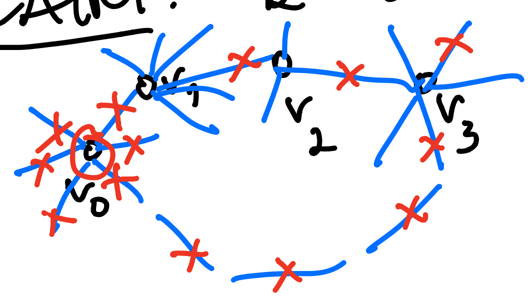
AND

(b) every vertex $v \in V$ has $\deg_G(v)$ even

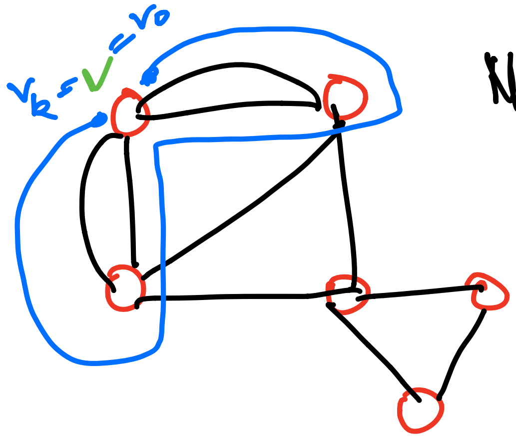
proof: $(\Leftarrow)$: Given a graph G that is connected and has all $\deg_G(v)$ even, let's give an algorithm to find an Euler tour.

Start at some vertex $v_0 \in V$, move along an edge to some v_1 and erase the edge you used. Then go from v_1 to some v_2 along an unused edge. Repeat until you get stuck at some vertex v_k where all its incident edges were erased.

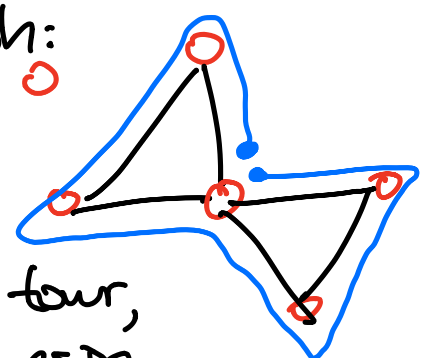
CLAIM: $v_k = v_0$



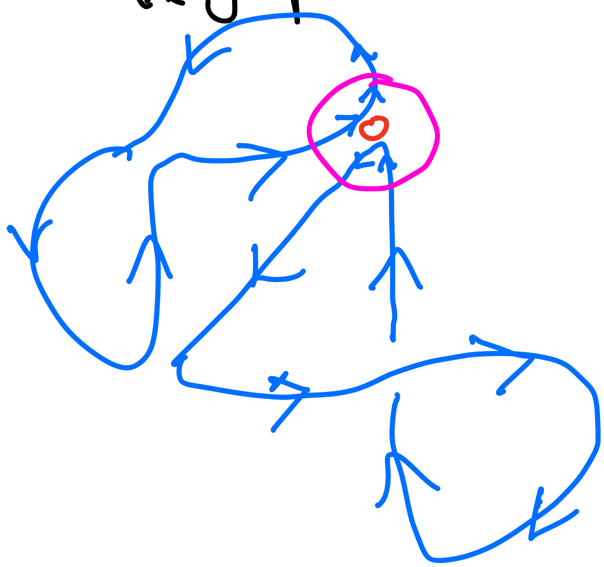
Only v_0 has odd degree in the erased edge graph, as all others maintain even degree as you enter and leave.



Now remove the tour just created, and repeat with the rest of the graph:



Similarly create another tour, and repeat until all edges in the graph are used.



Now suture them together into one big tour

∎