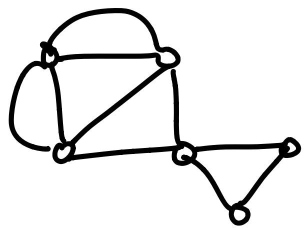
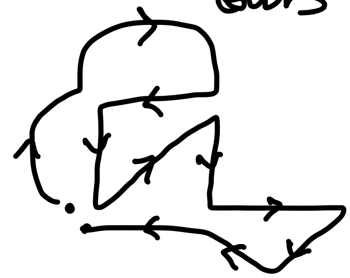
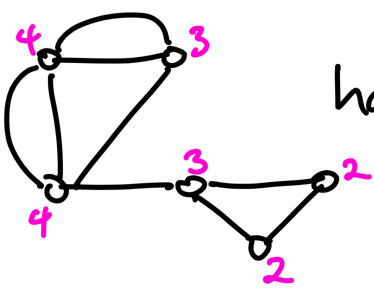
Math 4707 GROUP WORK on Euler paths/walks

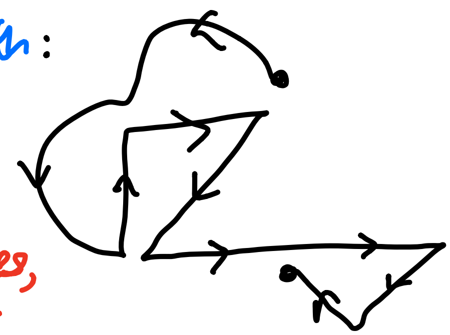
Recall that  has (many) Euler tours

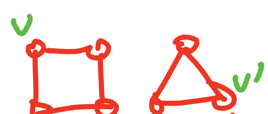
e.g. 

while



has no Euler tour, but does have an Euler path:





has no isolated vertices, but not connected.

QUESTIONS

(1) Try to come up with a conjecture that characterizes graphs $G=(V,E)$ having an Euler path, but no Euler tour, similar in spirit to the one we proved for graphs having an Euler Tour. Can you prove it?

THEOREM: G a graph with no isolated vertices has an Euler path but no Euler tour $\Longleftrightarrow$

(a) G is connected

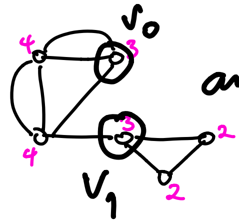
AND

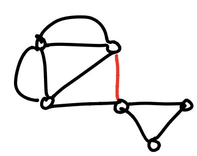
(b) exactly of its vertices v_0, v_1 have odd degree, all others even degree.

Furthermore, every Euler path in G will start & end at the two odd-degree vertices.

proof:

① Similar to Euler tour proof.

② Take $G=$ 

and add an edge v_0 to v_1 ,

and create the Euler tour, and the extra edge to get an Euler path. ∎

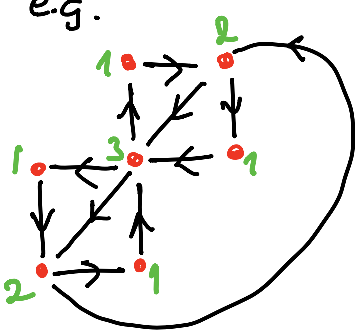
② Consider a directed graph $D = (V, A)$
(digraph)

V = vertices, A = directed arcs

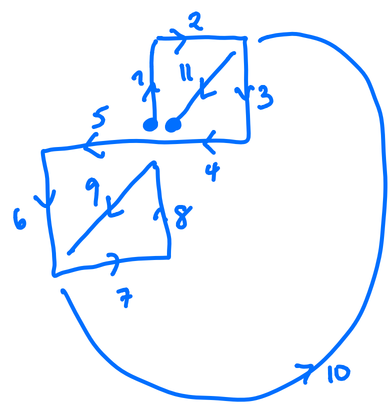
and directed Euler tours

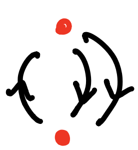
:= sequences of arcs that start at and end at a vertex v_0, follow the arrows along arcs and traverse each arc in A exactly once

e.g. indegs = outdegs

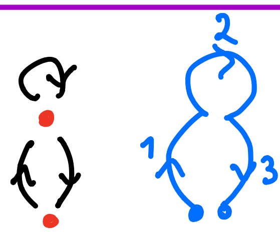


has directed Euler tour

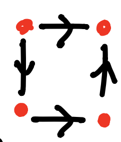




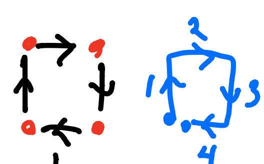
has none



has some



has none



has some

Can you write down a characterization of which digraphs $D = (V, A)$ have directed Euler tours, similar to the undirected case?

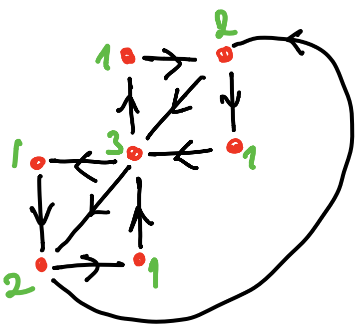
THEOREM: A digraph $D=(V,A)$ with no isolated vertices has a directed Euler tour $\iff$

(a) D is connected in the undirected or directed sense

AND

(b) every vertex $v \in V$ has

$$\underbrace{\text{outdeg}_D(v)}_{=\#\{\text{arcs } a\in A \text{ emanating from } v \xrightarrow{a} w\}} = \underbrace{\text{indeg}_D(v)}_{=\#\{\text{arcs } a \in A \text{ entering } v:\ w \rightarrow v\}}$$



proof: All the same proof as in the undirected Euler tour proof! $\blacksquare$

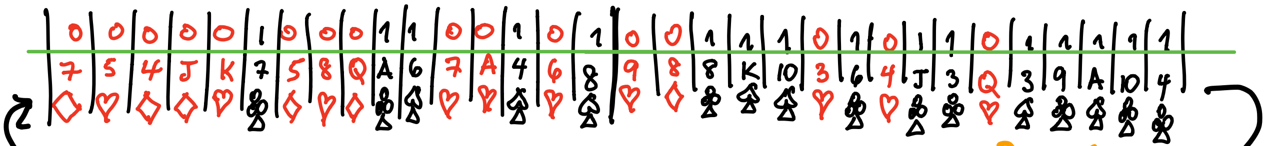
Math 4707 Oct. 19, 2020

Eulerian digraphs & De Bruijn sequences

FIRST: A card trick due to Persi Diaconis ... (mathematician-magician)

He throws a (prepared) deck of cards in a rubber band into the audience, asks several people to do usual cuts of the deck, then pass it to next 5 people, who draw the top 5 cards. Asks them to think about their cards, with those holding red cards to stand up. Then he guesses all 5 cards.

How does he do it? The deck starts out like this..

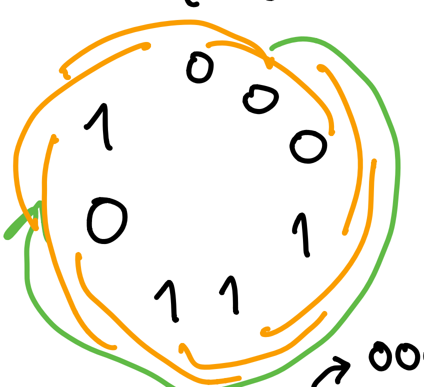


This is a DeBruijn sequence on $k=2$ letters of order $n=5$

DEFINITION:

A **DeBruijn sequence on k letters** $\{0,1,2,\dots,k-1\}$ **of order n** is a (circular) sequence of k^n letters having each possible word of length n appearing exactly once as a consecutive subword

e.g. $k=2, n=3$
$\{0,1\}$

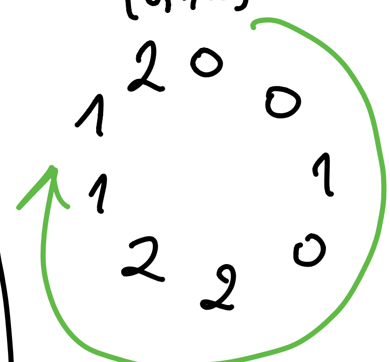


00011101

$2^3=8$

000
001
011
111
110
101
010
100

$k=3, n=2$
$\{0,1,2\}$



$3^2=9$

00
01
10
02
22
21
11
12
20

Q: Do DeBruijn sequences exist for every k and n? If so, how do we find them?

YES, and it's related to Eulerian digraphs!

Nice idea: Fixing k, n, make a digraph

$D_{k,n} = (V, A)$

V = {length $n-1$ sequences of letters $\{0,1,\dots,k-1\}$} $(a_1, a_2, \dots, a_{n-1})$

A = { initial $n-1$ subword $(a_1, \dots, a_{n-1})$ $\longrightarrow$ final $n-1$ subword $(a_2, \dots, a_n)$ }

$(a_1, a_2, \dots, a_{n-1}, a_n)$ n word

Nice idea: Fixing k, n, make a digraph $D_{k,n} = (V, A)$

V = {length $n-1$ sequences of letters $\{0,1,\dots,k-1\}$} $(a_1, a_2, \dots, a_{n-1})$

A = {initial $n-1$ subword $(a_1, \dots, a_{n-1})$ $\xrightarrow{}$ final $n-1$ subword $(a_2, \dots, a_n)$}, $(a_1, a_2, \dots, a_{n-1}, a_n)$ n word

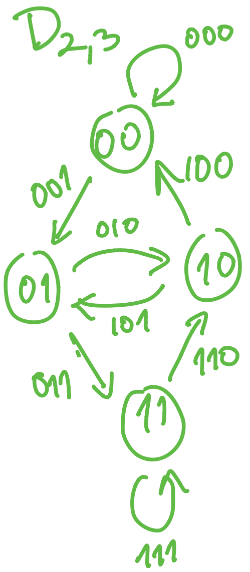
KEY POINT:

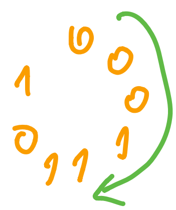
{DeBruijn sequences of order n k letters}

$\updownarrow$ bijection

{directed Euler tours in $D_{k,n}$}

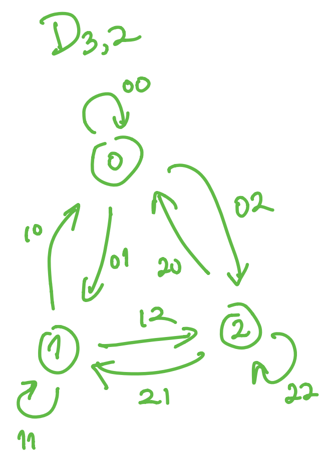
e.g. $k=2, n=3$

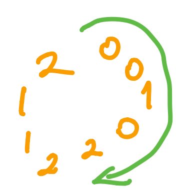




00011101

$k=3, n=2$

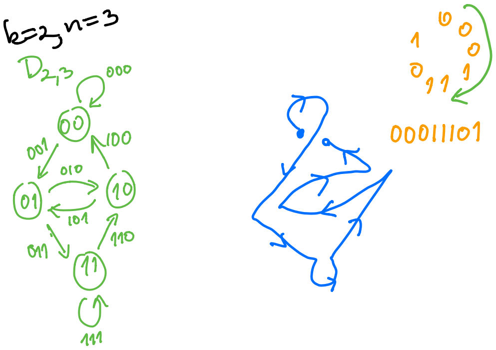




001022112

Math 4707 Oct. 21, 2020

e.g. $k=2, n=3$



$k=3,\ n=2$

<u>THEOREM</u>: DeBruijn sequences of order n on k letters $\{0,1,\ldots,k-1\}$ exist for all k,n since the digraphs $D_{k,n}$ are

(a) connected

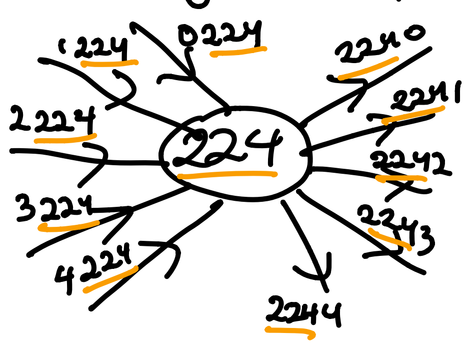
AND

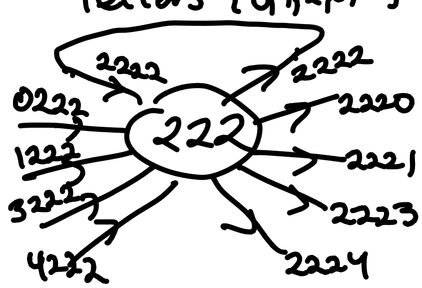
(b) have $\text{indeg}_{D_{k,n}}(v) = \text{outdeg}_{D_{k,n}}(v)\ \forall v \in V$

(in fact, both equal k)

<u>proof</u>: For (b), let's prove it by ~~a~~ rich enough example(s): Inside $D_{5,4}$

letters $\{0,1,2,3,4\}$





THEOREM: DeBruijn sequences of order n on k letters $\{0,1,\ldots,k-1\}$ exist for all k,n since the digraphs $D_{k,n}$ are

(a) connected

AND

(b) have $\text{indeg}_{D_{k,n}}(v) = \text{outdeg}_{D_{k,n}}(v)$ $\forall v \in V$

(in fact, both equal k)

proof: For (a), any two vertices v, v' have a path $v \to \cdots \to v'$ with $\leq n-1$ steps:

Proof by rich enough EXAMPLE $n=5$

$$\underset{v}{abcd} \xrightarrow{abcde} bcde \xrightarrow{bcdef} cdef \xrightarrow{cdefg} defg \xrightarrow{defgh} \underset{v'}{efgh}$$

$$1021 \xrightarrow{10214} 0214 \xrightarrow{02143} 2143 \xrightarrow{21431} 1431 \xrightarrow{14311} 4311 \quad \blacksquare$$

$$a_1 a_2 \cdots a_{n-1} \xrightarrow{a_1 a_2 \cdots a_{n-1} b} a_2 \cdots a_{n-1} b \to \qquad b_1 b_2 \cdots b_{n-1}$$

Hamiltonian tours/circuits and paths

DEF'N $:=$ walks from vertex to vertex in an undirected graph $G=(V,E)$ that visit every vertex $v \in V$ exactly once (but may miss some edges).

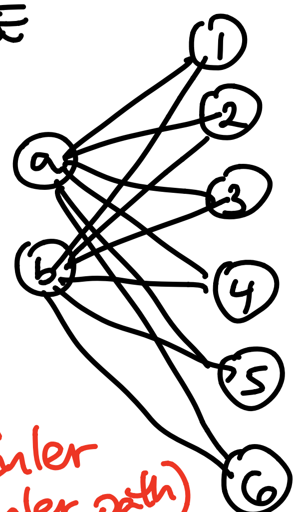
A <u>tour</u> if start vertex = end vertex
<u>path</u> if not.

Sounds close to Euler tours & paths, but is <u>not</u> <u>really</u>.

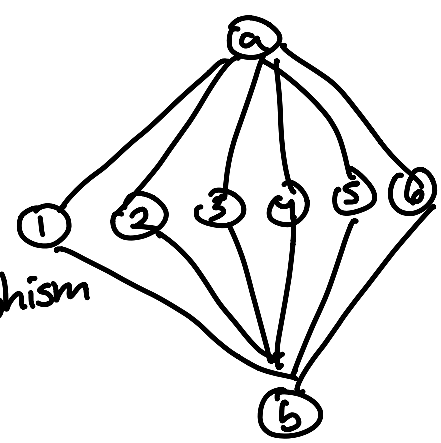
EXAMPLE

$K_{2,6}$

complete bipartite graph



$\cong$ graph isomorphism

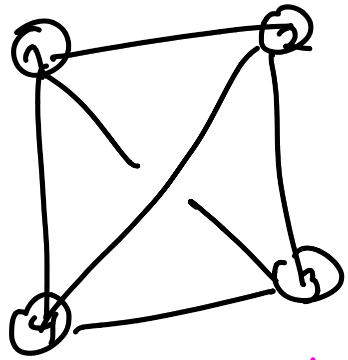


has an Euler tour (no Euler path)

and has no Hamiltonian tour, nor Hamiltonian path

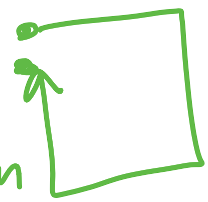
EXAMPLE

K_4
complete graph

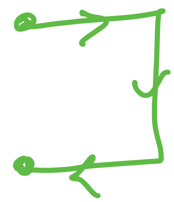


No Euler tour, nor Euler path.

But has a Hamiltonian tour;

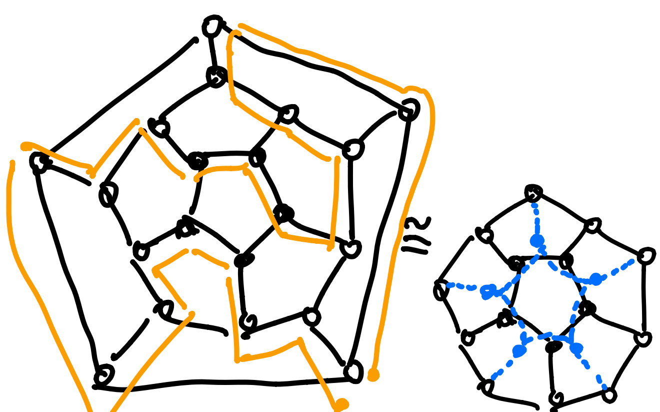


(and Hamiltonian path)

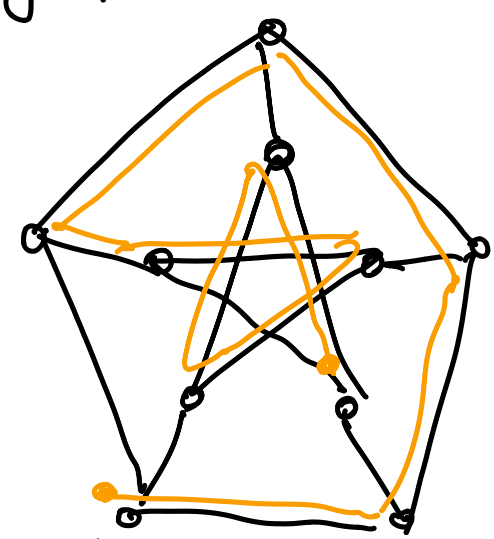


It's hard to decide if a graph G has a Hamiltonian tour/path.

Two examples from book:



Hamilton's game: find such a tour!



Petersen graph

Has a Hamilton path but no Ham. circuit (not obvious)

There is no known algorithm for deciding whether $G=(V,E)$ has a Hamiltonian circuit or path that does much better than checking all $(\#V)!$ orderings of V to see if any of them are a Hamiltonian circuit or path!

(We'll come back to this when discuss lack of good algorithms for T.raveling S.alesperson P.roblem)

There do exist theorems giving

- sufficient conditions for G to have such a circuit/path
- necessary conditions for having them

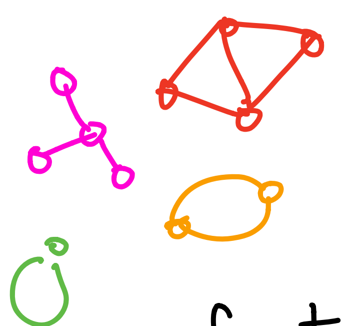
(see e.g. Bondy & Murty §4.2
West §7.2)

Essentially the same holds for directed versions of Hamiltonian circuits/paths.

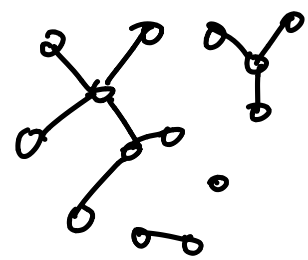
Chapter 8 Trees

DEFINITION: A graph $G=(V,E)$ with *no cycles* is called a *forest*; its connected components are called *trees*. That is, a *tree* is a *connected* graph with *no cycles*.

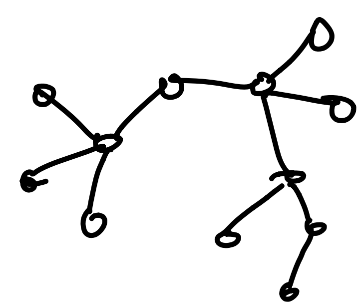
EXAMPLES



not a forest, with 4 connected components C_1, C_2, C_3, C_4

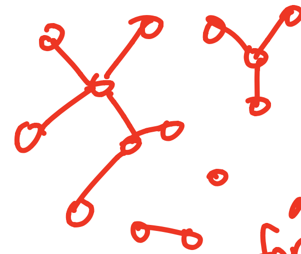


is a forest, but not a tree.



a tree

That is, a tree is a connected graph with no cycles.



a forest

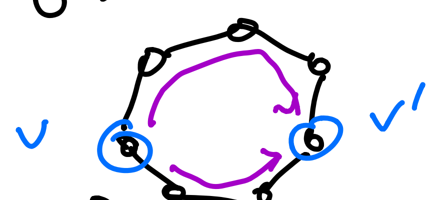
Another way to say it...

PROPOSITION:

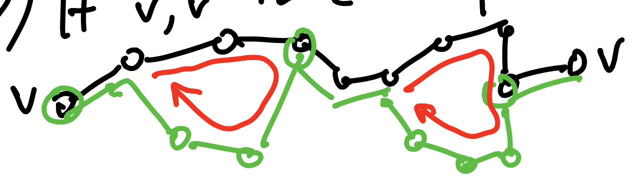
(a) A graph $G=(V,E)$ is a forest (has no cycles)
$\iff \exists$ at most one path v to v' $\forall v,v'\in V$ (possibly none)

(b) A graph $G=(V,E)$ is a tree (connected, no cycles)
$\iff \exists$ exactly one path v to v' $\forall v,v'\in V$

proof: (a) ($\Leftarrow$): If $\exists \leq 1$ path v to v' in G, then G must have no cycles:



($\Rightarrow$) If v, v' have two paths between them



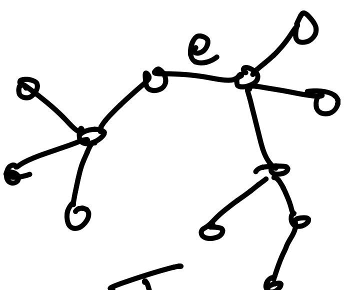
(b) Follows from (a) since connected means $\exists \geq 1$ path v to v' ∎

Another way to characterize trees...

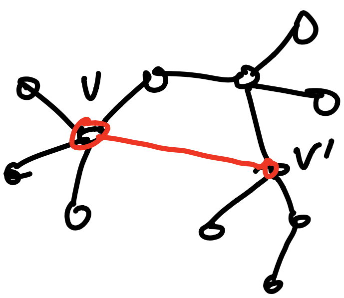
PROPOSITION

(a) G is a tree $\iff$ G is connected and removing any edge disconnects it.

(b) G is a tree $\iff$ G has no cycles, but adding any edge creates a cycle

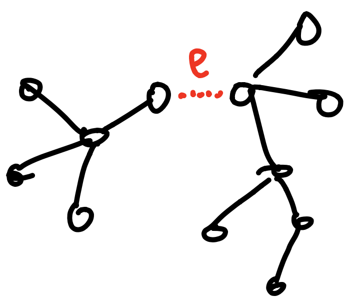


T



for part (b)

T + e' between $\{v, v'\}$



T − e

for part (a)

EXAM 1 Stats

median 98
average 93
standard deviation 8

GUESSED GRADE : ≥ 90 predicts an A of some kind

80–90 predicts some kind of B

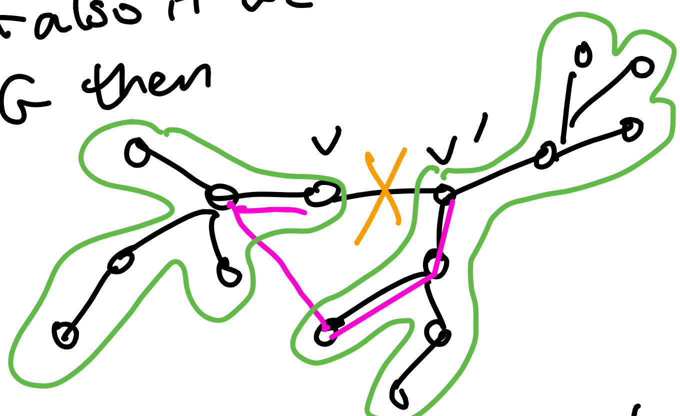
< 80 C or below.

PROPOSITION

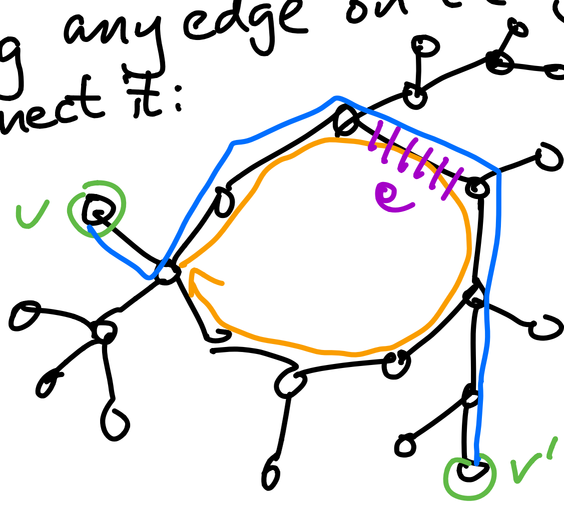
(a) G is a tree $\iff$ G is connected and removing any edge disconnects it.

(b) G is a tree $\iff$ G has no cycles, but adding any edge creates a cycle

proof: ($\Rightarrow$): G a tree implies it's connected, but also if we remove any edge $\{v,v'\}$ in G then

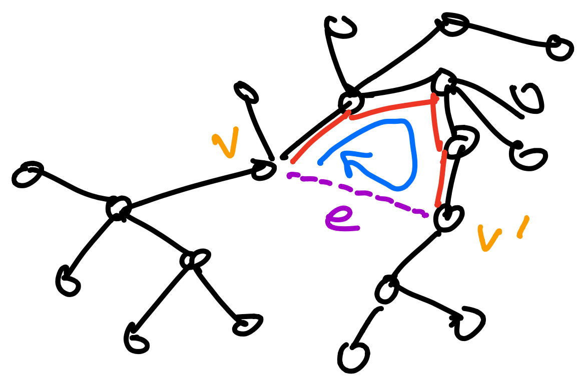


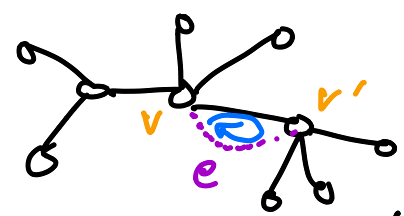
($\Leftarrow$): If G is connected and removing any edge disconnects it, then it must have no cycles because removing any edge on the cycle cannot disconnect it:



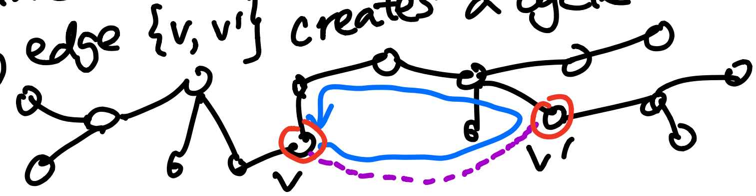
(b) G is a tree $\iff$ G has no cycles, but adding any edge creates a cycle

proof: ($\implies$): G a tree implies it has no cycles but also, adding any edge creates a cycle say from v to v' by



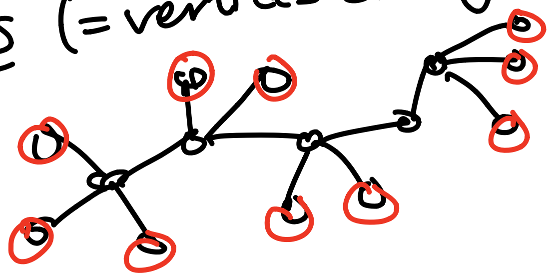


($\impliedby$): Assuming G has no cycles, but adding any edge creates one, since G is a forest we need to know it's connected. So given $v, v' \in V$ adding edge $\{v, v'\}$ creates a cycle

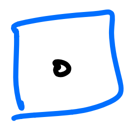


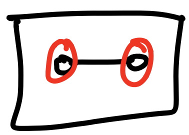
NEXT let's prove

PROPOSITION: Every finite tree with at least 2 vertices has at least 2 leaves (= vertices of degree one)

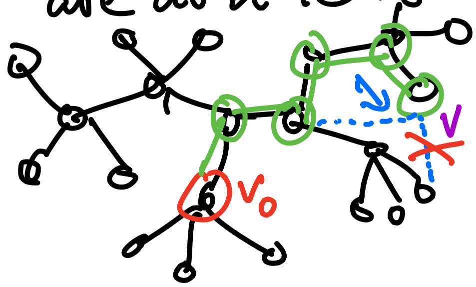


leaves in red

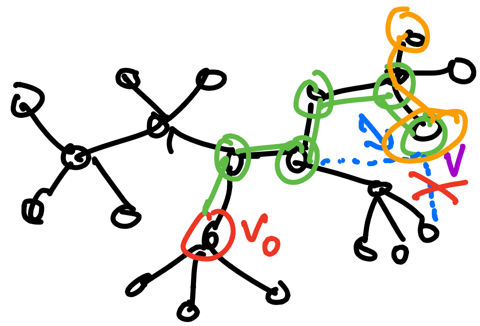
proof: (NOTE: This tree  has no edge, no leaves

 has two leaves)

Start at any vertex $v_0 \in V$ in the tree and walk along edges to new (unvisited) vertices until you get stuck: then you are at a leaf, v because if v has another edge aside from the one you entered to some visited vertex, and this creates a cycle, or two paths v_0 to v ∎

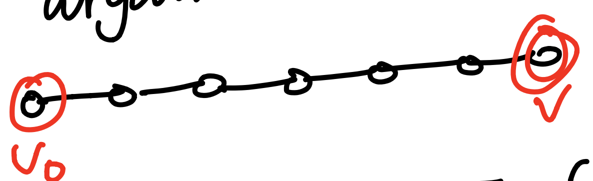


To get at least two leaves, do the same argument starting from the leaf v that you just found.

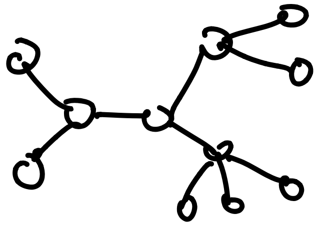


$\blacksquare$

Q: Why can't we find more than 2 leaves by iterating this argument?

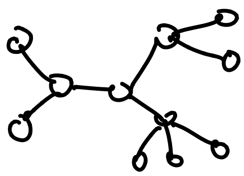


COROLLARY: Any tree $T=(V,E)$ with $n=|V|$ has $|E|=n-1$ edges, and hence $\sum_{v\in V} \deg_T(v) = 2|E| = 2(n-1)$
$= 2n-2$.



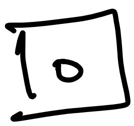
$|V|=11$
$|E|=10=|V|-1$

COROLLARY: Any tree $T=(V,E)$ with $n=|V|$ has $|E|=n-1$ edges, and hence $\sum_{v\in V}\deg_T(v)=2|E|=2(n-1)$
$=2n-2.$



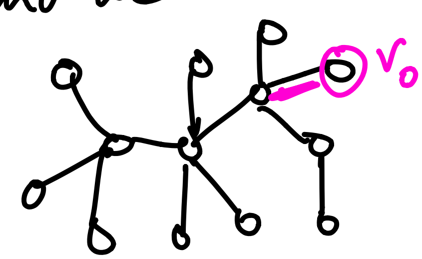
proof: Prove it by induction on $n=|V|$.

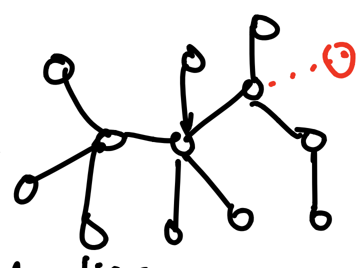
BASE CASE: $|V|=1$ has no edges



so $|E|=0=|V|-1$. ✓

INDUCTIVE STEP: Since we can assume $|V|=n\geq 2$, $\exists$ some leaf vertex v_0 and we can remove it to get $T' = T-\{v_0\}$

$T=$ 

$T-v_0=$ 

By induction,
$|E(T-v_0)|=|V(T-v_0)|-1$
$=(n-1)-1$
$=n-2$ ∎

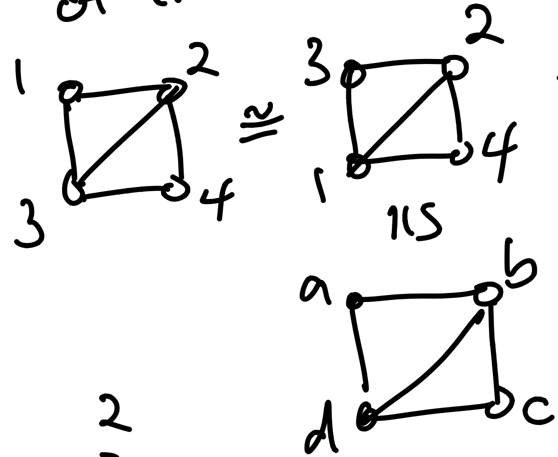
$|E(T)|=|E(T-v_0)|+1$
$=n-2+1$
$=n-1$ $\Leftarrow$

Need $T-v_0$ is still a tree! (EXERCISE)

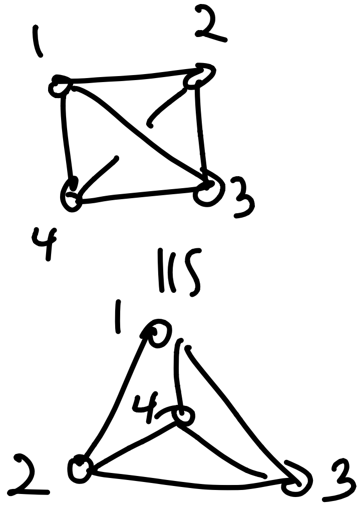
Q: How many labeled and unlabeled trees are there on n vertices $\{1, 2, \dots, n\}$?

"Labeled" means we distinguish $\{1, 2, \dots, n\}$ from each other

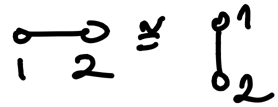
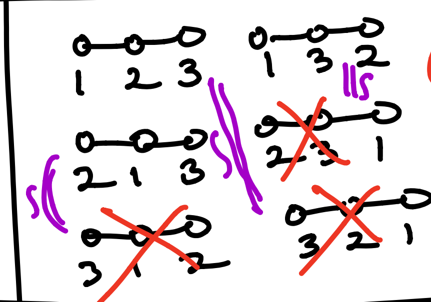
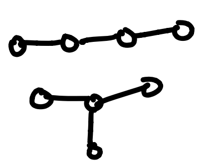
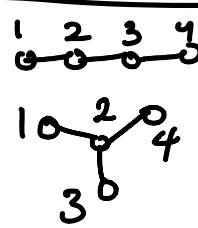
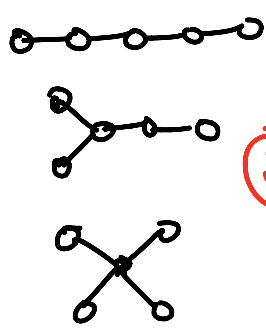
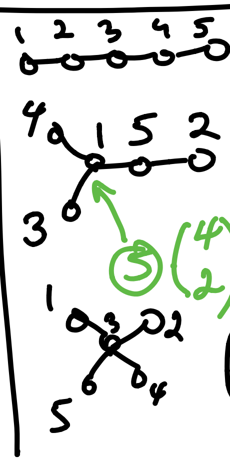
"Unlabeled" means isomorphism classes of trees



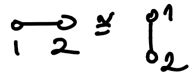
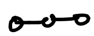
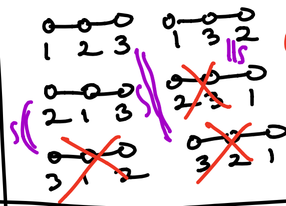
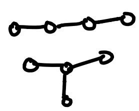
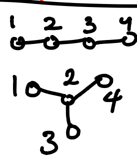
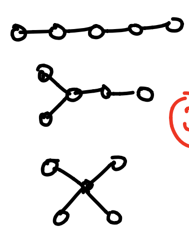
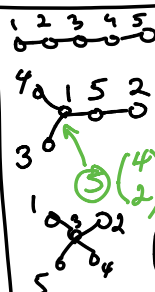
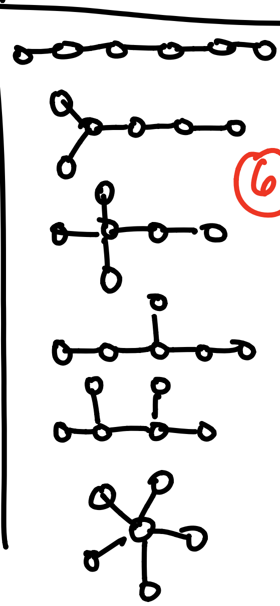
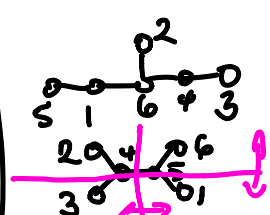
are isomorphic graphs



Q: How many labeled and unlabeled trees are there on n vertices $\{1,2,\dots,n\}$?

n	unlabeled trees on n vertices		labeled	
1		1	1o	1
2		1		1
3		1		3
4		2	 $\frac{4!}{2}=12$ + $\binom{4}{1}=4$	16
5		3	 $\frac{5!}{2}=60$ + $\frac{5!}{2}=60$ + $\binom{5}{1}=5$	125

$5\binom{4}{2}2=60$

n	unlabeled trees on n vertices	labeled
1	 1	1 1 $= 1^{-1}$
2	 1	 1 $= 2^0$
3	 1	 3 $= 3^1$
4	 2	 $\frac{4!}{2} = 12$ $+$ $\binom{4}{1} = 4$ 16 $= 4^2$
5	 3	 $\frac{5!}{2} = 60$ $+$ $\frac{5!}{2} = 60$ $3\binom{4}{2}2 = 60$ $+$ $\binom{5}{1} = 5$ 125 $= 5^3$
6	 6	$6!/2 = 360$ $+$ $6!/2 = 360$ $+$ $6!/3! \overset{?}{=} \binom{6}{3} \cdot 3! = 120$  $+$ $6!/2 = 360$ $+$ $6! / 2 \cdot 2 \cdot 2 = 90$ $+$ $\binom{6}{1} = 6$ 1296 $= 6^4$

THEOREM (Cayley) 1890?

labeled trees on n vertices

$= n^{n-2}$