MATH 4707 Nov. 16, 2020

A non-trivial corollary...

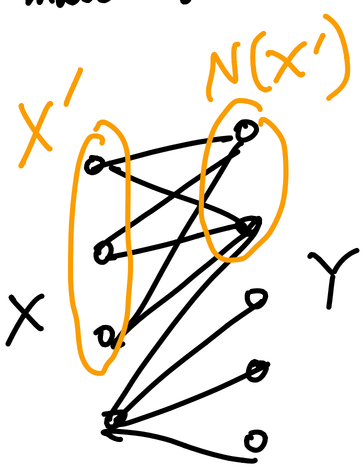
THEOREM (Hall's Marriage Theorem)

In a bipartite graph $G = (\underbrace{X \sqcup Y}_{V}, E)$ there will be a matching M that matches all of X (i.e. $\#M = \#X$)

$\iff$ $\forall$ subsets $X' \subseteq X$

$$\#\underbrace{N(X')}_{\substack{\text{neighbors of } X' \\ := \text{neighbors of at least one } x' \in X'}} \geq \#X'$$

EXAMPLE:
This G has **no** matching of all of X:



proof: ~~next time~~, using Hungarian algorithm...

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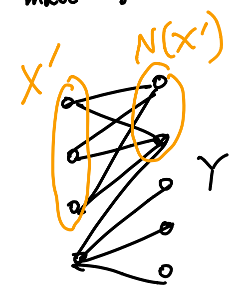


Proof: ($\Longrightarrow$) is easy enough: if we had a matching M that matched all of X, then matching M that matched all of X, then $\forall X' \subseteq X$, M gives an injective map

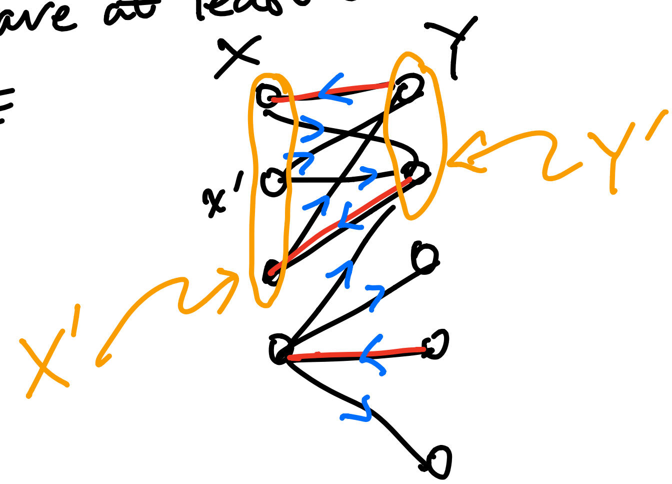
$$X' \longrightarrow N(X')$$
$$x \longmapsto x\text{'s neighbor in } M$$

($\Longleftarrow$): Assume $\nexists$ a matching M of all of X. Use Hungarian algorithm to find a max-sized matching M. There must be at least one vertex $x' \in X$ unmatched in M.

<u>($\Leftarrow$)</u>: Assume $\nexists$ a matching M of all of X. Use Hungarian algorithm to find a max-sized matching M. There must be at least one vertex $x' \in X$ unmatched in M.

Given this $x' \in X$ unmatched in M, define $X' \sqcup Y'$ to be the vertices in G that have at least one directed path from x'.

EXAMPLE



<u>Goal</u>: Show $\underset{(b)}{Y' = N(X')}$ and $\underset{(a)}{\#Y' < \#X'}$.

<u>To show (a)</u>: Note Y' only contains M-matched vertices from Y, because otherwise we would have an M-augmenting path.

But then, M gives an injective map from

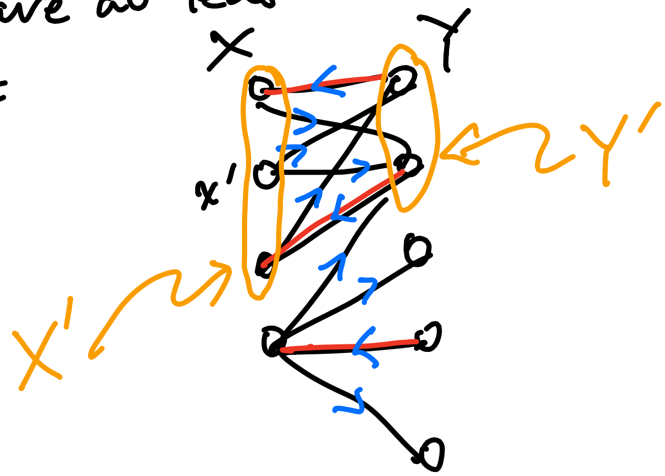
$$Y' \longrightarrow X' - \{x'\}$$

since x' is M-unmached

$\Rightarrow \#Y' < \#X'$

Given this $x' \in X$ unmatched in M,
define $X' \sqcup Y'$ to be the vertices in G
that have at least one directed path from x'

EXAMPLE



Goal: Show $Y' \underset{(b)}{=} N(X')$ and $\#Y' \underset{(a)}{<} \#X'$.

To show (b): $Y' \subseteq N(X')$ since every $y' \in Y'$ had a path from x' to y', and the 2^{nd}-to-last vertex on this path is a vertex in X'.
But also any neighbor y' of some $x'' \in X'$ has a path from x' to x'' and then to y', so it lies in Y'. Hence $N(X') \subseteq Y'$.
Thus $Y' = N(X')$. ∎

COROLLARY: If in a bipartite graph $G = (X \sqcup Y, E)$ has all vertices of same degree $d \geq 1$, then it has a perfect matching.

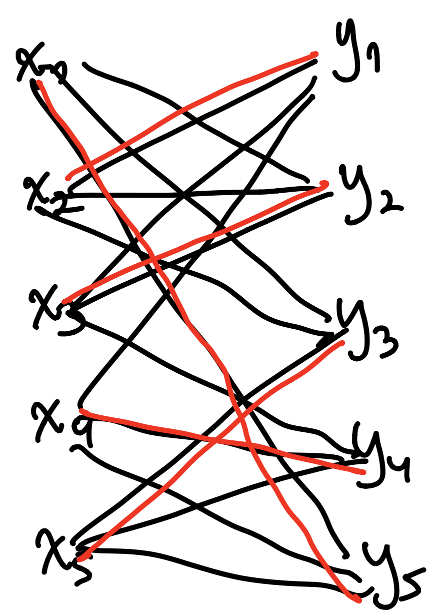
COROLLARY to COROLLARY (!):

In the above setting, E is the disjoint union of d perfect matchings $M_1, M_2, \dots, M_d$.

EXAMPLE:

$d = 3$

$\# X = \# Y = 5$



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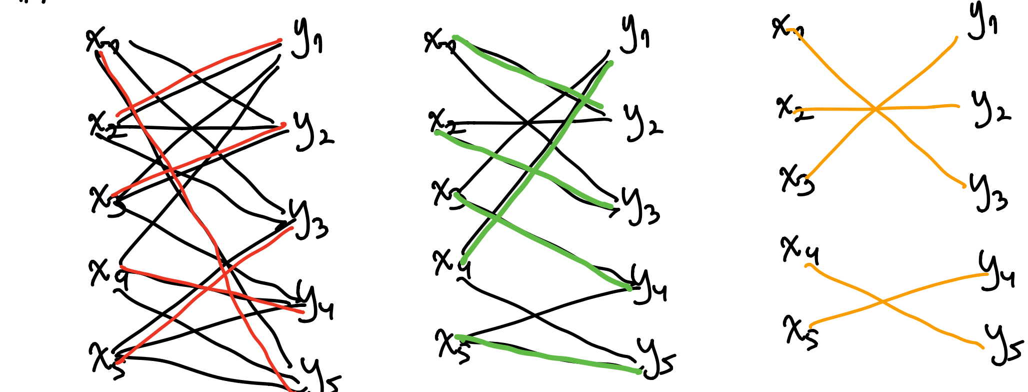
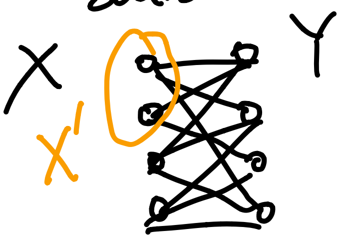


$M_1 \sqcup M_2 \sqcup M_3 = E$

proof of COROLLARY:

Assume that $G=(X \sqcup Y, E)$ has $\deg(x_i)=\deg(y_j)=d$ $\forall\, x_i \in X,\ y_j \in Y$

and let's check Hall's condition is satisfied: given $X' \subseteq X$, consider the subgraph on $X' \sqcup N(X')$ and the edges between them...



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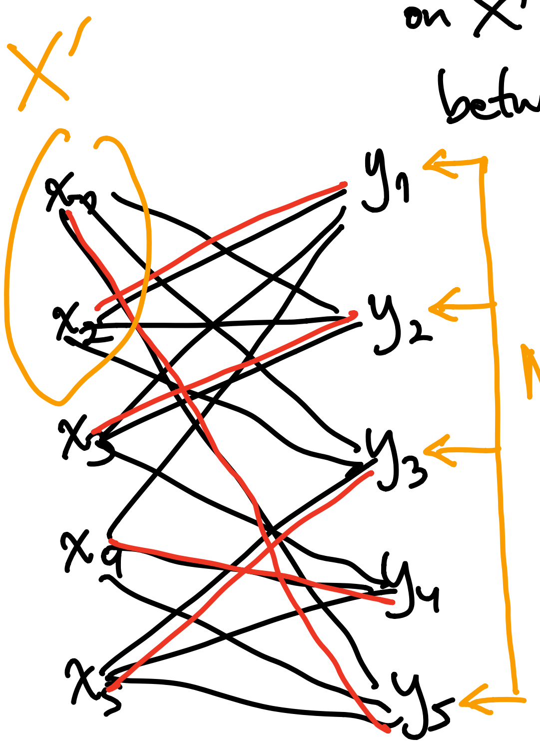
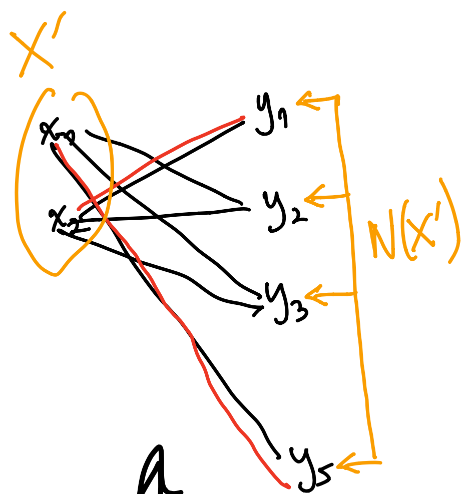




Count in two ways the edges in this subgraph:

On one hand, it is exactly $\sum_{x\in X'} \deg_G(x) = (\#X')\cdot d$

On the other hand, it is $\leq \sum_{y\in N(X')} \deg_G(x) = \#N(X')\cdot d$

So $d(\#X') \leq d\cdot \#N(X')$. Cancel the d's. $\blacksquare$

Similarly...

COROLLARY:

If a square matrix $A = (a_{ij})_{\substack{i=1,\dots,n\\ j=1,\dots,n}}$ has $a_{ij} \geq 0$ and all row sums and columns are equal to $s > 0$,

then (a) $\exists$ a permutation σ of $\{1, 2, \dots, n\}$ so that $a_{i,\sigma(i)} > 0$ for $i = 1, 2, \dots, n$

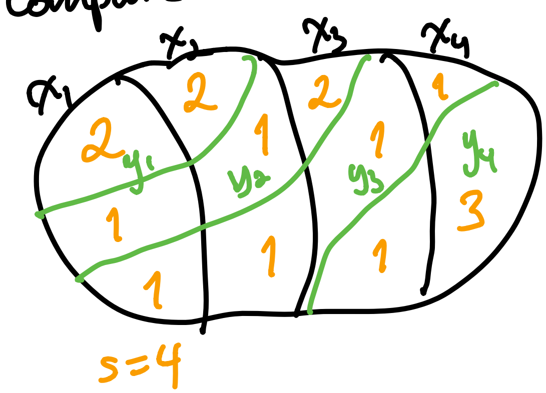
(b) one can write

$$A = s_1 P_1 + s_2 P + \dots + s_m P_m$$ for some $s_i \geq 0$ with $s_1 + \dots + s_m = s$

and P_i are each permutation matrices so they have exactly one nonzero entry equal to 1 in each row and column,

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Compare with the book's "totem problem"



$$\rightsquigarrow A = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \end{array}$$

COROLLARY:

If a square matrix $A = (a_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$ has $a_{ij} \geq 0$ and all row sums and columns are equal to $s > 0$,

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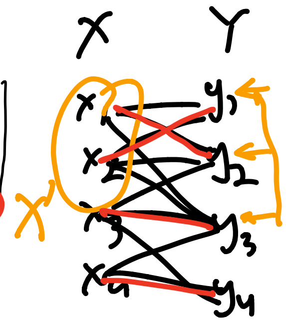
$$A = \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 2 & 1 & 1 & 0 \\ x_2 & 2 & 1 & 1 & 0 \\ x_3 & 0 & 2 & 1 & 1 \\ x_4 & 0 & 0 & 1 & 3 \end{array}$$



proof of (a):

Create from such a matrix A a bipartite graph $G = (X \sqcup Y, E)$ with edges

$X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_n\}$, $E = \left\{ \{x_i, y_j\} \text{ for } a_{ij} > 0 \right\}$

and let's check Hall's condition is satisfied:

Given $X' \subseteq X$, consider the submatrix whose rows are indexed by X' and columns are indexed by $N(X')$, e.g.

$$\begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 2 & 1 & 1 \\ x_2 & 2 & 1 & 1 \end{array}$$

Compute in two ways the sum of all entries in this submatrix:

Summing across rows, it is $s \cdot \#X'$

Summing down columns, it is $\leq s \cdot \#N(X')$

So $s \cdot \#X' \leq s \cdot \#N(X')$,

and we cancel the s to get $\#X' \leq \#N(X')$.

(b) one can write
$A = s_1 P_1 + s_2 P_2 + \dots + s_m P_m$ for some $s_i \geq 0$
with $s_1 + \dots + s_m = s$

To prove (b), first find σ such that $a_{i,\sigma(i)} > 0$
for each $i = 1, 2, \dots, n$

so let $s_1 := \min\{ a_{i,\sigma(i)} \}_{i=1,\dots,n}$

and let $\hat{A} = A - s_1 \cdot \left(\begin{matrix} \text{permutation} \\ \text{matrix for } \sigma \end{matrix} \right)$

$$A = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \end{array} \mapsto \hat{A} = A - 1 \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$s_1 = \min\{1, 2, 1, 3\}$
$= 1$

has row sums, column sums all equal to $s - s_1$
$= 4 - 1 = 3$

Repeat with $\hat{A}$ replacing A, which has strictly fewer nonzero entries.
Induct on # of nonzero entries to conclude the proof. ∎

Math 4707 Nov. 18, 2020

Chapter 12 Planar graphs

DEF'N: A graph $G = (V, E)$ is **planar** if it can be drawn in the plane $\mathbb{R}^2$ with no edges crossing (or self-intersecting)

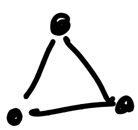
a slippery term!

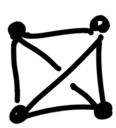
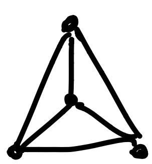
EXAMPLES:

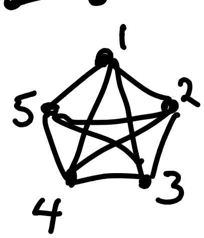
① K_n = complete graph on n vertices

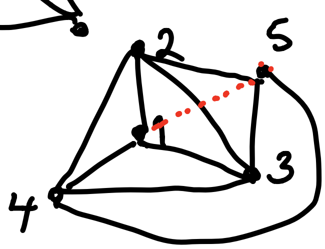
$K_1 =$  planar

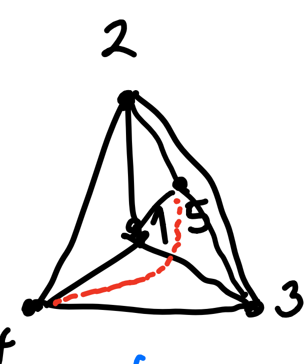
$K_2 =$  planar

$K_3 =$  planar

$K_4 =$  $\cong$  planar

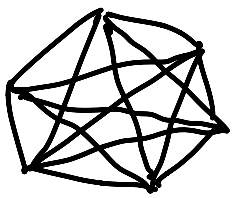
$K_5 =$ 

$\cong$ 

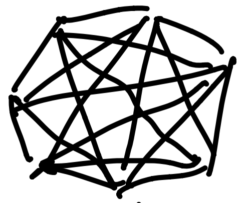
$\cong$ 

Suspect K_5 is not planar

K_6 $\qquad$ K_7 $\qquad$ . . .



suspect not planar, since it contains a K_5 as an edge-subgraph

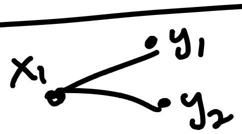


also not planar

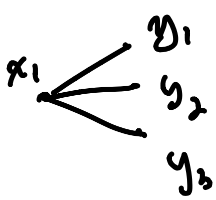
② $K_{m,n}$ = complete bipartite graph

$K_{1,1}$

$K_{1,2}$

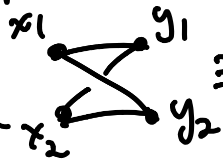


$K_{1,3}$

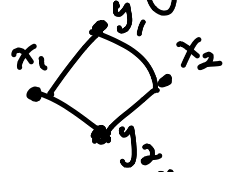


$K_{1,n}$ is planar $\forall n$

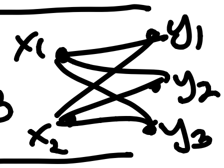
$K_{2,2}$



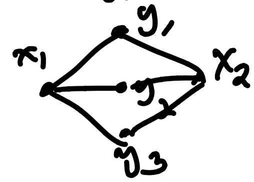
$\cong$



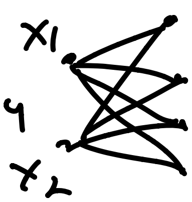
$K_{2,3}$



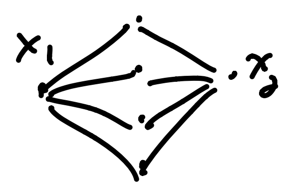
$\cong$



$K_{2,4}$

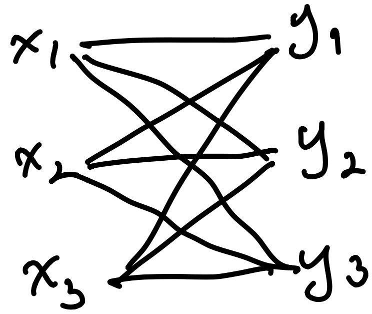


$\cong$



$K_{2,n}$ is planar $\forall n$

$K_{3,3}$



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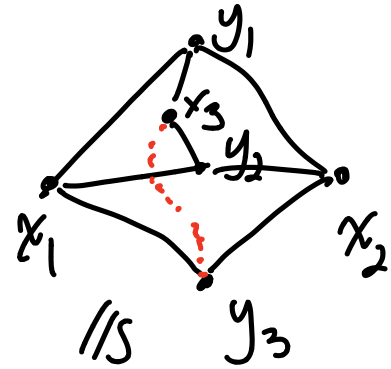
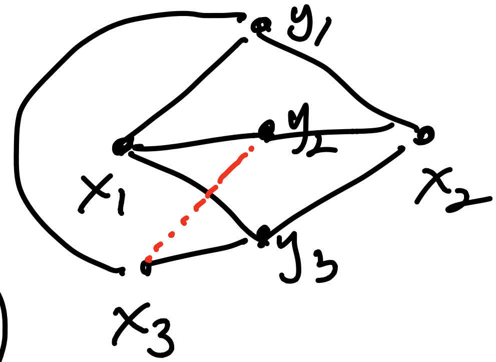


$\cong$



suspect $K_{3,3}$ is not planar

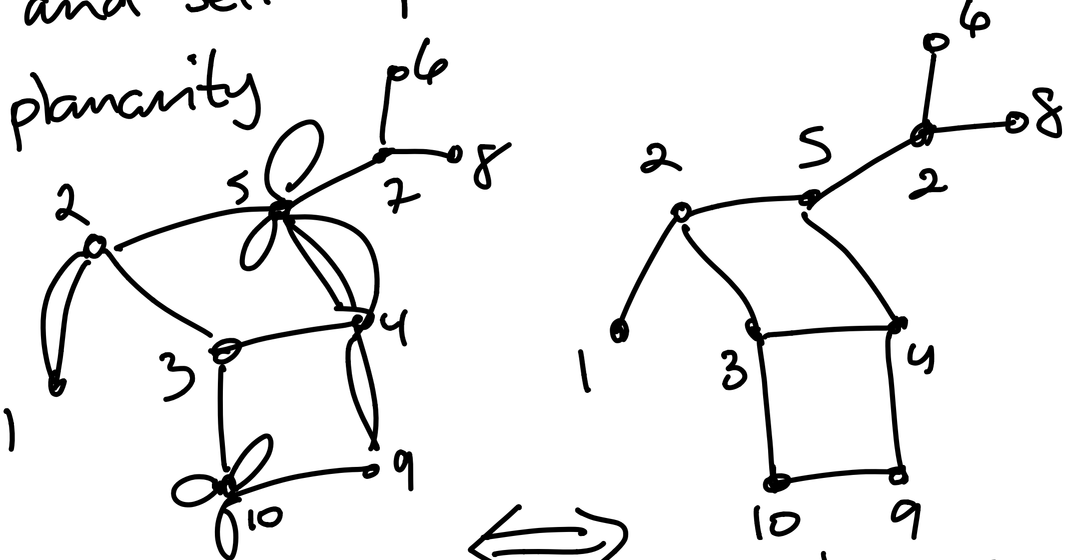
(and neither are $K_{m,n}$ for $m,n \geq 3$)

We think planar graphs can't have too many edges compared to number of vertices?

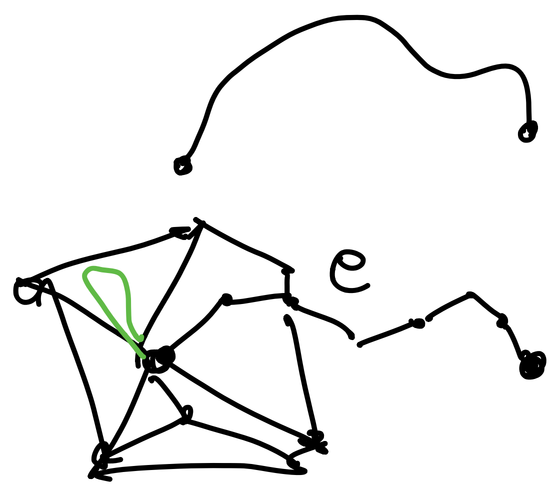
← suspect!

We'll make this precise using Euler's formula + some inequalities

REMARK: Note that multiple edges and self-loops do not affect planarity

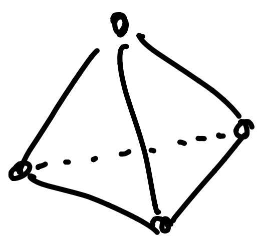
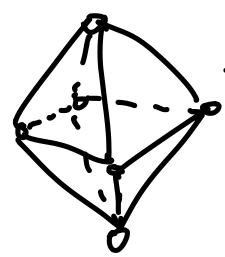
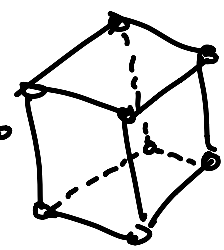


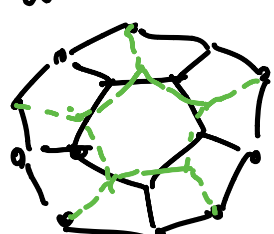
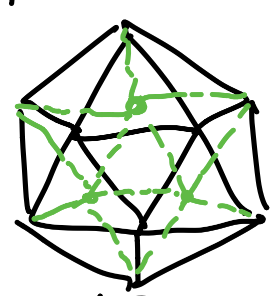
planar $\Longleftrightarrow$ planar



Euler's formula – is a relation between the number of vertices v, edges e and regions f of a planar graph or faces of a polyhedron

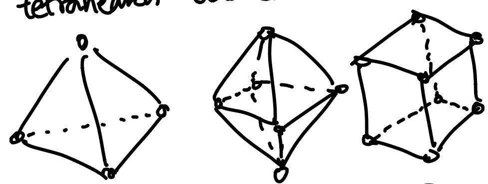
e.g. famous polyhedra are Platonic solids

	tetrahedron	octahedron	cube = hexahedron
			
v = # of vertices	4	6	8
e = # of edges	6	12	12
f = # of faces	4	8	6

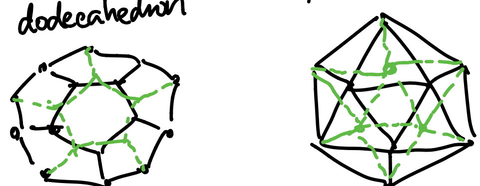
	dodecahedron	icosahedron
		
v	20	12
e	30	30
f	12	20

Euler:

$v - e + f = 2$



	tetrahedron	octahedron	cube = hexahedron
v = # of vertices	4	6	8
e = # of edges	6	12	12
f = # of faces	4	8	6

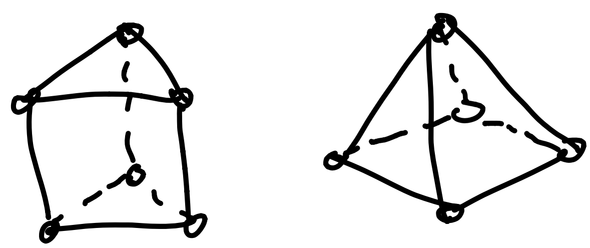


	dodecahedron	icosahedron
v	20	12
e	30	30
f	12	20

Euler:
$v - e + f = 2$

Euler's formula: For 3-dimensional polyhedra, $v - e + f = 2$

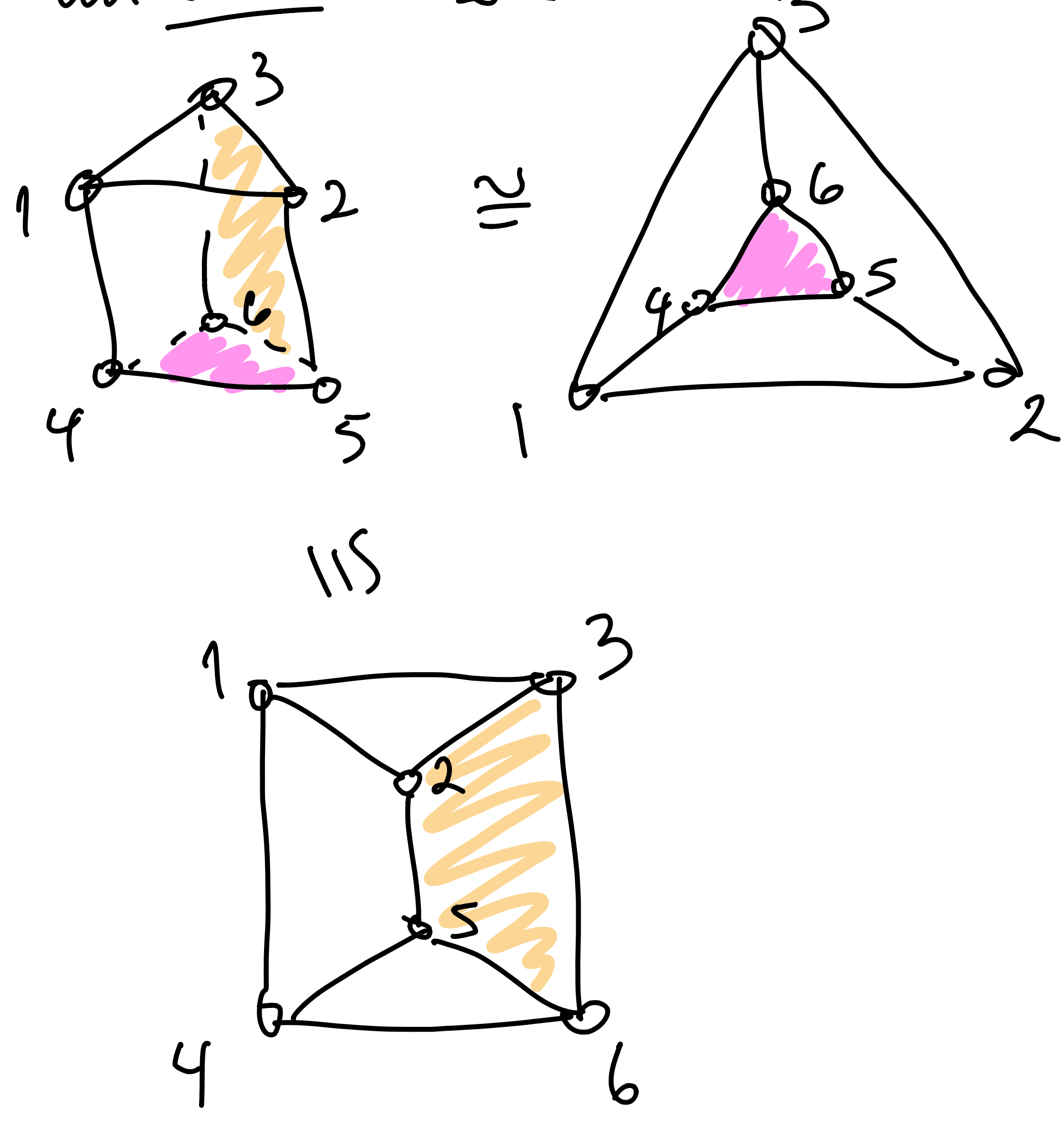
EXAMPLES that are not regular, Platonic...



v	6	5
e	9	8
f	5	5

Still $v - e + f = 2$

Really, Euler's formula is more general, holding for (connected) planar graphs, if we count as <u>faces</u> the <u>regions</u>, including the <u>unbounded region</u>:



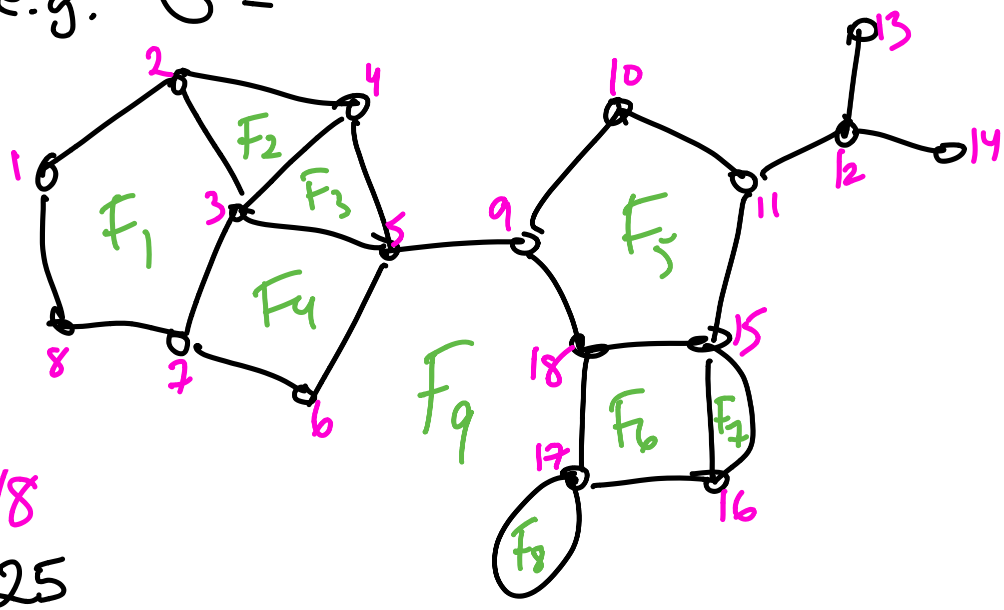
(Euler's) **THEOREM**: In any connected planar graph G, one has $\boxed{v - e + f = 2}$

where $v = \#V$

$e = \#E$

$f = \#$regions/faces, including the unbounded face/region

e.g. $G =$



$v = 18$

$e = 25$

$f = 9$

$v - e + f = 18 - 25 + 9 = 2$ ✓

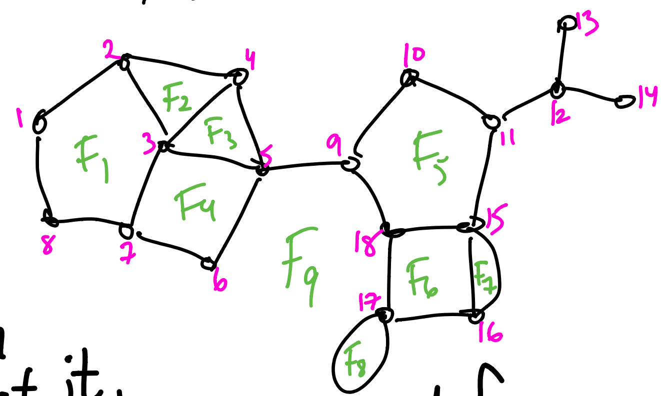
THEOREM : In any connected planar graph G,

one has $v - e + f = 2$

where $v = \#V$

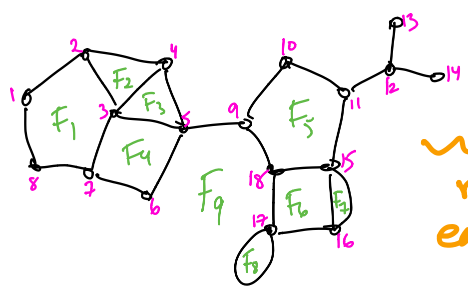
$e = \#E$

$f = \#$regions/faces, including the unbounded face/region

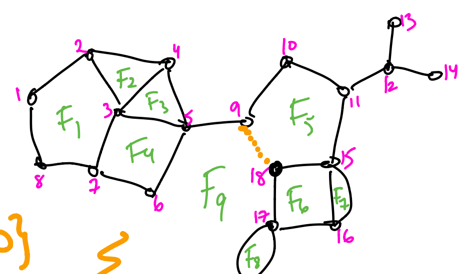


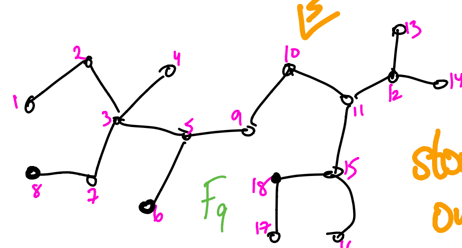
proof:

One-by-one, remove edges from G that do not disconnect it, and rather separate one of its bounded faces from the unbounded face.



e.g. remove edge $\{9,10\}$





At the end, we have $\#$edges $= v-1$, and we erased exactly $f-1$ edges, one for each bounded face.

stop when we have only one (unbounded) region, so left with a tree.

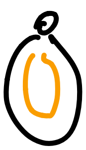
Hence $e = (f-1)+(v-1) = v+f-2$ $\blacksquare$

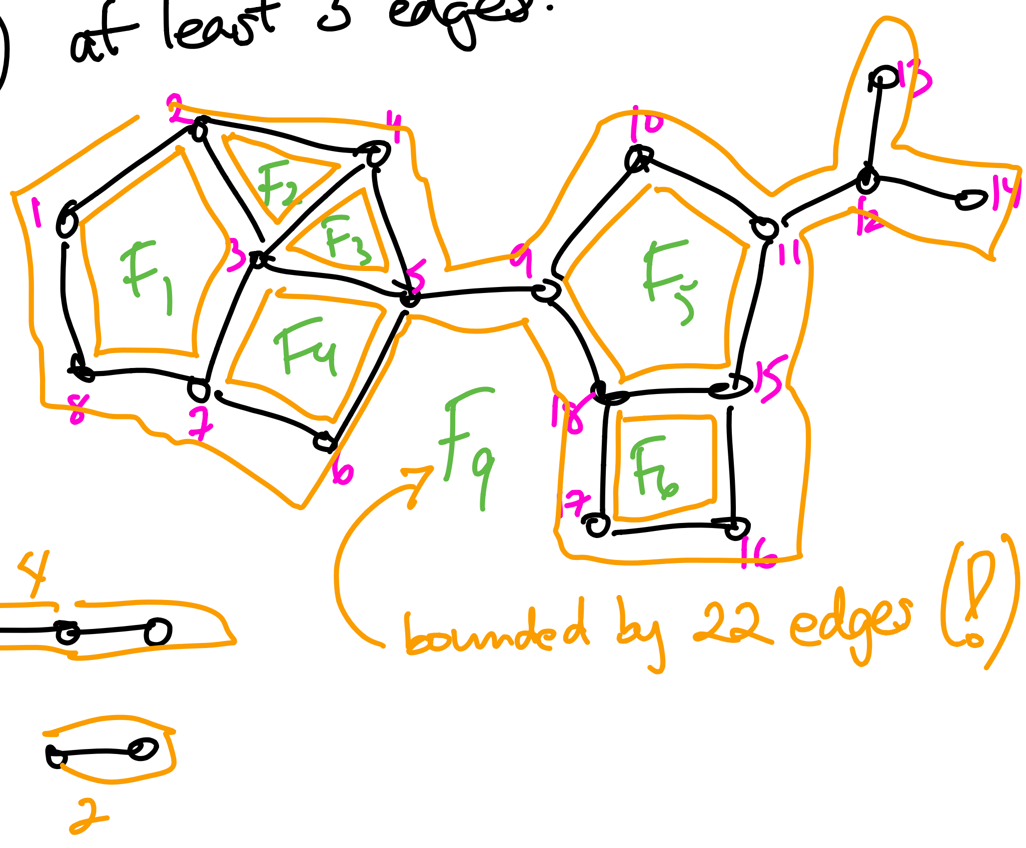
We'll use (Euler's) THEOREM: In any connected planar graph G, one has $\boxed{v-e+f=2}$

to get an inequality for planar graphs relating e and v.

For K_5, $K_{3,3}$, which are simple graphs (no loops, multiple edges)

note that any (connected) graph G with no loops, no multiple edges, and at least two edges, every face is bounded by at least 3 edges:





COROLLARY: If G is planar, connected and has at least two edges, then

$$2e \geq 3f$$

and hence $\boxed{e \leq 3v - 6}$

K_5 fails this!

proof: Next time ...