COROLLARY: If G is planar, connected and has at least two edges, then

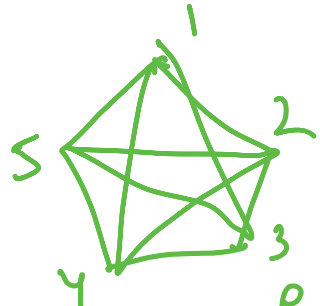
$$2e \ge 3f$$

and hence $\boxed{e \le 3v-6}$

K_5 fails this!

proof: Next time ...

... Since



K_5

$e = \binom{5}{2} = \frac{5\cdot 4}{2} = 10$

$v = 5$

$$\underset{10}{e} \not\le \underset{\substack{\| \\ 3\cdot 5-6 \\ \| \\ 15-6 \\ \| \\ 9}}{3v-6}$$

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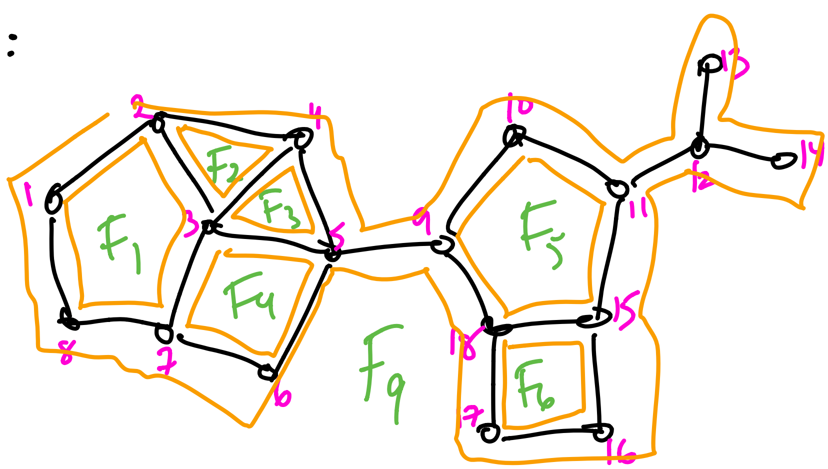
COROLLARY: If G is simple (no loops, no multiple edges) planar, connected and has at least two edges, then

$$\boxed{2e \geq 3f}$$

and hence $\boxed{e \leq 3v - 6}$

K_5 fails this!

proof:



To show $2e \geq 3f$, let's count the number of orange edges/segments above in two ways:

$$2 \cdot e = \#\text{orange segments} = \sum_{\text{faces } F} \underbrace{\left(\#\text{edges bounding } F\right)}_{\boxed{\geq 3}}$$

$$\boxed{2e \geq 3f}$$

Starting with $\boxed{2e \geq 3f}$, one has $\frac{2}{3}e \geq f$

so Euler's formula

$$v - e + f = 2$$

$$\Rightarrow \quad v - e + \frac{2}{3}e \geq 2$$

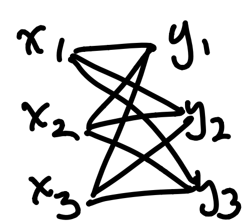
$$v - \frac{1}{3}e \geq 2$$

$$v - 2 \geq \frac{1}{3}e$$

$$\boxed{3v - 6 \geq e} \quad \blacksquare$$

So we understand why K_5 can't be planar (and $K_6, K_7, \ldots$)

but what about K_3



?

having $v = 6$

$e = 3 \cdot 3 = 9$

which satisfies $\underbrace{e}_{9} \leq \underbrace{3v - 6}_{3 \cdot 6 - 6 \,=\, 12}$

Let's use the bipartiteness of $K_{3,3}$ ....

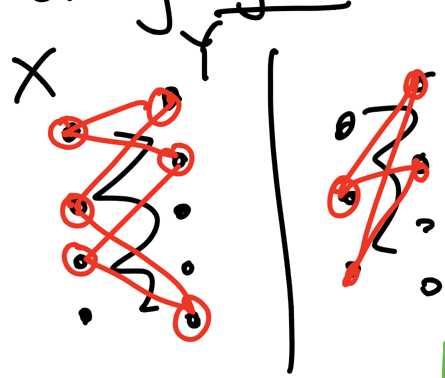
PROPOSITION: If G is a bipartite, simple graph with at least 2 edges, (no loops, no multiple edges) then $\boxed{2e \geq 4f}$

and $\boxed{e \leq 2v-4}$

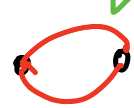
$K_{3,3}$ fails this, so is not planar:

$$\underbrace{e}_{\substack{\| \\ 9=3\cdot 3}} \not\leq \underbrace{2v-4}_{\substack{\| \\ 2\cdot 6-4 \\ \| \\ 12-4 \\ \| \\ 8}}$$

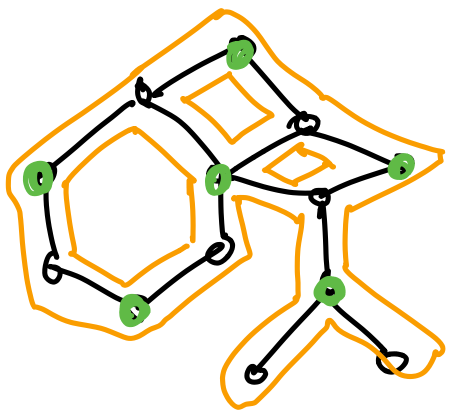
proof: In a bipartite graph, every cycle has an even number of edges:



And it it's simple, it is at least 4 (not 2)



So the faces/regions in a bipartite simple planar graph, with ≥ 2 edges, are quadrangles, hexagons, octagons, ... having ≥ 4 edges bounding them:



Counting the orange segments in two ways gives

$$2e = \#\text{orange segments} = \sum_{\text{faces } F} \underbrace{\left(\#\text{edges bounding } F\right)}_{\geq 4}$$

so $\boxed{2e \geq 4f} \Rightarrow \frac{e}{2} \geq f$

Then Euler's formula $v - e + f = 2$

gives

$$v - e + \frac{e}{2} \geq 2$$

$$v - \frac{e}{2} \geq 2$$

$$v - 2 \geq \frac{e}{2}$$

$$\boxed{2v - 4 \geq e} \quad \blacksquare$$

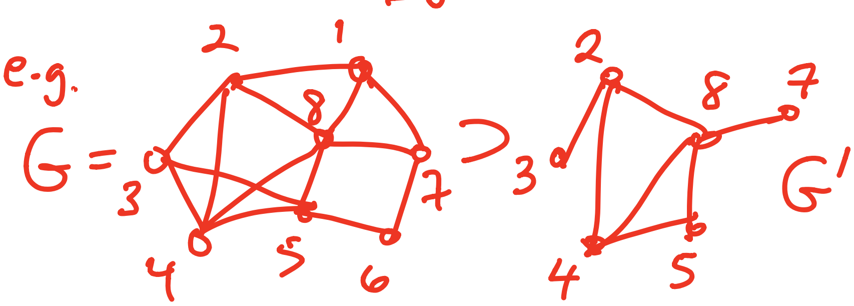
We've seen K_5, $K_{3,3}$ are not planar, and also any graph G having them as an edge-subgraph would also not be planar.

$$G' \subset G$$
$$G' = (V', E') \qquad G = (V, E)$$

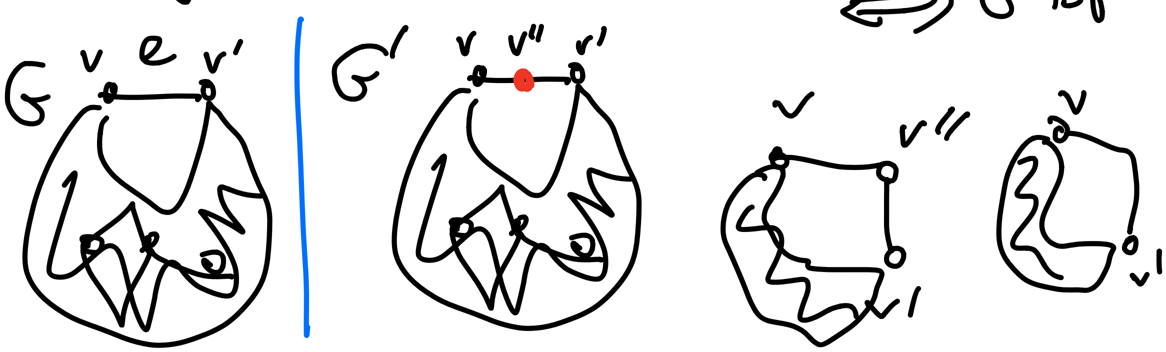
with $V' \subset V$
$E' \subset E$

is called an edge-subgraph

e.g.



Note also, that if G and G' differ by an edge-subdivision then G is planar $\iff$ G' is planar

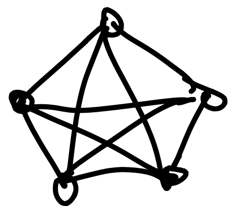


One can further subdivide edges, and call G an edge-subdivision of G' if it is obtained by iterating this process:

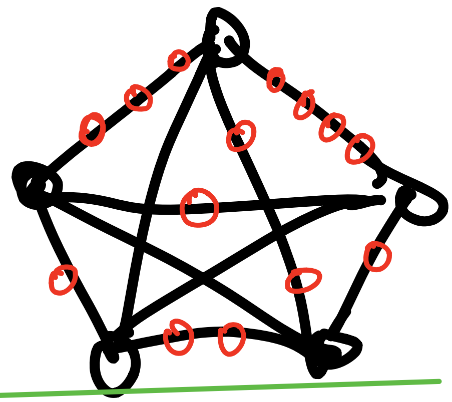
e.g. $G' =$

$\cong$

K_5



$\rightsquigarrow$ G



THEOREM (Kuratowski 1980):

G a graph is planar $\iff$ G contains no edge-subgraph G' that is an edge-subdivision of K_5 or of $K_{3,3}$

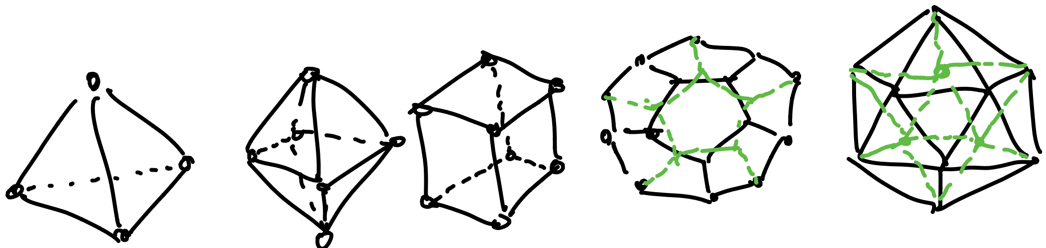
Not obvious, quite surprising!

Not so hard to prove, a little tedious; see Bondy & Murty Chapter 9.

REMARK: $\exists$ fast algorithms running $\leq c \cdot \#V$ steps to test if G is planar; first came in 1974 by Hopcroft & Tarjan

Platonic solids = 3-dimensional polyhedra with • every 2-dimensional face has the same number p of sides, so is a p-gon or p-sided polygon

and • every vertex has same degree or valence q

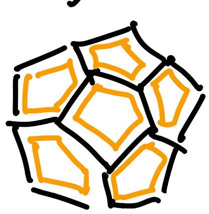


$p=$	3	3	4	5	3
$q=$	3	4	3	3	5

Why are these the only 5 Platonic solids?
Let's see why these are the only (p,q) possible.

As before, faces being p-gons $\Rightarrow \boxed{2e = p \cdot f}$
or $\frac{1}{p} = \frac{f}{2e}$

$q=5$



Long ago we saw $2e = \sum_{v_0 \in V} \deg(v_0) = q \cdot \#V = q \cdot v$

$\Rightarrow \boxed{2e = q \cdot v}$

or $\frac{1}{q} = \frac{v}{2e}$

$\boxed{2e = p \cdot f}$

or $\frac{1}{p} = \frac{f}{2e}$

$\boxed{2e = q \cdot v}$

or $\frac{1}{q} = \frac{v}{2e}$

From Euler's formula,

$$v - e + f = 2$$

$$v + f = e + 2$$

divide by $2e$

$$\underbrace{\frac{v}{2e}}_{\frac{1}{q}} + \underbrace{\frac{f}{2e}}_{\frac{1}{p}} = \frac{1}{2} + \frac{1}{e} > \frac{1}{2}$$

so $\boxed{\frac{1}{q} + \frac{1}{p} = \frac{1}{2} + \frac{1}{e}}$

Conclusion: $\boxed{\frac{1}{q} + \frac{1}{p} > \frac{1}{2}}$ and $p, q \geq 3$

forces $\frac{1}{q} > \frac{1}{2} - \frac{1}{p} \geq \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

i.e. $q < 6$

$q \in \{3, 4, 5\}$

Similarly $\frac{1}{p} > \frac{1}{2} - \frac{1}{q} \geq \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

i.e. $p < 6$

$p \in \{3, 4, 5\}$

Note $(p,q) = (4,4), (4,5), (5,4), (5,5)$ <u>all disobey</u> $\frac{1}{q} + \frac{1}{p} > \frac{1}{2}$,

e.g. $\frac{1}{4} + \frac{1}{4} = \frac{1}{2} \not> \frac{1}{2}$

This only leaves as possibilities

$(p,q) = (3,3), (3,4), (4,3), (3,5), (5,3)$

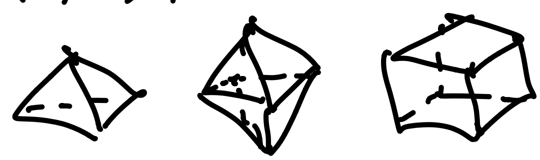
and from each of these, one deduces the unique value of e from

$$\boxed{\frac{1}{q} + \frac{1}{p} = \frac{1}{2} + \frac{1}{e}}$$

and then the unique values of v, f from $v = \frac{2e}{q}$, $f = \frac{2e}{p}$.

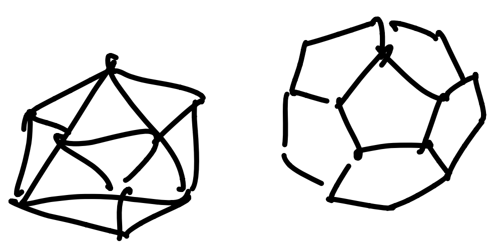
For each of $(p,q) = (3,3), (3,4), (4,3)$

it's not too hard to convince yourself it looks like



but I personally find it a lot more tedious to show $(p,q) = (3,5), (5,3)$

force



but it can be done

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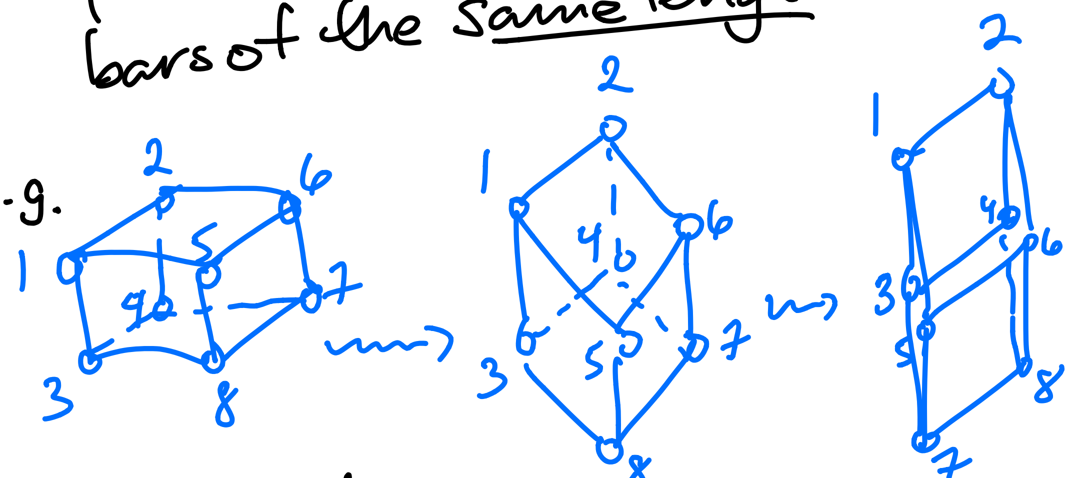
Rigidity (of bar-node frameworks)

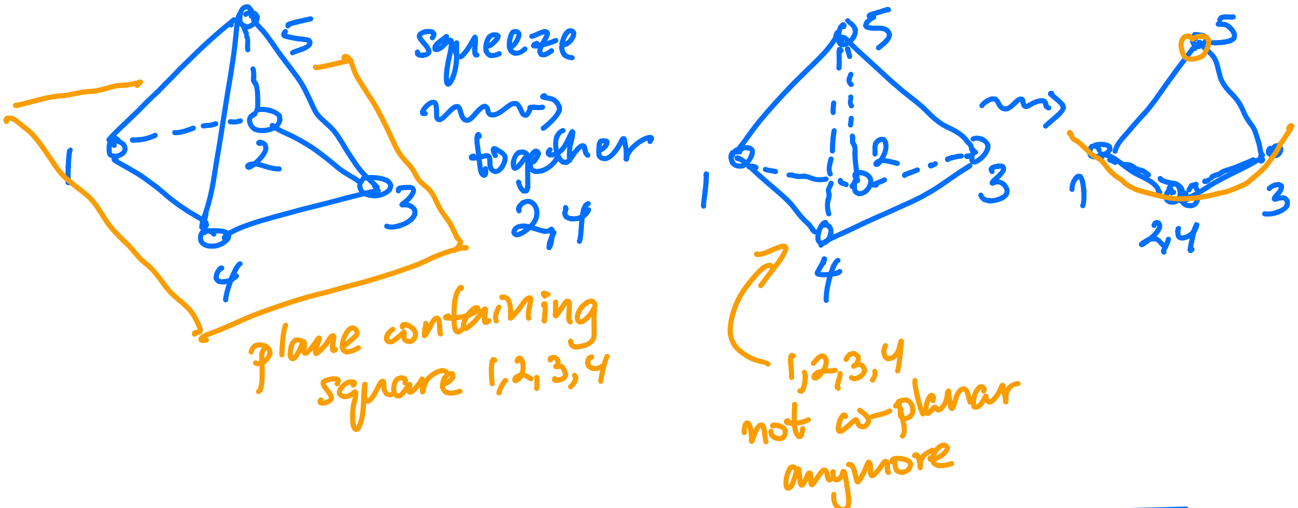
QUESTION: Which of our 3-dimensional polyhedra are rigid when built from nodes and bars
(vertices) (edges),

meaning that they don't have extra motions that keep the (or collapsing) bars of the same length.

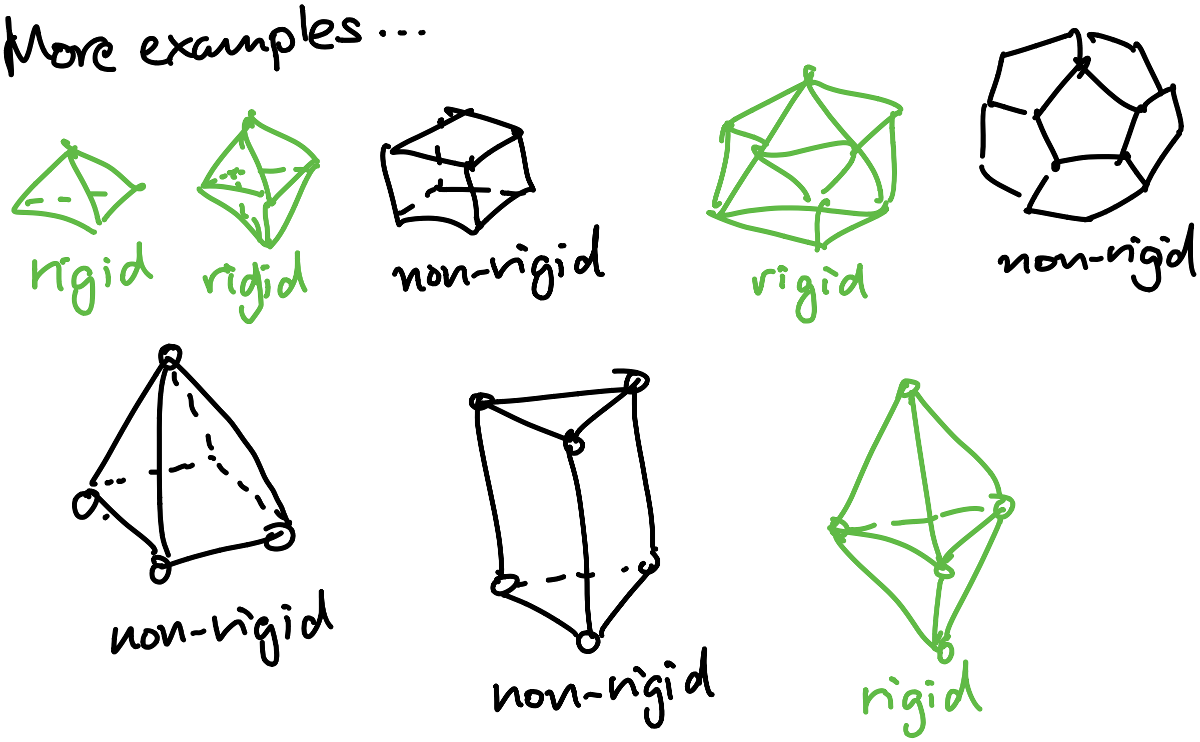
e.g.



A non-rigid motion of the cube



More examples...



The rigid ones seem to be the ones whose faces are all <u>triangles</u>.

Let's understand <u>why</u>, roughly.

Remember, we showed the vertices, edges of a polyhedron satisfy

$$e \leq 3v - 6 \Leftarrow \left\{ \begin{array}{l} v - e + f = 2 \\ \text{AND} \\ 2e \geq 3f \\ \text{with equality if and only if all faces are triangular} \end{array} \right.$$

or $\boxed{3v - e \geq 6}$

with equality here if and only if all the faces are triangular

Let's re-interpret $3v - e$ and the 6 in terms of the (informal) notion of <u>degrees of freedom</u> (d.o.f.) for objects in $\mathbb{R}^3$

$:=$ # of real number parameters needed to specify the object's exact location and configuration in $\mathbb{R}^3$

DEF'N

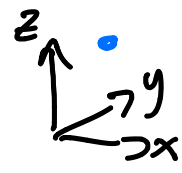
degrees of freedom (d.o.f.) for objects in $\mathbb{R}^3$

$:=$ # of real number parameters needed to specify the object's exact location and configuration in $\mathbb{R}^3$

DEF'N

EXAMPLES

① A ~~point~~ vertex in $\mathbb{R}^3$ has 3 d.o.f.'s,



namely its (x,y,z) coordinates in space

② A robot arm, with hinge fixed on a table, but rotatable has 2 d.o.f's: we need to angles (θ, φ) to specify it

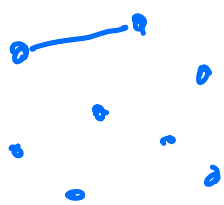
③ v vertices in $\mathbb{R}^3$ floating freely have $3v$ d.o.f's

v of these total

③ v vertices in $\mathbb{R}^3$ floating freely have $3v$ d.o.f.'s

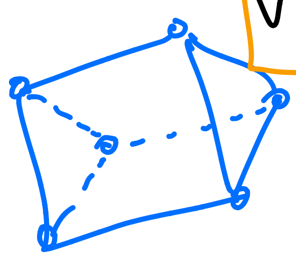
v of these total

④ v vertices with 1 edge/bar added
of fixed length



has $3v-1$ d.o.f.'s

⑤

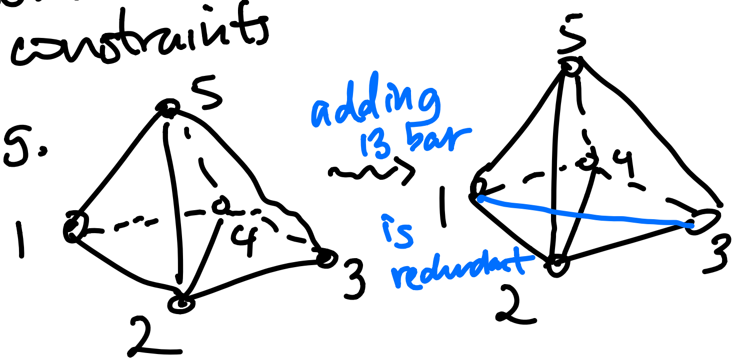


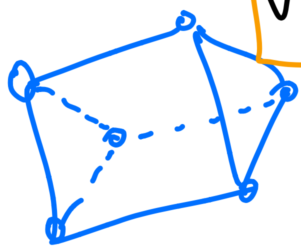
v vertices in $\mathbb{R}^3$ with e edges/bar added have

$\geq 3v-e$ d.o.f.'s

(not <u>exactly</u> $3v-e$ necessarily, since sometimes bars impose redundant constraints

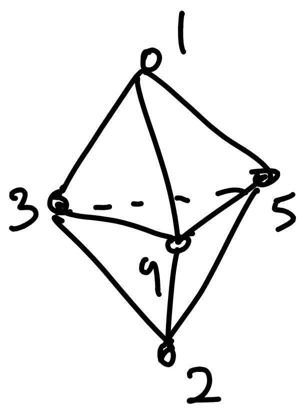
e.g.



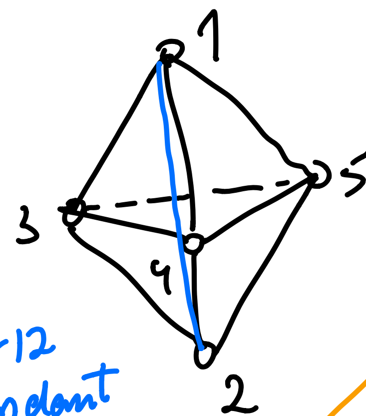


v vertices in $\mathbb{R}^3$ with e edges/bar added have

$\geq 3v - e$ d.o.f.'s



adding the bar 12 is redundant



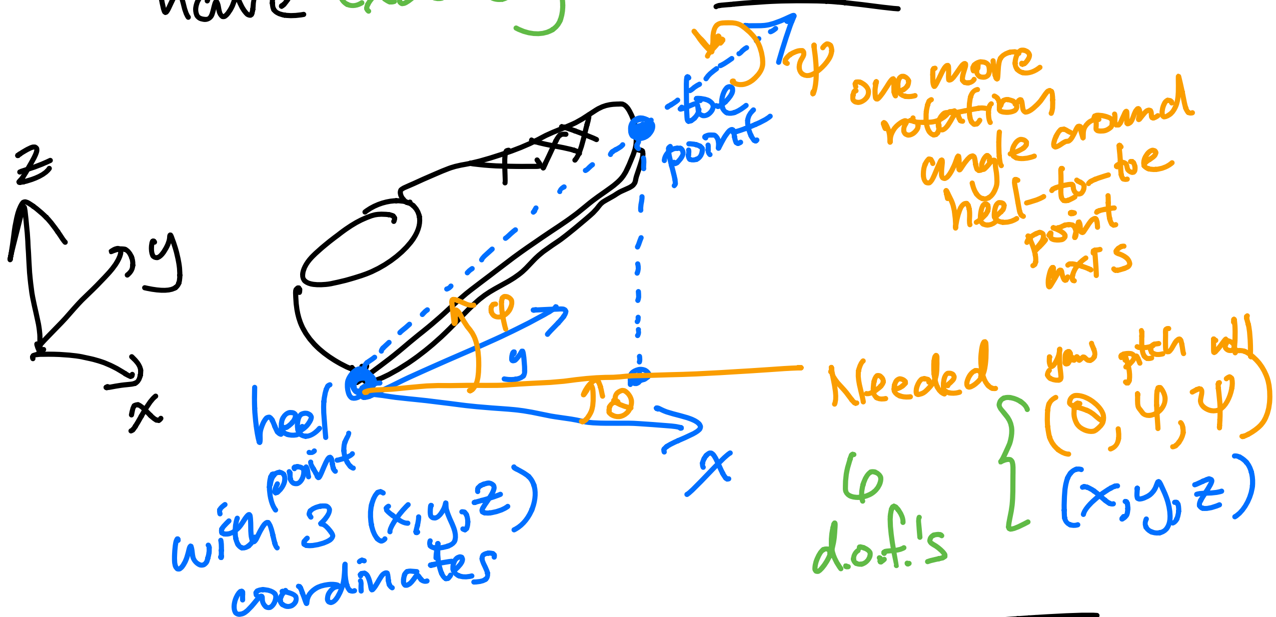
So our 3-dimensional polyhedra satisfied $\boxed{3v - e \geq 6}$

with equality here if and only if all the faces are <u>triangular</u>

and have at least <u>3v−e d.o.f.'s</u>

<u>Q</u>: What is this important # of 6 d.o.f.'s?

⑥ Rigid 3-dimensional objects have exactly 6 d.o.f.'s :



So rigidity means having exactly 6 d.o.f.'s

Polyhedra have $\geq 3v - e$ d.o.f.'s

and have $3v - e \geq 6$ with equality $\Longleftrightarrow$ all faces triangular.

CONCLUSION:

A convex polyhedron in $\mathbb{R}^3$ cannot be rigid unless all faces are triangular.

But, it doesn't quite make it clear that there is a converse.

THEOREM (Cauchy 1813 incorrect proof; Steinitz 1928; Alexandrov 1950 correct proofs)

Convex polyhedra with all triangular faces are rigid.

EXAMPLE (first one by Connelly 1977) Look up flexible polyhedron on Wikipedia)

There exist non-convex spheres built from triangular faces which are non-rigid

convex polygon

a non-convex polygon

a convex sphere/ball

a non-convex sphere

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Planar duality (not in book)

There was a hidden symmetry between

$v = \#$vertices

$f = \#$faces/region

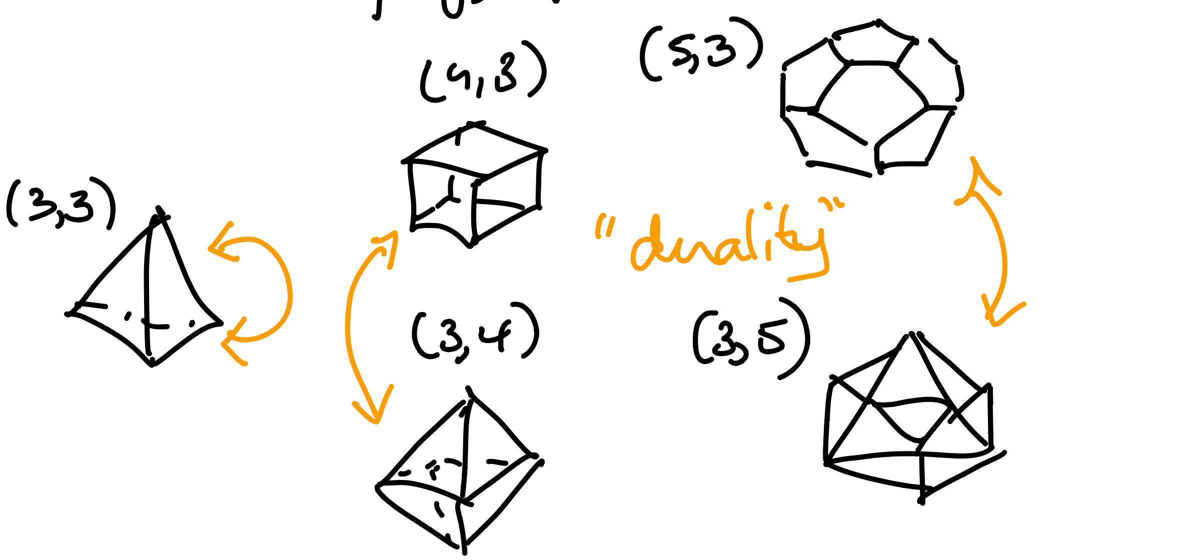
in our discussion of Euler's formula

$$v - e + f = 2$$

proven $(v-1) + (f-1) = e$

and in our discussion of regular polyhedra

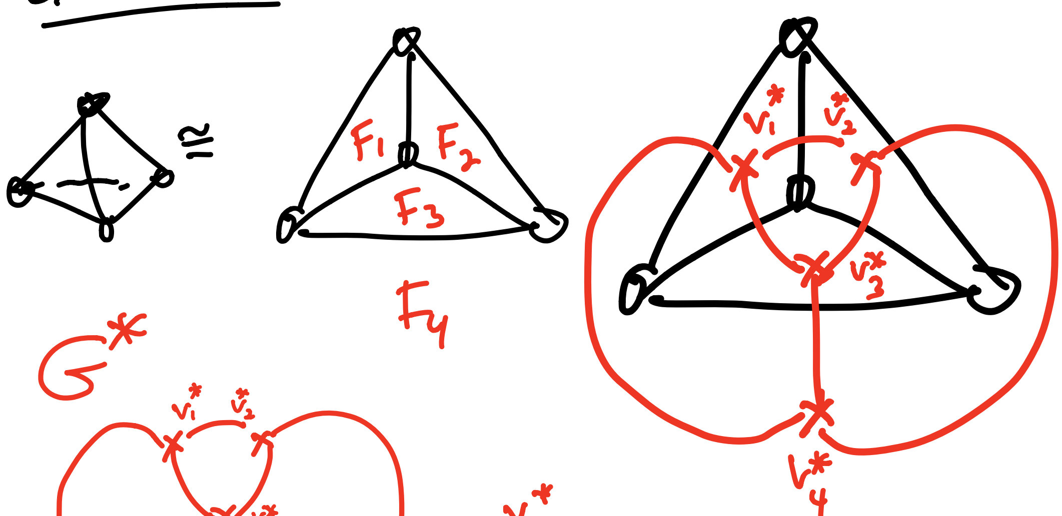
(p,q) played symmetric roles



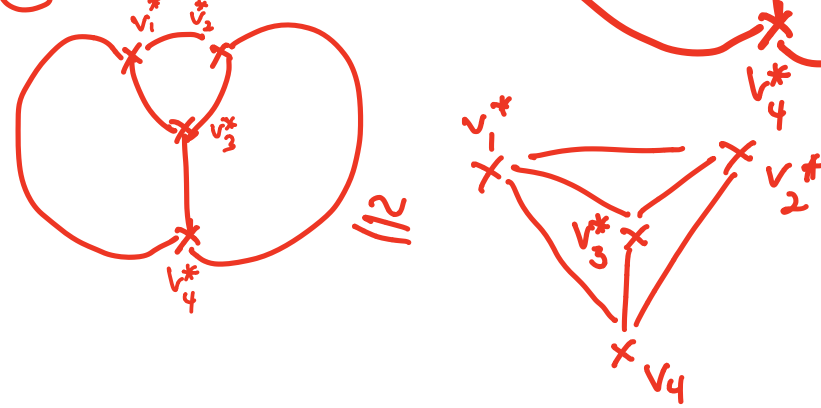
What is this symmetry?

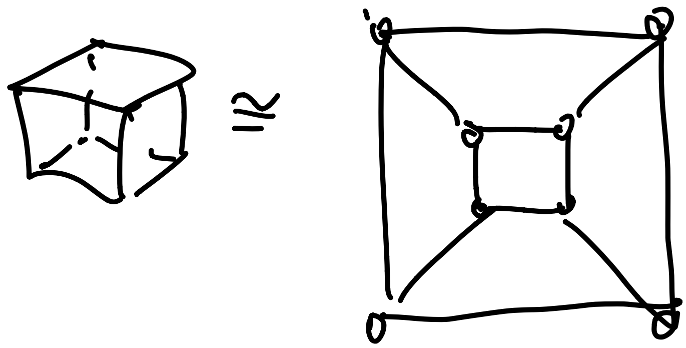
<u>DEF'N</u>: Given $G=(V,E)$ a planar graph <u>embedded</u> <u>in</u> <u>the</u> <u>plane</u>, create its planar dual graph $G^*=(V^*,E^*)$ by letting V^* have a vertex v^* in the middle of each face of G, and an edge e^* for every edge $e \in E$ connecting v_1^*, v_2^* corresponding to the faces F_1, F_2 on the two sides of e in G.

<u>EXAMPLES</u>: G

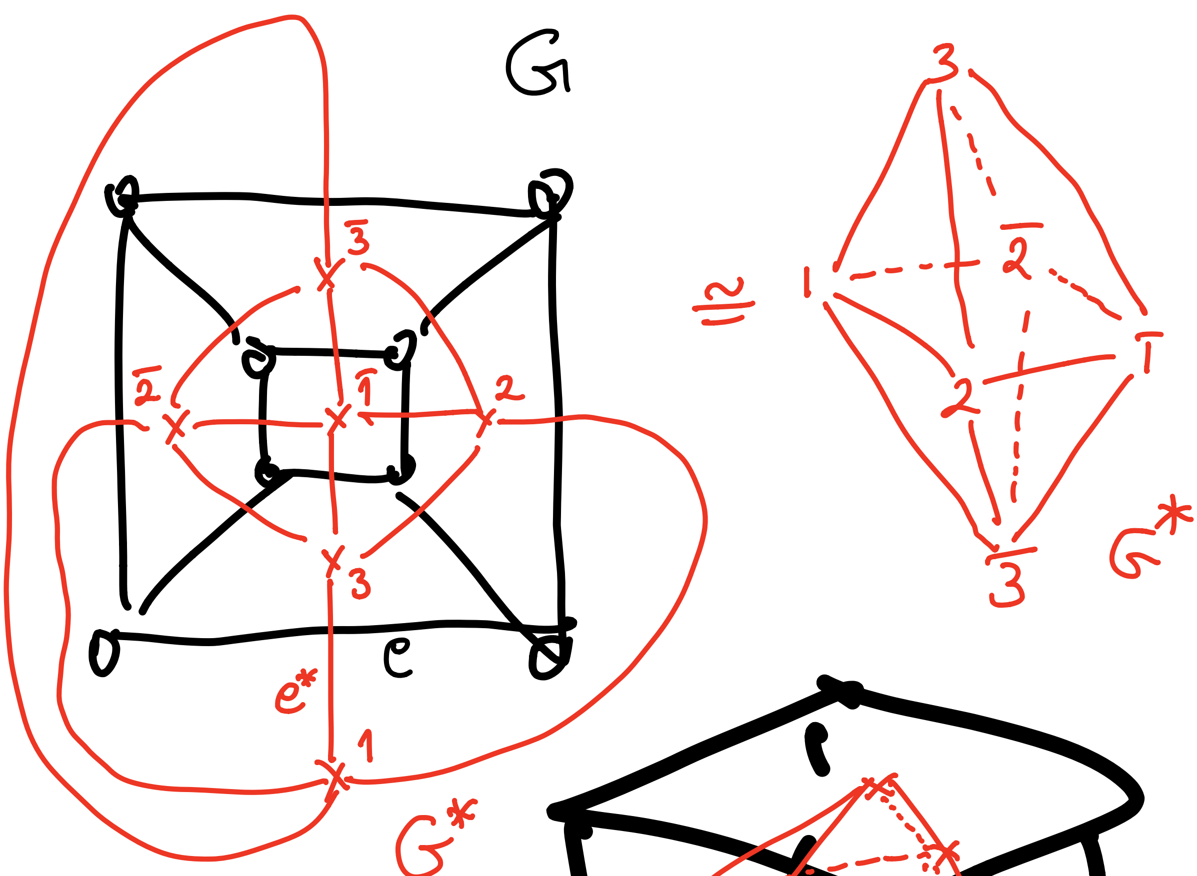


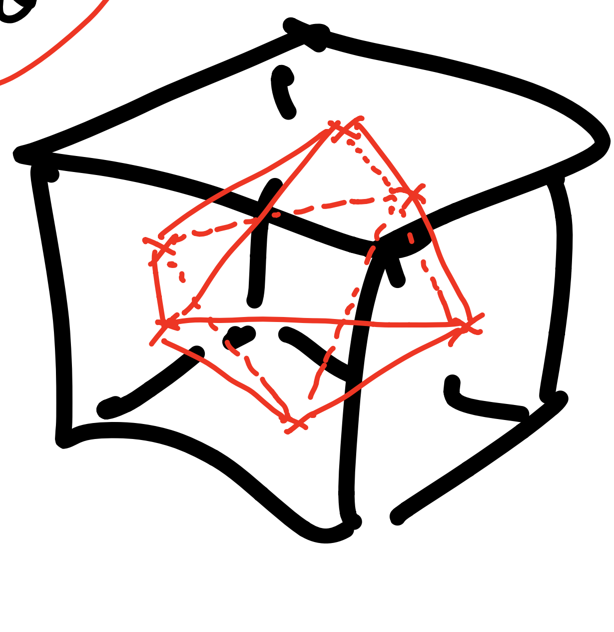
G^*

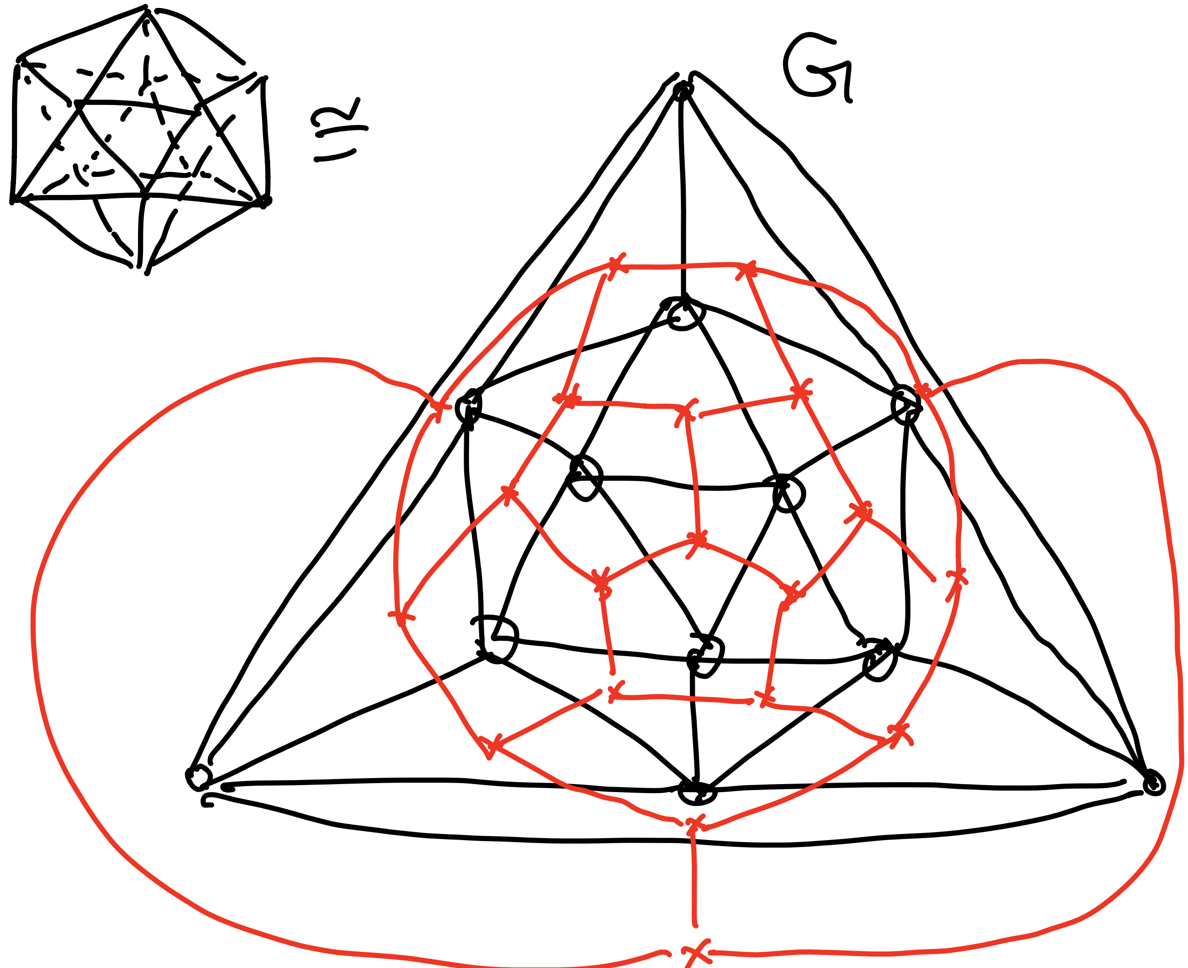


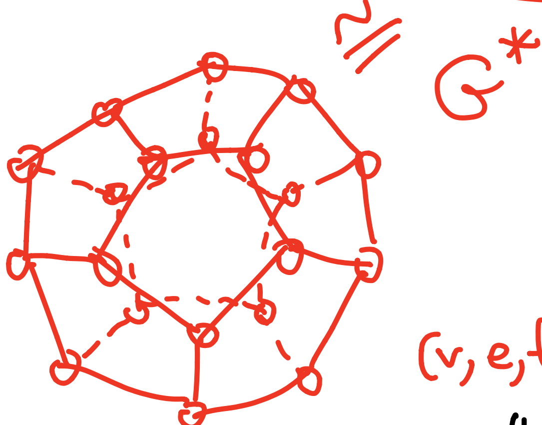


G









(v, e, f) for G^*

$\|$

(f, e, v) for G

(p, q) for G^*

$\|$

(q, p) for G

THEOREM: The duality map

$$\left\{\begin{matrix}\text{connected}\\ \text{planar}\\ \text{graphs } G\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{connected}\\ \text{planar}\\ \text{graphs } G^*\end{matrix}\right\}$$

$$G \longmapsto G^*$$

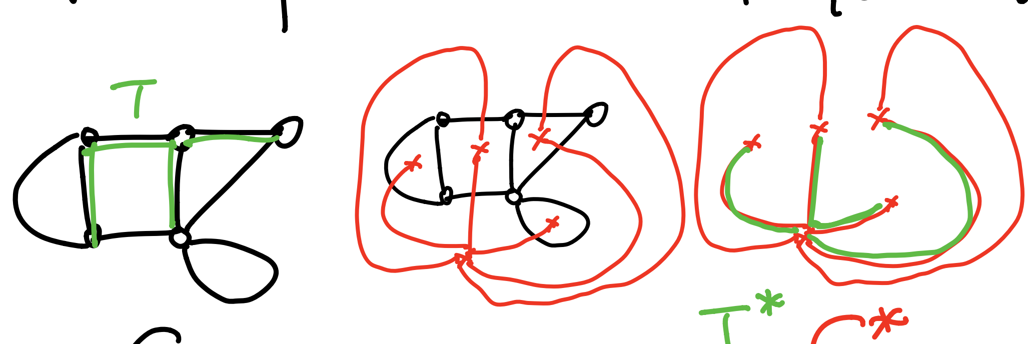
has these properties

(1) $(G^*)^* = G$ if we assume a bit more about G, namely G is 3-connected (requires removing 3 vertices to disconnect it)

(2) vertices of G $\longleftrightarrow$ faces of G^*
faces of G $\longleftrightarrow$ vertices of G^*

(3) spanning trees of G $\longleftrightarrow$ spanning trees of G^*
T $\qquad\qquad$ $T^* = \{e^* : e \notin T\}$

G



G

T^* G^*

(4) For an edge $e \in G$ which is neither a loop, nor a cut-edge

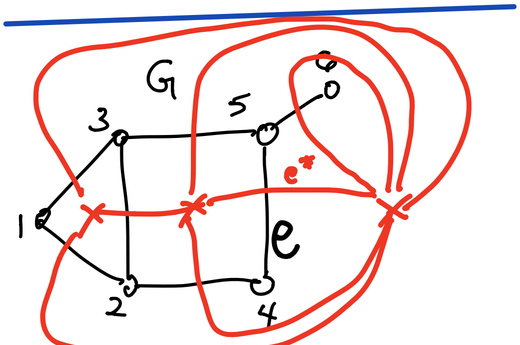
means $G-e$ is disconnected

$$(G \setminus e)^* \cong G^*/e^*$$

$$(G/e)^* \cong G^* \setminus e^*$$

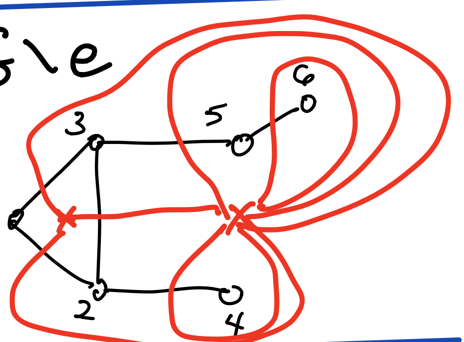
That is, duality exchanges deletion & contraction!

duality



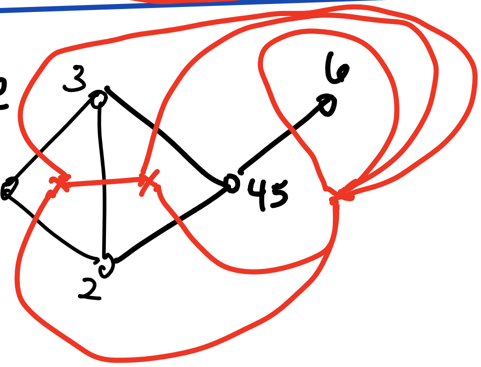
G^*

$G \setminus e$



G^*/e^*

G/e



$G^* \setminus e^*$

Chapter 13 Coloring maps & graphs

<u>DEF'N</u>: Given $G = (V, E)$ a graph, a <u>proper</u> <u>(vertex-) coloring</u> with k colors is an assignment $f : V \longrightarrow \{k \text{ colors}\}$

e.g. $\{1, 2, \ldots, k\}$

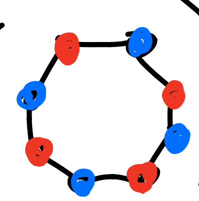
such that for every edge $e \in E$, $e = \{v, v'\}$, its two endpoints v, v' receive different colors $f(v) \neq f(v')$.

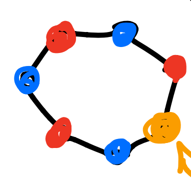
Say G is <u>k-colorable</u> if $\exists$ it has a proper vertex k-coloring;

and $\chi(G)$ = <u>chromatic number</u> of G

$:= \min\{k : G \text{ is } k\text{-colorable}\}$

EXAMPLES:

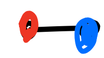
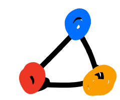
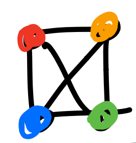
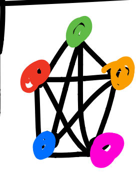
① $\chi\Big($  $\Big) = 2$ for even sized cycles

$\chi\Big($  $\Big) = 3$ for odd sized cycles

needed a 3rd color!

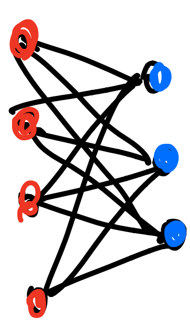
② $\chi(K_n) = n$

complete graph

n	K_n	$\chi(K_n)$
2		2
3		3
4		4
5		5

③ $\chi(K_{m,n}) = 2$

$K_{4,3}$



and same for all bipartite graphs (with at least one edge).

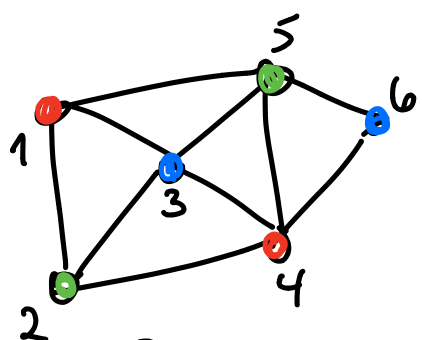
APPLICATIONS:

① Scheduling –

V = tasks to be done, each taking 1 unit of time

E = pairs $\{v, v'\}$ of tasks that can't be done at the same time

proper k-coloring = schedulings using k time units



G

$\chi(G) = 3$

● = time unit 1

● = time unit 2

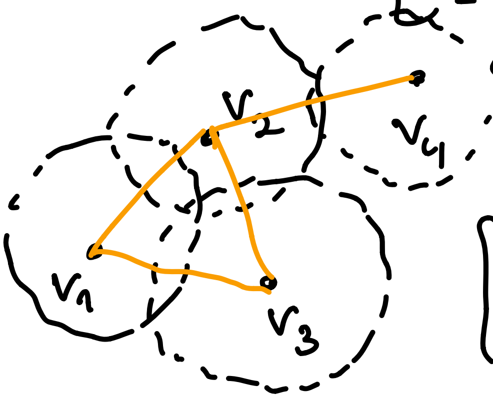
● = time unit 3

$\chi(G)$ = minimum # of time slots needed to complete the tasks

② Frequency assignments

V = cell phones

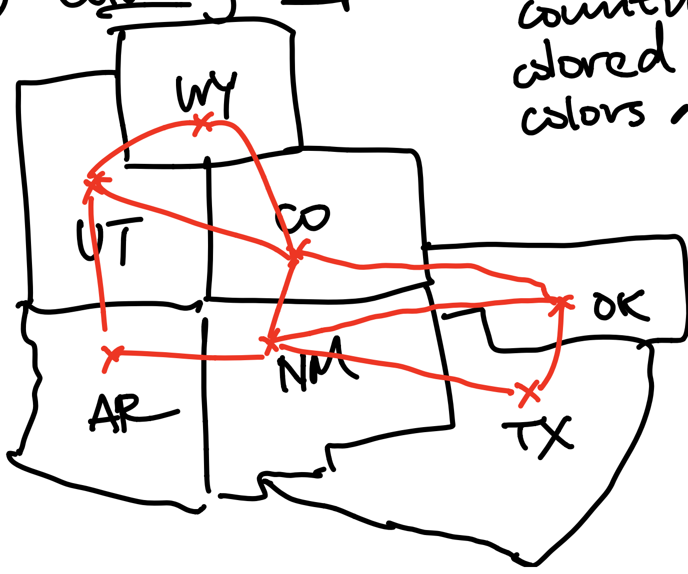
E = pairs $\{v,v'\}$ of phones that are sometimes close enough to interfere



$\{$proper k-colorings of $G\}$ $\longleftrightarrow$ $\{$frequency assignments with k frequencies$\}$

$\chi(G)$ = minimum number of frequencies

③ Coloring maps – A map with (simply) connected countries needs to be colored with contrasting colors along each border



V = countries

E = pairs $\{v,v'\}$ sharing a boundary

$\chi(G)$ = min # of colors needed.