Probability of Poker Hands

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In a standard deck of cards, there are 4 possible suits (clubs, diamonds, hearts, spades), and 13 possible values (2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace). Let A, J, Q, K represent Ace, Jack, Queen and King, respectively. Every card has a suit and value, and every combination is possible. Hence a standard deck contains $13 \cdot 4 = 52$ cards.

A “poker hand” consists of 5 unordered cards from a standard deck of 52. There are $\binom{52}{5} = 2,598,960$ possible poker hands. Below, we calculate the probability of each of the standard kinds of poker hands.

**Royal Flush.** This hand consists of values 10, J, Q, K, A, all of the same suit. Since the values are fixed, we only need to choose the suit, and there are $\binom{4}{1} = 4$ ways to do this.

**Straight Flush.** A straight flush consists of five cards with values in a row, all of the same suit. Ace may be considered as high or low, but not both. (For example, A, 2, 3, 4, 5 is a straight, but Q, K, A, 2, 3 is not a straight.) The lowest value in the straight may be A, 2, 3, 4, 5, 6, 7, 8 or 9. (Note that a straight flush beginning with 10 is a royal flush, and we don’t want to count those.) So there are 9 choices for the card values, and then $\binom{4}{1} = 4$ choices for the suit, giving a total of $9 \cdot 4 = 36$.

**Straight.** A straight consists of five values in a row, not all of the same suit. The lowest value in the straight could be A, 2, 3, 4, 5, 6, 7, 8, 9 or 10, giving 10 choices for the card values. Then there are $\binom{4}{1}^5 = 4^5$ ways to choose the suits of the five cards, for a total of $10 \cdot 4^5 = 10,240$ choices. But this value also includes the straight flushes and royal flushes which we do not want to include. Subtracting the 40 straight and royal flushes, we get $10,240 - 40 = 10,200$.

**Flush.** A flush consists of five cards, all of the same suit. There are $\binom{13}{1} = 4$ ways to choose the suit, then given that there are 13 cards of that suit, there are $\binom{13}{5}$ ways to choose the hand, giving a total of $4 \cdot \binom{13}{5} = 5,148$ flushes. But note that this includes the straight and royal flushes, which we don’t want to include. Subtracting 40, we get a grand total of $5,148 - 40 = 5,108$.

**Four of a Kind.** This hand consists of four cards of one value, and a fifth card of a different value. There are $\binom{13}{1} = 13$ ways to choose the value for the quadruple. Then, among the cards
of this value, there are \( \binom{4}{1} = 1 \) ways to choose the quadruple. After this, there are \( \binom{12}{1} = 12 \) ways to choose a value for the single from the remaining values, and \( \binom{4}{1} = 4 \) ways to choose the single from the four cards of this value, for a grand total of

\[
\binom{13}{1} \binom{4}{4} \binom{12}{1} \binom{4}{1} = 13 \cdot 1 \cdot 12 \cdot 4
\]

\[= 624.
\]

**Full House.** This hand consists of three cards of one value, and two cards of a different value. There are \( \binom{13}{1} \) ways to choose a value for the triple, then \( \binom{4}{3} \) ways to choose the triple from the four cards of this value. Then, there are \( \binom{12}{1} \) ways to choose the value of the double from the remaining values, and \( \binom{4}{2} \) ways to choose the double from the four cards of this value, for a grand total of

\[
\binom{13}{1} \binom{4}{3} \binom{12}{1} \binom{4}{2} = 13 \cdot 4 \cdot 12 \cdot 6
\]

\[= 3,744.
\]

**Three of a Kind.** This hand consists of three cards of one value, and two more cards, each of different values. There are \( \binom{13}{1} \) ways to choose the value for the triple, and \( \binom{4}{3} \) ways to choose the triple from the four cards of this value. Then there are \( \binom{12}{2} \) ways to choose two (unordered) values for the remaining singles, and \( \binom{4}{1} \binom{4}{1} \) to choose the singles from their respective values, for a grand total of

\[
\binom{13}{1} \binom{4}{3} \binom{12}{2} \binom{4}{1} \binom{4}{1} = 13 \cdot 4 \cdot 66 \cdot 4 \cdot 4
\]

\[= 54,912.
\]

**Two Pairs.** This hand consists of two pairs of different values, and a fifth card of another different value. There are \( \binom{13}{2} \) ways to choose two (unordered) values for the two pairs, then \( \binom{4}{2} \binom{4}{2} \) to choose the pairs from the cards of these values. Then there are \( \binom{11}{1} \) ways to choose a remaining value for the single, and \( \binom{4}{1} \) ways to choose the single from the four cards of this value, for a grand total of

\[
\binom{13}{2} \binom{4}{2} \binom{4}{2} \binom{11}{1} \binom{4}{1} = 78 \cdot 6 \cdot 6 \cdot 11 \cdot 4
\]

\[= 123,552.
\]

**One Pair.** This hand consists of a pair of one value, and three additional cards, each of different value. There are \( \binom{13}{1} \) ways to choose a value for the pair, then \( \binom{4}{2} \) ways to choose the
pair from the four cards of this value. Then there are \( \binom{12}{3} \) ways to choose three (unordered) values for the remaining three singles, and \( \binom{4}{1}^3 \) to choose suits for the singles, for a grand total of

\[
\binom{13}{1} \binom{4}{2} \binom{12}{3} \binom{4}{1}^3 = 13 \cdot 6 \cdot 220 \cdot 4^3 = 1,098,240.
\]

Putting all of this together, we obtain the following ranking of poker hands:

<table>
<thead>
<tr>
<th>Poker Hand</th>
<th>Number of Ways to Get This</th>
<th>Probability of This Hand</th>
</tr>
</thead>
<tbody>
<tr>
<td>Royal Flush</td>
<td>4</td>
<td>0.000154%</td>
</tr>
<tr>
<td>Straight Flush</td>
<td>36</td>
<td>0.00139%</td>
</tr>
<tr>
<td>Four of a Kind</td>
<td>624</td>
<td>0.0240%</td>
</tr>
<tr>
<td>Full House</td>
<td>3,744</td>
<td>0.144%</td>
</tr>
<tr>
<td>Flush</td>
<td>5,108</td>
<td>0.197%</td>
</tr>
<tr>
<td>Straight</td>
<td>10,200</td>
<td>0.392%</td>
</tr>
<tr>
<td>Three of a Kind</td>
<td>54,912</td>
<td>2.11%</td>
</tr>
<tr>
<td>Two Pairs</td>
<td>123,552</td>
<td>4.75%</td>
</tr>
<tr>
<td>One Pair</td>
<td>1,098,240</td>
<td>42.3%</td>
</tr>
<tr>
<td>Nothing</td>
<td>1,302,540</td>
<td>50.1%</td>
</tr>
</tbody>
</table>

Wait, how did I compute the probability of getting “Nothing”?

How would you answer the question: “What is the probability of getting Three of a Kind or better?”