Graph coloring, UMN Math 4707, Spr. 2020

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Recall that for a graph G, the *chromatic number* of G, denoted $\chi(G)$, is the smallest number of colors needed to (properly) color the vertices of G.

If G has a subgraph isomorphic to K_d, the complete graph on d vertices, then $\chi(G) \geq d$ because it requires d colors to color even that subgraph.

1. Give an example of a graph G which does not contain a subgraph isomorphic to K_3 but with $\chi(G) \geq 3$. Can you give infinitely many examples?

Remark: for any $d \geq 3$, there exists a graph G which does not contain any K_d's but which has $\chi(G) \geq d$ (see Figure 13.6 in the book for $d = 4$). In fact, much more is true. The *girth* of a graph G is the size of the smallest cycle in G. A classic result of Erdös (beyond what we'll prove in this class) says that for any g, d, there exists a graph G with girth $\geq g$ and $\chi(G) \geq d$.

In lecture/the book, we saw a simple proof by induction that if the maximum degree of G is $\leq d$ then $\chi(G) \leq d + 1$.

2. Show that the bound just mentioned is sharp: for each $d \geq 1$, give an example of a graph with maximum degree $\leq d$ and $\chi(G) = d + 1$.

3. For each $d \geq 1$, give an example of a graph G for which the *minimum degree* of G is $\geq d$ but with $\chi(G) = 2$.