Coherence of $f$-Monotone Paths on Zonotopes.

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An Analogy: The Secondary Polytope

Definition (Polytope)

*A polytope is a convex hull of finitely many points in* \( \mathbb{R}^d \). *Combinatorially a polytope can be defined by its face lattice.*

Definition (Polyhedral Subdivision)

*A polyhedral subdivision is a decomposition of* \( P \) *into subpolytopes. A subdivision is a triangulation when each subpolytope is a simplex.*
Remark

Subdivisions of \( P \) form a poset called the refinement poset of \( P \).
Remark

In this example, the refinement poset is the face lattice of a polytope.
Some bad triangulations are not regular or are incoherent.

Coherence is a linear inequality condition.

\( \Sigma(P) \) is an example of a *Fiber Polytope*.

**Theorem (GKZ)**

The refinement poset of all regular subdivisions of \( P \) is the face lattice of a polytope \( \Sigma(P) \).
Our Work: Monotone Paths

Our version of triangulations are \( f \)-monotone edge paths of \( P \).

\( f \) must be generic, non-constant on each edge of \( P \).

The refinement poset consists of cellular strings.

**Definition**

An \( f \)-monotone edge path is a path from the \( f \)-minimal vertex \( -z \) to the \( f \)-maximal vertex \( z \) along the edges of \( P \).
Definition

- The vertices graph $G_2(P, f)$ is formed from all elements on the bottom level levels of the refinement poset.

- In this example every $f$-monotone path is coherent.
Question

*When does $P$ have incoherent $f$-monotone paths?*
Definition (Coherent)

An \( f \)-monotone path \( \gamma \) is coherent if there exists \( g \in (\mathbb{R}^d)^* \) making \( \gamma \) the lower face of the polytope \( P = \text{Conv} \{(f(p_i), g(p_i))\} \subset \mathbb{R}^2 \).

Remark

The refinement poset of coherent cellular strings is the fiber polytope \( \Sigma(P, f) \).
Theorem (Billera & Sturmfels)

Every $f$-monotone path of a cube is coherent.
Definition

- A zonotope is the image of the $n$-cube in $\mathbb{R}^d$ under a projection $A : C_n \to \mathbb{R}^d$ specified by a $d \times n$ matrix $A = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \end{pmatrix}$.

- The zonotope $Z(A) = \sum[-a_i, +a_i]$ is the Minkowski of the columns of $A$.

- The vertices of $Z(A)$ are sign vectors.
Proposition

- Every $f$-monotone path of $Z(A)$ is of length $n$.
- The function $f$ is generic when $f(a_i) > 0$ for all $i$.
- The choice of $f$ corresponds to the choice of a $f$-minimal vertex $z$.
- But not all vertices are symmetric, so we will have to consider multiple options for $z$.
- The corank of $Z$ is $n - d$. 
Proposition

A \( f \)-monotone path \( \gamma \) is coherent if there exists a \( g \in (\mathbb{R}^d)^* \) so that:

\[
\frac{g_{\gamma(1)}}{f_{\gamma(1)}} < \frac{g_{\gamma(2)}}{f_{\gamma(2)}} < \ldots < \frac{g_{\gamma(n)}}{f_{\gamma(n)}}
\]
Corank 1

\[ Z(4, 3) = \begin{bmatrix}
    1 & 1 & 1 & 1 \\
    1 & 2 & 3 & 4 \\
    1 & 4 & 9 & 16
\end{bmatrix} \]

Remark

- Every \( f \)-monotone path is coherent for \(- + ++\).
- \(+ ++ +\) has an incoherent \( f \)-monotone path for every \( f \).
Corank 2 (cyclic)

\[
Z(5, 3) = \begin{pmatrix}
a_1 & a_2 & a_3 & a_4 & a_5 \\
1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 & 5 \\
1 & 4 & 9 & 16 & 25
\end{pmatrix}
\]

Remark

- Has incoherent \( f \)-monotone path for every \( f \).
- \(+ + + + +\) is an important geometric counterexample.
Definition (Pointed hyperplane arrangement)

The normal fan of the zonotope, is a hyperplane arrangement, 
\( \mathcal{A} = \{ a_i^\perp, \ldots, a_n^\perp \} \). The choice of a chamber \( c \) of \( \mathcal{A} \) corresponds to the choice of \( f \).

- Easy to draw under stereographic projection
- \( k \)-faces of \( Z \) \( \Longleftrightarrow d - k \) intersections of hyperplanes.
- \( L_2(\mathcal{A}) \) are the codimension 2 intersections of hyperplanes.
Reflection Arrangements

Remark

- Does not depend on the choice of a base chamber $c$.
- Paths corresponds to reduced words.
- Dual hyperplane configuration is a \((n - d) \times n\) matrix.
- Functions on \(\mathcal{A}\) correspond to dependencies of \(\mathcal{A}^*\).
- When \(n - d\) is small, this makes things easy.

\[
\begin{pmatrix}
a_1 & a_2 & a_3 & a_4 \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1 \\
-1 \\
\end{pmatrix}
= 0 \quad \mathcal{A}^* = \begin{pmatrix}
a_1^* & a_2^* & a_3^* & a_4^* \\
1 & 1 & 1 & -1 \\
\end{pmatrix}
\]

**Example**

| + | + | + | + | \(f(x, y, z) = x + y + z\) | \(a_1^* + a_2^* + a_3^* + 3a_4^* = 0\) |
| - | + | + | + | \(f(x, y, z) = -x + y + z\) | \(-a_1^* + a_2^* + a_3^* + a_4^* = 0\) |
| + | + | + | - | ? | ? |

Affine Gale duals replace \((\mathcal{A}, f)\) with a picture.
Proposition

- Extensions preserve dimension.
- Liftings preserve corank; if $f$ is generic on $A$ then there exists $\hat{f}$ is generic on $\hat{A}$.
Proposition

If \( \gamma \) is an \( f \)-monotone path of \( A \) and \( \hat{A} \) a single-element lifting of \( A \), then any \( \hat{\gamma} \) with \( \hat{\gamma}/(n+1) = \gamma \) is an \( \hat{f} \)-monotone path of \( \hat{A} \).
Proposition

If $A^+$ is a single-element extension of $A$, and $\gamma^+$ is an $f$-monotone path of $A^+$ then any $\gamma \setminus (n + 1)$ is an $f$-monotone path of $A$. 
Findings: Reflection Arrangements

| \( \mathcal{A} \) | \( |\Gamma(\mathcal{A})| \) |
|-----------------|------------------|
| \( H_3 \)       | 152              |
| \( D_4 \)       | 2316             |
| \( D_5 \)       | 12985968         |
| \( D_6 \)       | 3705762080       |
| \( F_4 \)       | 2144892          |

Proposition

\( H_3 \) has exactly 4 \( L_2 \)-accessible nodes.
Findings: Diameter

There is an \((\mathcal{A}, f)\) pair with no \(L_2\)-accessible nodes.

Example

\(Z(8, 4)\), cyclic arrangement of 8 vectors in \(\mathbb{R}^4\) has \(\text{Diam } G_2(\mathcal{A}, c) = 30\) but \(|L_2| = 28\) for \(c = (-)^4(+)^4\).

Theorem

When \(n - d = 1\) \(G_2(\mathcal{A}, f)\) has diameter \(|L_2|\) and always has an \(L_2\)-accessible node.
Findings: Classification of \((\mathcal{A}, f)\) in corank 1.

- The purple \((\mathcal{A}, f)\) pair is a \textit{minimal obstruction}, all other \((\mathcal{A}, f)\) containing incoherent \(f\)-monotone paths are liftings of it.

- Really remarkable: Coherence depends only on the oriented matroid structure, not on the particular \(f\).

\textbf{Theorem}

\textit{When} \(n - d = 1\) \textit{there is a unique family of all-coherent} \((\mathcal{A}, f)\) \textit{pairs and all other} \((\mathcal{A}, f)\) \textit{pairs have incoherent paths.}
Findings: Classification of \((\mathcal{A}, f)\) in corank 2.

Theorem

When \(n - d = 2\) there are two all-coherent families and 9 minimal obstructions. Of the 9 minimal obstructions 8 are single-element lifting of the corank 1 minimal obstruction.
Findings: Minimal obstructions for Cyclic Zonotopes

\[ A(n, d) = \begin{pmatrix}
a_1 & a_2 & \cdots & a_n \\
1 & 1 & \cdots & 1 \\
t_1 & t_2 & \cdots & t_n \\
\vdots & \vdots & & \vdots \\
t_1^{d-1} & t_2^{d-1} & \cdots & t_n^{d-1}
\end{pmatrix}, \]

**Theorem**

*When* \( d > 2 \) and *f* realizing *c*, the monotone path graph

- *When* \( n - d = 1 \), every *f*-monotone path of \((A(n, d), f)\) is coherent when *c* is a reorientation of a certain hyperplane arrangement, and has incoherence *f*-monotone paths for all other *c*.

- *When* \( n - d \geq 2 \), \((A(n, d), f)\) has incoherent galleries for every *f*. 
Lemma (4.17)

Suppose $\mathcal{A}^+ = \{a_i, \ldots, a_{n+1}\}$ is a single-element extension of $\mathcal{A}$ and $f$ is a generic function on both $Z(\mathcal{A})$ and $Z(\mathcal{A}^+)$. If $\gamma^+$ is a coherent $f$-monotone path of $(\mathcal{A}^+, f)$ then $\gamma = \gamma^+ \setminus (n + 1)$ is a coherent $f$-monotone path of $(\mathcal{A}, f)$.

Lemma (4.22)

Let $\mathcal{A}$ be a hyperplane arrangement and $\hat{\mathcal{A}}$ a single element lifting of $\mathcal{A}$. Suppose

\[
\hat{\gamma}_g = (n + 1, 1, 2, \ldots, n) \\
\hat{\gamma}_h = (1, 2, \ldots, n, n + 1)
\]

are coherent $\hat{f}$-monotone paths of $(Z(\hat{\mathcal{A}}), \hat{f})$ for some $\hat{f}$. Then there is a generic functional $f$ on $Z(\mathcal{A})$ for which $\gamma$ is a coherent $f$-monotone path.
Questions?
Thank You.

Committee Members

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