Critical groups of McKay-Cartan matrices

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Plan

- Motivation: chip firing with avalanche-finite matrices
- McKay-Cartan matrices
- Theorem 1: size and structure of critical groups
- Theorem 2: critical groups for the reflection representation of $\mathfrak{S}_n$. 
Chip firing

A matrix $C = (c_{ij})$ in $\mathbb{Z}^{\ell \times \ell}$ with $c_{ij} \leq 0$ for all $i \neq j$ is called a $Z$-matrix.

Given a $Z$-matrix $C$, we call the elements $v = (v_1, ..., v_\ell)^t \in \mathbb{N}^\ell$ chip configurations, and we define a dynamical system on the set of such configurations as follows:

- A configuration $v$ is stable if $v_i < c_{ii}$ for $i = 1, ..., \ell$.
- If $v$ is unstable, then choose some $i$ so that $v_i \geq c_{ii}$ and form a new configuration $v' = (v'_1, ..., v'_\ell)^t$ where $v'_j = v_j - c_{ij}$ for $j = 1, ..., \ell$. The result $v'$ is called the $C$-toppling of $v$ at position $i$.

A $Z$-matrix is called an avalanche-finite matrix if any chip configuration can be brought to a stable one by a sequence of these topplings.
Example: chip firing on graphs

If $C$ is the (reduced) Laplacian matrix of a graph $\Gamma$, then toppling at position $i$ corresponds to sending chips along the edges incident to vertex $i$. One of the vertices (corresponding the the removed row and column in the Laplacian matrix) is a “black hole”.

Reduced Laplacian is

$$
\begin{pmatrix}
2 & -1 & -1 & 0 \\
-1 & 2 & -1 & 0 \\
-1 & -1 & 3 & -1 \\
0 & 0 & -1 & 1
\end{pmatrix}
$$
Example: chip firing on graphs

Give each node some number of chips, this corresponds to picking the configuration $\nu$. In this case $\nu = (1, 1, 3, 0)$.

\[ \begin{array}{c}
1 \\
\downarrow \\
3 \\
\downarrow \\
1
\end{array} \quad \begin{array}{c}
2 \\
\downarrow \\
0 \\
\downarrow \\
1
\end{array} \quad \begin{array}{c}
2 \\
\downarrow \\
0 \\
\downarrow \\
1
\end{array} \]

Subtracting the 3rd row of the Laplacian:

\[ \nu = (1, 1, 3, 0) - (-1, -1, 3, -1) = (2, 2, 0, 1) = \nu' \]
Critical groups

If $C$ is an avalanche-finite matrix the critical group of $C$ is

$$K(C) := \text{coker } (\mathbb{Z}^\ell \overset{C}{\to} \mathbb{Z}^\ell) = \mathbb{Z}^\ell / \text{im } (C)$$

This gives a finite group, since avalanche-finite matrices are invertible.

- It turns out that critical configurations (those which are stable and recurrent) form a set of coset representatives in $K(C)$.
- Another special set of configurations, the superstable configurations also form a set of coset representatives.
Smith normal form basics

Let $R$ be a ring and $A \in R^{n \times n}$ be a matrix. A matrix $S$ is called the Smith normal form of $A$ if:

- There exist invertible matrices $P, Q \in R^{n \times n}$ such that $S = PAQ$.
- $S$ is a diagonal matrix $S = \text{diag}(s_1, ..., s_n)$ with $s_i | s_{i+1}$ for $i = 1, ..., n - 1$.

Proposition

Let $A \in R^{n \times n}$ be a matrix and suppose $A$ has Smith normal form $S = \text{diag}(s_1, ..., s_n)$. Then $\text{coker } (A : R^n \to R^n) \cong \bigoplus_{i=1}^n R/(s_i)$

Proposition

Let $A \in R^{n \times n}$ be a matrix. If $R$ is a PID then $A$ has a Smith normal form.
Notation

For the rest of the talk:

- $G$ is a finite group
- $1_G = \chi_0, \chi_1, ..., \chi_\ell$ are its irreducible complex characters
- $\gamma$ is a faithful (not-necessarily-irreducible) $n$-dimensional representation of $G$ with character $\chi_\gamma$
**McKay-Cartan matrices**

Let $M$ be the $(\ell + 1) \times (\ell + 1)$ integer matrix with entries $m_{ij}$ defined by

$$\chi_\gamma \cdot \chi_i = \sum_{j=0}^{\ell} m_{ij} \chi_j$$

The *extended McKay-Cartan matrix* $\tilde{C}$ is

$$\tilde{C} := nI - M$$

and the *McKay-Cartan matrix* $C$ is the $\ell \times \ell$ submatrix formed by removing the row and column corresponding to $\chi_0$ from $\tilde{C}$. The *critical group* is $K(\gamma) = K(C)$.

**Theorem (G. Benkart, C. Klivans, and V. Reiner)**

*The McKay-Cartan matrix associated to a faithful representation $\gamma$ is an avalanche-finite matrix.*
Let $G = S_4$ and $\gamma$ be the reflection representation. This corresponds to the partition $(3, 1)$.

<table>
<thead>
<tr>
<th></th>
<th>$\chi_0$</th>
<th>(12)</th>
<th>(123)</th>
<th>(1234)</th>
<th>(12)(34)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_\gamma = \chi_1$</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>
Example: McKay-Cartan matrix for $\mathfrak{S}_4$

Let $G = \mathfrak{S}_4$ and $\gamma$ be the reflection representation.

$$
M = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\quad
\tilde{C} = \begin{pmatrix}
3 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & -1 & 0 \\
0 & -1 & 3 & -1 & 0 \\
0 & -1 & -1 & 2 & -1 \\
0 & 0 & 0 & -1 & 3
\end{pmatrix}
$$

$$
C = \begin{pmatrix}
2 & -1 & -1 & 0 \\
-1 & 3 & -1 & 0 \\
-1 & -1 & 2 & -1 \\
0 & 0 & -1 & 3
\end{pmatrix}
$$
Example: McKay-Cartan matrix for $\mathfrak{S}_4$

\[ C = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix} \xrightarrow{SNF} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \]

Thus $K(\gamma) = \text{coker} \ (C) \cong \mathbb{Z}/4\mathbb{Z}$. 
Theorem 1
Let $e = c_0, c_1, \ldots, c_\ell$ be a set of conjugacy class representatives for $G$, then:

i. 
$$\prod_{i=1}^{\ell} (n - \chi_\gamma(c_i)) = |K(\gamma)| \cdot |G|$$

ii. If $\chi_\gamma$ is real-valued, and $\chi_\gamma(c)$ is an integer character value achieved by $m$ different conjugacy classes, then $K(\gamma)$ contains a subgroup isomorphic to $(\mathbb{Z}/(n - \chi_\gamma(c))\mathbb{Z})^{m-1}$.
Example: checking Theorem 1

We will use the reflection representation of $\mathcal{S}_4$ again. Recall that $K(\gamma) \cong \mathbb{Z}/4\mathbb{Z}$.

<table>
<thead>
<tr>
<th>$\chi_\gamma = \chi_1$</th>
<th>$e$</th>
<th>(12)</th>
<th>(123)</th>
<th>(1234)</th>
<th>(12)(34)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>

i. In this case we have

$$\ell \prod_{i=1}^\ell (n - \chi_\gamma(c_i)) = 2 \cdot 3 \cdot 4 \cdot 4 = 96$$

$$|K(\gamma)| \cdot |G| = 4 \cdot 4! = 96$$

ii. $\chi_\gamma$ is real-valued and has the repeated character value $-1$ with multiplicity 2. Thus $K(\gamma)$ should have a subgroup isomorphic to $(\mathbb{Z}/4\mathbb{Z})^1$. 
Critical groups for reflection representations of $\mathfrak{S}_n$.

In the previous example, Theorem 1 was enough to uniquely determine $K(\gamma)$. However, this does not happen in general. It turns out we can use the following proposition:

**Proposition (A. Miller and V. Reiner)**

*Suppose an $(\ell + 1) \times (\ell + 1)$ integer matrix $A$ has a Smith normal form over $\mathbb{Z}[t]$ for $tl - A$, then this Smith form must be*

$$
\begin{pmatrix}
    s_{\ell+1}(t) \\
    \vdots \\
    s_1(t)
\end{pmatrix}
$$

*where*

$$
    s_i(t) = \prod_{\mu(\lambda) \geq i} (t - \lambda)
$$

*and where $\mu(\lambda)$ denote the dimension of the $\lambda$-eigenspace for $A$.***
Why is this proposition useful?

- It turns out that $tl - \tilde{C}$ has a Smith form over $\mathbb{Z}[t]$ when $\gamma$ is the reflection representation of $\mathfrak{S}_n$.
- Setting $t = 0$ then gives us the critical group $K(\gamma)$. 
Two facts

1. For all $\lambda \vdash n$:

$$\chi_{(n-1,1)} \cdot \chi_\lambda = C(\lambda)\chi_\lambda + \sum \chi_\mu$$

Where the sum is over those partitions $\mu$ which can be obtained from $\lambda$ by removing and then adding a box. And $C(\lambda)$ is one less than the number of corners of $\lambda$ (Ballantine and Orellana).

2. The map $UD - tl$ in Young’s lattice has a Smith normal form over $\mathbb{Z}[t]$ (Cai and Stanley).
Putting it together

Theorem 2
Let $\gamma$ be the reflection representation of $\mathfrak{S}_n$ and let $\tilde{C}$ be the associated extended McKay-Cartan matrix. Let $p(k)$ denote the number of partitions of the integer $k$. Then

$$K(\gamma) \cong \bigoplus_{i=2}^{p(n)-p(n-1)} \mathbb{Z}/q_i\mathbb{Z}$$

where

$$q_i = \prod_{1 \leq k \leq n, p(k) - p(k-1) \geq i} k$$
Example: checking Theorem 2

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(k)$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>11</td>
</tr>
</tbody>
</table>

For $n = 4$, we have

$$q_2 = \left( \prod_{1 \leq k \leq 4} \frac{k}{p(k) - p(k-1) \geq 2} \right) = 4$$

Thus $K(\gamma) \cong \mathbb{Z}/4\mathbb{Z}$. 
Example: using Theorem 2

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
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</table>

For a more interesting example, let’s try the case $n = 6$:

\[ q_2 = \left( \prod_{1 \leq k \leq 6} \frac{k}{p(k) - p(k-1) \geq 2} \right) = 4 \cdot 5 \cdot 6 = 120 \]

\[ q_3 = \left( \prod_{1 \leq k \leq 6} \frac{k}{p(k) - p(k-1) \geq 3} \right) = 6 \]

\[ q_4 = 6 \]

Thus $K(\gamma) \cong \mathbb{Z}/120\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$. 
Thanks for coming

Are there any questions?
References


