The Critical Group of a Line Graph: The Bipartite Case

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Overview

Give a graph $G = (V, E)$ the critical group $K(G)$ is a finite abelian group whose order is $\kappa(G)$, the number of spanning forests of the graph. Here $G$ is an undirected graph without self loops, though multiple edges are allowed. There is a known relationship between the critical group of $G$ and the critical group of the line graph $\text{line} \ G$ when $G$ is nonbipartite. Our task is to explore the relationship when $G$ is bipartite.
On Dr. Vic Reiner’s web page www.math.umn.edu/~reiner/:

- REU
- math latin honors theses
- “The Critical Group of a Line Graph” (Berget, Manion, Maxwell, Potechin, and Reiner)
Definition

Let $G = (V, E)$ be finite graph without self loops. The graph Laplacian $L(G)$ is the singular positive semidefinite $|V| \times |V|$ matrix given by

$$L(G)_{i,j} = \begin{cases} \deg_G(i) & \text{if } i = j \\ -m_{i,j} & \text{otherwise}, \end{cases}$$

where $m_{i,j}$ is the multiplicity of the edge $\{i, j\}$ in $E$.

Note $L(G) = D - A$ where $D$ is the degree matrix and $A$ is the adjacency matrix.
Kirchhoff’s Matrix Tree Theorem

We notice the rank of $L(G)$ is $|V| - c$ if $G$ has $c$ connected components. Assuming $G$ is connected denote the eigenvalues of $L(G)$ by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > \lambda_n = 0$ where $|V| = n$. Also let $L(G)^{i,j}$ be the reduced graph Laplacian obtained from $L(G)$ by striking out row $i$ and column $j$.

**Theorem (Kirchhoff’s Matrix Tree Theorem)**

$$\kappa(G) = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n}$$

$$= (-1)^{i+j} \det L(G)^{i,j}$$
The critical group $K(G)$ of a graph $G$ is a finite abelian group whose order is $\kappa(G)$ the number of spanning forests of the graph. If $G$ has $c$ connected components then

$$\mathbb{Z}^{|V|}/\text{im } L(G) \cong \mathbb{Z}^c \oplus K(G).$$

If $G$ is connected, then we have

$$\mathbb{Z}^{|V|-1}/\text{im } L(G)^{i,j} \cong K(G).$$

**Remark**

The *Smith normal form* of $L(G)$ gives us $K(G)$. 
We have the following alternative presentation of critical group

\[ K(G) \cong \mathbb{Z}^E / (B \oplus Z). \]

Where \( B \) is the bond lattice and \( Z \) is the cycle lattice.

**Remark**

Here we fix an arbitrary orientation of the edges and the edge set \( E \) becomes a basis for \( \mathbb{R}^E \cong \mathbb{R}^m \) where \( |E| = m \).
Remark

The single vertex cuts like the one of the right give a spanning set for $B$. 

Example Bonds
Example Cycle

Remark
Recall all cycles in bipartite graphs have even length.
The Edge Subdivision Graph

**Definition**

The *edge subdivision graph* for $G$ denoted $\text{sd} \ G$ is obtained by placing a new vertex at the midpoint of every edge in $G$.

*Figure: $G$*  
*Figure: $\text{sd} \, G$*
The line graph for $G$ denoted $\text{line } G = (V_{\text{line } G}, E_{\text{line } G})$ is defined by $V_{\text{line } G} = E$ where there is an edge in $E_{\text{line } G}$ corresponding to each pair of edges in $E$ incident on a vertex in $V$.
Let $\beta(G)$ be the number of independent cycles in $G$. It is known the number of generators of $K(G)$ is bounded by $\beta(G)$. We also have the following simple relationship between $G$ and $sd\ G$.

**Theorem (Lorenzini)**

$$K(G) = \bigoplus_{i=1}^{\beta(G)} \mathbb{Z}_{d_i}$$

$$K(sd\ G) = \bigoplus_{i=1}^{\beta(G)} \mathbb{Z}_{2d_i}$$
Theorem (Sachs)

If $G$ is $d$-regular, then

$$\kappa(\text{line } G) = d^{\beta(G)} - 2^{\beta(G)} \kappa(G)$$

$$= d^{\beta(G)} - 2 \kappa(\text{sd } G).$$

Theorem (Berget et al.)

If a simple graph $G$ is 2-edge-connected, then the critical group $K(\text{line } G)$ can be generated by $\beta(G)$ elements.

Question

Can we say anything about the relationship between $K(G)$ and $K(\text{line } G)$?
A Homomorphism

Theorem (Berget et al.)

For any connected $d$-regular simple graph $G$ with $d \geq 3$ there is a natural group homomorphism $f : K(\text{line } G) \rightarrow K(\text{sd } G)$ whose kernel-cokernel exact sequence takes the form

$$0 \rightarrow \mathbb{Z}_d^{\beta(G) - 2} \oplus C \rightarrow K(\text{line } G) \xrightarrow{f} K(\text{sd } G) \rightarrow C \rightarrow 0$$

in which the cokernel $C$ is the following cyclic $d$-torsion group:

$$C = \begin{cases} 
0 & \text{if } G \text{ non-bipartite and } d \text{ is odd} \\
\mathbb{Z}_2 & \text{if } G \text{ non-bipartite and } d \text{ is even} \\
\mathbb{Z}_d & \text{if } G \text{ bipartite} 
\end{cases}$$
Corollary (Berget et al.)

For $G$ a simple, connected, $d$-regular graph with $d \geq 3$ which is nonbipartite, after uniquely expressing

$$K(G) \cong \bigoplus_{i=1}^{\beta(G)} \mathbb{Z}_{d_i}$$

with $d_i$ dividing $d_{i+1}$, one has

$$K(\text{line } G) \cong \bigoplus_{i=1}^{\beta(G)-2} \mathbb{Z}_{2dd_i} \oplus \begin{cases} \mathbb{Z}_{2d\beta(G)-1} \oplus \mathbb{Z}_{2d\beta(G)} & \text{if } |V| \text{ even} \\ \mathbb{Z}_{4d\beta(G)-1} \oplus \mathbb{Z}_{d\beta(G)} & \text{if } |V| \text{ odd} \end{cases}$$

Proof.

Follow from previous theorem on exact sequence and a technical lemma on the $p$-primary component.
An Example

Let $G = K_4$, then $\beta(G) = 3$, $d = 3$, and $|V|$ is even and we have

$$K(G) \cong \mathbb{Z}_1 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$$

$$K(\text{line } G) \cong \mathbb{Z}_6 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_8 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{24}$$
The goal in this thesis was to collect data from various infinite families of regular bipartite graphs $G$ on the relation between $K(G)$ and $K(\text{line } G)$, in the hope that they might lead us to some conjecture(s) as precise as the previous corollary.
The Complete Bipartite Graph

**Theorem (Lorenzini, Berget)**

Let \( G = K_{n,n} \), then

\[
\begin{align*}
K(G) &\cong \mathbb{Z}_n^{2n-4} \oplus \mathbb{Z}_{n^2} \\
K(\text{line } G) &\cong \mathbb{Z}_{2n}^{(n-2)^2+1} \oplus \mathbb{Z}_{2n^2}^{2n-4}.
\end{align*}
\]
Almost Complete Bipartite Graph

Theorem

Let \( G = K_{n,n} - M \) where \( M \) is a complete matching and \( n \geq 4 \), then

\[
K(G) \cong \mathbb{Z}_{n-2} \oplus \mathbb{Z}_{n(n-2)}^{n-3} \oplus \mathbb{Z}_{n(n-1)(n-2)}
\]

\[
K(\text{line } G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{2(n-1)}^{(n-2)^2-3} \oplus \mathbb{Z}_{2(n-1)(n-2)} \oplus \mathbb{Z}_{2n(n-1)(n-2)}^{n-2}
\]

Proof.

- Use Smith Normal Form reduction to obtain \( K(G) \).
- Use known relationships to obtain \( K(\text{line } G) \).
Circulant Graphs

We denote circulant graphs by $C_n(a_1, a_2, \ldots, a_m)$. We note that a circulant graph is always regular, and it is bipartite if and only if $n$ is even and $a_i$ is odd for each $i$.

Figure: $C_8(1, 3)$
A Bipartite Circulant Graph

Conjecture

Let $G = C_{2(2l+1)}(1, 2l+1)$ where $2l + 1 = 3^k m$ with $\gcd(3, m) = 1$, then we have

$$K(G) \cong \mathbb{Z}_{3^k} \oplus \mathbb{Z}_{3^k d_1} \oplus \mathbb{Z}_{3^k d_2}$$

$$K(\text{line } G) \cong \mathbb{Z}_{6^{2l-1}} \oplus \mathbb{Z}_{2 \cdot 3^k} \oplus \mathbb{Z}_{2 \cdot 3^k d_1} \oplus \mathbb{Z}_{2 \cdot 3^k d_2}$$

where $3$ does not divide $d_1$ or $d_2$. 
Another Bipartite Circulant Graph

**Conjecture**

Let $G = C_{2.2l}(1, 2l - 1)$, then we have

$$K(G) \cong \begin{cases} \mathbb{Z}_4 \oplus \mathbb{Z}_8^{2l-4} \oplus \mathbb{Z}_{8l} & \text{if } l \text{ is even} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_8^{2l-2} \oplus \mathbb{Z}_{8l} & \text{if } l \text{ is odd} \end{cases}$$

$$K(\text{line } G) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_8^{2l} \oplus \mathbb{Z}_{16}^2 \oplus \mathbb{Z}_{64}^{2l-3} \oplus \mathbb{Z}_{64l} \quad \text{if } l \text{ is odd.}$$
The relationship between $K(G)$ and $K(\text{line } G)$ is known for $G$ regular and nonbipartite. Both $K(G)$ and $K(\text{line } G)$ have been explicitly computed for the special cases $K_{n,n}$ and $K_{n,n} - M$. We have conjectures for $K(G)$ and $K(\text{line } G)$ in other cases, but nothing conclusive has emerged yet.
Recall the following corollary:

**Corollary (Berget et al.)**

For $G$ a simple, connected, $d$-regular graph with $d \geq 3$ which is nonbipartite, after uniquely expressing

\[
K(G) \cong \bigoplus_{i=1}^{\beta(G)} \mathbb{Z}_{d_i}
\]

with $d_i$ dividing $d_{i+1}$, one has

\[
K(\text{line } G) \cong \bigoplus_{i=1}^{\beta(G)-2} \mathbb{Z}_{2dd_i} \oplus \begin{cases} 
\mathbb{Z}_{2d_{\beta(G)-1}} \oplus \mathbb{Z}_{2d_{\beta(G)}} & \text{if } |V| \text{ even} \\
\mathbb{Z}_{4d_{\beta(G)-1}} \oplus \mathbb{Z}_{d_{\beta(G)}} & \text{if } |V| \text{ odd}
\end{cases}
\]
The Bipartite Relationship?

Let $G = K_{n,n}$, then

$$K(G) \cong \mathbb{Z}_n \oplus \mathbb{Z}^{2n-5}_n \oplus \mathbb{Z}_{n^2}$$

$$K(sd\ G) \cong \mathbb{Z}_2^{(n-2)^2} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}^{2n-5}_2 \oplus \mathbb{Z}_{2n^2}$$

$$K(line\ G) \cong \mathbb{Z}_2^{(n-2)^2} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}^{2n-5}_2 \oplus \mathbb{Z}_{2n^2}$$

Let $G = K_{n,n} - M$, then

$$K(G) \cong \mathbb{Z}(n-2) \oplus \mathbb{Z}^{n-3}_{n(n-2)} \oplus \mathbb{Z}_{n(n-1)(n-2)}$$

$$K(sd\ G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}^{n^2-4n+1}_2 \oplus \mathbb{Z}_{2(n-2)} \oplus \mathbb{Z}^{n-3}_{2n(n-2)} \oplus \mathbb{Z}_{2n(n-1)(n-2)}$$

$$K(line\ G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}^{n^2-4n+1}_2 \oplus \mathbb{Z}_{2(n-1)(n-2)} \oplus \mathbb{Z}^{n-3}_{2n(n-1)(n-2)} \oplus \mathbb{Z}_{2n(n-1)(n-2)}$$
Let $G = C_{2 \cdot 2^l}(1, 2^l - 1)$ for $l$ odd, then conjecturally

$$K(G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_8^{2^l - 3} \oplus \mathbb{Z}_8^l$$

$$K(\text{sd } G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2^{2^l - 1} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_{16}^{2^l - 3} \oplus \mathbb{Z}_{16}^l$$

$$K(\text{lineG}) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_8^{2^l - 1} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_{64}^{2^l - 3} \oplus \mathbb{Z}_{64}^l$$