The Critical Group of a Line Graph: The Bipartite Case

John M. Machacek

University of Minnesota - Twin Cities

macha052@umn.edu

December 14, 2011

Overview

Give a graph $G = (V, E)$ the critical group $K(G)$ is a finite abelian group whose order is $\kappa(G)$, the number of spanning forests of the graph. Here G is an undirected graph without self loops, though multiple edges are allowed. There is a known relationship between the critical group of G and the critical group of the line graph $\mathrm{line}\, G$ when G is nonbipartite. Our task is to explore the relationship when G is bipartite.

Previous Work

On Dr. Vic Reiner's web page `www.math.umn.edu/~reiner/`:

- REU
- math latin honors theses
- "The Critical Group of a Line Graph" (Berget, Manion, Maxwell, Potechin, and Reiner)

The Graph Laplacian Matrix

Definition

Let $G = (V, E)$ be finite graph without self loops. The *graph Laplacian* $L(G)$ is the singular positive semidefinite $|V| \times |V|$ matrix given by

$$L(G)_{i,j} = \begin{cases} \deg_G(i) & \text{if } i = j \\ -m_{i,j} & \text{otherwise,} \end{cases}$$

where $m_{i,j}$ is the multiplicity of the edge $\{i, j\}$ in E.

Note $L(G) = D - A$ wher D is the *degree* matrix and A is the *adjacency* matrix.

Kirchhoff's Matrix Tree Theorem

We notice the rank of $L(G)$ is $|V| - c$ if G has c connected components. Assuming G is connected denote the eigenvalues of $L(G)$ by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > \lambda_n = 0$ where $|V| = n$. Also let $\overline{L(G)}^{i,j}$ be the reduced graph Laplacian obtained from $L(G)$ by striking out row i and column j.

Theorem (Kirchhoff's Matrix Tree Theorem)

$$\begin{aligned}\kappa(G) &= \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n} \\ &= (-1)^{i+j} \det \overline{L(G)}^{i,j}\end{aligned}$$

The Critical Group I

The *critical group* $K(G)$ of a graph G is a finite abelian group whose order is $\kappa(G)$ the number of spanning forests of the graph. If G has c connected components then

$$\mathbb{Z}^{|V|}/\operatorname{im} L(G) \cong \mathbb{Z}^c \oplus K(G).$$

If G is connected, then we have

$$\mathbb{Z}^{|V|-1}/\operatorname{im} \overline{L(G)}^{i,j} \cong K(G).$$

Remark

The *Smith normal form* of $L(G)$ gives us $K(G)$.

The Critical Group II

We have the following alternative presentation of critical group

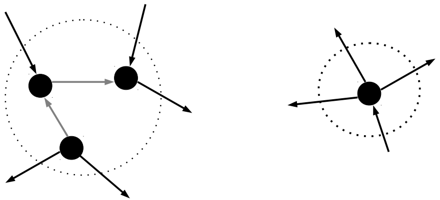
$$K(G) \cong \mathbb{Z}^E/(B \oplus Z).$$

Where B is the *bond lattice* and Z is the *cycle lattice*.

Remark

Here we fix are arbitrary orientation of the edges and the edge set E becomes a basis for $\mathbb{R}^E \cong \mathbb{R}^m$ where $|E| = m$.

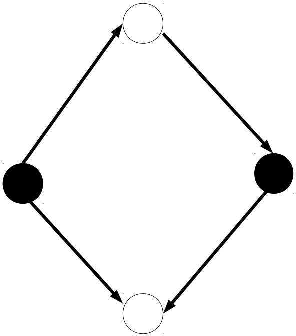
Example Bonds



Remark

The single vertex cuts like the one of the right give a spanning set for B.

Example Cycle



Remark

Recall all cycles in bipartite graphs have even length.

The Edge Subdivision Graph

Definition

The *edge subdivision graph* for G denoted $\mathrm{sd}\, G$ is obtained by placing a new vertex at the midpoint of every edge in G.

Figure: G

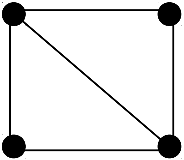
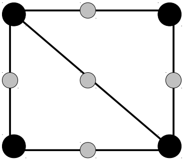


Figure: $\mathrm{sd}\, G$



The Line Graph

Definition

The *line graph* for G denoted $\operatorname{line} G = (V_{\operatorname{line} G}, E_{\operatorname{line} G})$ is defined by $V_{\operatorname{line} G} = E$ where there is an edge in $E_{\operatorname{line} G}$ corresponding to each pair of edges in E incident on a vertex in V.

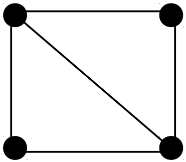


Figure: G

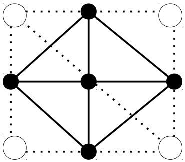


Figure: $\operatorname{line} G$

G and $\mathrm{sd}\, G$

Let $\beta(G)$ be the number of *independent cycles* in G. It is know the number of generators of $K(G)$ is bounded by $\beta(G)$. We also have the following simple relationship between G and $\mathrm{sd}\, G$.

Theorem (Lorenzini)

$$K(G) = \bigoplus_{i=1}^{\beta(G)} \mathbb{Z}_{d_i}$$

$$K(\mathrm{sd}\, G) = \bigoplus_{i=1}^{\beta(G)} \mathbb{Z}_{2d_i}$$

G and $\mathrm{line}\, G$

Theorem (Sachs)

If G is d-regular, then

$$\begin{aligned}\kappa(\mathrm{line}\, G) &= d^{\beta(G)-2} 2^{\beta(G)} \kappa(G) \\ &= d^{\beta(G)-2} \kappa(\mathrm{sd}\, G).\end{aligned}$$

Theorem (Berget et al.)

If a simple graph G is 2-edge-connected, then the critical group $K(\mathrm{line}\, G)$ can be generated by $\beta(G)$ elements.

Question

Can we say anything about the relationship between $K(G)$ and $K(\mathrm{line}\, G)$?

A Homomorphism

Theorem (Berget et al.)

For any connected d-regular simple graph G with $d \geq 3$ there is a natural group homomorphism $f : K(\mathrm{line}\, G) \to K(\mathrm{sd}\, G)$ whose kernel-cokernel exact sequence takes the form

$$0 \to \mathbb{Z}_d^{\beta(G)-2} \oplus C \to K(\mathrm{line}\, G) \xrightarrow{f} K(\mathrm{sd}\, G) \to C \to 0$$

in which the cokernel C is the following cyclic d-torsion group:

$$C = \begin{cases} 0 & \textit{if } G \textit{ non-bipartite and } d \textit{ is odd} \\ \mathbb{Z}_2 & \textit{if } G \textit{ non-bipartite and } d \textit{ is even} \\ \mathbb{Z}_d & \textit{if } G \textit{ bipartite} \end{cases}$$

Nonbipartite Graphs

Corollary (Berget et al.)

For G a simple, connected, d-regular graph with $d \geq 3$ which is nonbipartite, after uniquely expressing

$$K(G) \cong \bigoplus_{i=1}^{\beta(G)} \mathbb{Z}_{d_i}$$

with d_i dividing d_{i+1}, one has

$$K(\text{line}\, G) \cong \bigoplus_{i=1}^{\beta(G)-2} \mathbb{Z}_{2dd_i} \oplus \begin{cases} \mathbb{Z}_{2d_{\beta(G)-1}} \oplus \mathbb{Z}_{2d_{\beta(G)}} & \text{if } |V| \text{ even} \\ \mathbb{Z}_{4d_{\beta(G)-1}} \oplus \mathbb{Z}_{d_{\beta(G)}} & \text{if } |V| \text{ odd} \end{cases}$$

Proof.

Follow from previous theorem on exact sequence and a technical lemma on the p-primary component. □

An Example

Let $G = K_4$, then $\beta(G) = 3$, $d = 3$, and $|V|$ is even and we have

$$K(G) \cong \mathbb{Z}_1 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$$

$$K(\text{line}\, G) \cong \mathbb{Z}_6 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_8 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{24}$$

Figure: G

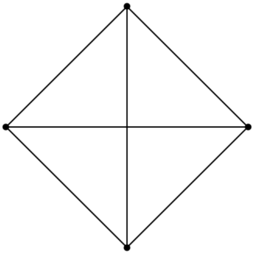
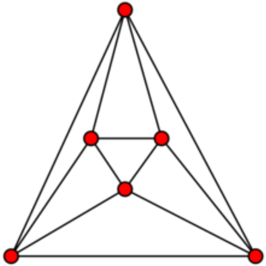


Figure: $\text{sd}\, G$



Goal

The goal in this thesis was to collect data from various infinite families of regular *bipartite graphs* G on the relation between $K(G)$ and $K(\mathrm{line}\, G)$, in the hope that they might lead us to some conjecture(s) as precise as the previous corollary.

The Complete Bipartite Graph

Theorem (Lorenzini, Berget)

Let $G = K_{n,n}$, *then*

$$K(G) \cong \mathbb{Z}_n^{2n-4} \oplus \mathbb{Z}_{n^2}$$

$$K(\text{line } G) \cong \mathbb{Z}_{2n}^{(n-2)^2+1} \oplus \mathbb{Z}_{2n^2}^{2n-4}.$$

Almost Complete Bipartite Graph

Theorem

Let $G = K_{n,n} - M$ where M is a complete matching and $n \geq 4$, then

$$K(G) \cong \mathbb{Z}_{n-2} \oplus \mathbb{Z}_{n(n-2)}^{n-3} \oplus \mathbb{Z}_{n(n-1)(n-2)}$$

$$K(\mathrm{line}\, G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{2(n-1)}^{(n-2)^2-3} \oplus \mathbb{Z}_{2(n-1)(n-2)} \oplus \mathbb{Z}_{2n(n-1)(n-2)}^{n-2}.$$

Proof.

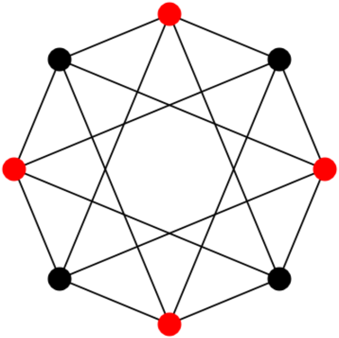
- Use Smith Normal Form reduction to obtain $K(G)$.
- Use known relationships to obtain $K(\mathrm{line}\, G)$.

□

Circulant Graphs

We denote circulant graphs by $C_n(a_1, a_2, \ldots, a_m)$. We note that a circulant graph is always regular, and it is bipartite if and only n is even and a_i is odd for each i.

Figure: $C_8(1, 3)$



A Bipartite Circulant Graph

Conjecture

Let $G = C_{2(2l+1)}(1, 2l+1)$ *where* $2l + 1 = 3^k m$ *with* $\gcd(3, m) = 1$, *then we have*

$$K(G) \cong \mathbb{Z}_{3^k} \oplus \mathbb{Z}_{3^k d_1} \oplus \mathbb{Z}_{3^{k+1} d_2}$$

$$K(\text{line } G) \cong \mathbb{Z}_6^{2l-1} \oplus \mathbb{Z}_{2 \cdot 3^k} \oplus \mathbb{Z}_{2 \cdot 3^{k+1} d_1} \oplus \mathbb{Z}_{2 \cdot 3^{k+1} d_2}$$

where 3 *does not divide* d_1 *or* d_2.

Another Bipartite Circulant Graph

Conjecture

Let $G = C_{2 \cdot 2l}(1, 2l-1)$, *then we have*

$$K(G) \cong \begin{cases} \mathbb{Z}_4^4 \oplus \mathbb{Z}_8^{2l-4} \oplus \mathbb{Z}_{8l} & \text{if } l \text{ is even} \\ \mathbb{Z}_2^2 \oplus \mathbb{Z}_8^{2l-2} \oplus \mathbb{Z}_{8l} & \text{if } l \text{ is odd} \end{cases}$$

$$K(\mathrm{line}\, G) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_8^{2l} \oplus \mathbb{Z}_{16}^2 \oplus \mathbb{Z}_{64}^{2l-3} \oplus \mathbb{Z}_{64l} \quad \text{if } l \text{ is odd.}$$

Summary

The relationship between $K(G)$ and $K(\mathrm{line}\, G)$ is known for G regular and nonbipartite. Both $K(G)$ and $K(\mathrm{line}\, G)$ have been explicitly computed for the special cases $K_{n,n}$ and $K_{n,n} - M$. We have conjectures for $K(G)$ and $K(\mathrm{line}\, G)$ in other cases, but nothing conclusive has emerged yet.

The Nonbipartite Relationship Revisited

Recall the following corollary:

Corollary (Berget et al.)

For G a simple, connected, d-regular graph with $d \geq 3$ which is nonbipartite, after uniquely expressing

$$K(G) \cong \bigoplus_{i=1}^{\beta(G)} \mathbb{Z}_{d_i}$$

with d_i dividing d_{i+1}, one has

$$K(\text{line}\, G) \cong \bigoplus_{i=1}^{\beta(G)-2} \mathbb{Z}_{2dd_i} \oplus \begin{cases} \mathbb{Z}_{2d_{\beta(G)-1}} \oplus \mathbb{Z}_{2d_{\beta(G)}} & \text{if } |V| \text{ even} \\ \mathbb{Z}_{4d_{\beta(G)-1}} \oplus \mathbb{Z}_{d_{\beta(G)}} & \text{if } |V| \text{ odd} \end{cases}$$

The Bipartite Relationship?

Let $G = K_{n,n}$, then

$$\begin{aligned} K(G) &\cong \mathbb{Z}_n \oplus \mathbb{Z}_n^{2n-5} \oplus \mathbb{Z}_{n^2} \\ K(\text{sd } G) &\cong \mathbb{Z}_2^{(n-2)^2} \oplus \mathbb{Z}_{2n} \oplus \mathbb{Z}_{2n}^{2n-5} \oplus \mathbb{Z}_{2n^2} \\ K(\text{line } G) &\cong \mathbb{Z}_{2n}^{(n-2)^2} \oplus \mathbb{Z}_{2n} \oplus \mathbb{Z}_{2n^2}^{2n-5} \oplus \mathbb{Z}_{2n^2} \end{aligned}$$

Let $G = K_{n,n} - M$, then

$$\begin{aligned} K(G) &\cong \mathbb{Z}_{(n-2)} \oplus \mathbb{Z}_{n(n-2)}^{n-3} \oplus \mathbb{Z}_{n(n-1)(n-2)} \\ K(\text{sd } G) &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2^{n^2-4n+1} \oplus \mathbb{Z}_{2(n-2)} \oplus \mathbb{Z}_{2n(n-2)}^{n-3} \oplus \mathbb{Z}_{2n(n-1)(n-2)} \\ K(\text{line } G) &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_{2(n-1)}^{n^2-4n+1} \oplus \mathbb{Z}_{2(n-1)(n-2)} \oplus \mathbb{Z}_{2n(n-1)(n-2)}^{n-3} \oplus \mathbb{Z}_{2n(n-1)(n-2)} \end{aligned}$$

The Bipartite Relationship?

Let $G = C_{2\cdot 2l}(1, 2l-1)$ for l odd, then conjecturally

$$
\begin{aligned}
K(G) &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_8^{2l-3} \oplus \mathbb{Z}_{8l} \\
K(\mathrm{sd}\, G) &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2^{2l-1} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_{16}^{2l-3} \oplus \mathbb{Z}_{16l} \\
K(lineG) &\cong \mathbb{Z}_4 \oplus \mathbb{Z}_8^{2l-1} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_{64}^{2l-3} \oplus \mathbb{Z}_{64l}
\end{aligned}
$$