THE CRITICAL GROUP OF A LINE GRAPH: THE BIPARTITE CASE

JOHN MACHACEK

Abstract. The critical group $K(G)$ of a graph G is a finite abelian group whose order is the number of spanning forests of the graph. Here we investigate the relationship between the critical group of a regular bipartite graph G and its line graph $\operatorname{line} G$. The relationship between the two is known completely for regular nonbipartite graphs.

We compute the critical group of a graph closely related to the complete bipartite graph and the critical group of its line graph. We also discuss general theory for the critical group of regular bipartite graphs. We close with various examples demonstrating what we have observed through experimentation. The problem of classifying the the relationship between $K(G)$ and $K(\operatorname{line} G)$ for regular bipartite graphs remains open.

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1. Introduction

We investigate the *critical group* of graph. Here we want to emphasize that we will assume that all graphs do **not** have *self loops*, however multiple edges are allowed. The critical group is a graph invariant that is a finite abelian group whose order is the number of spanning trees of the graph. The critical group of a graph is closely related to a *chip-firing game* played on vertices of a graph. In physics literature the critical group is called the *abelian sandpile model*. Other aliases for the critical group include the *Jacobian group* and the *Picard group*. The critical group has been computed in certain special cases, but in general the relationship to of the critical group the structure of the graph is unknown. Here we are particularly interested in the relationship between the critical groups of a graph and its associated line graph for regular bipartite graphs. Such a relationship has been determined for regular nonbipartite graphs [3].

2. Preliminaries

Here we introduce concepts and definitions necessary for our work. We will also state existing results that our work builds upon.

2.1. **The graph Laplacian.** The critical group $K(G)$ of a graph G is a finite abelian group with order $\kappa(G)$, the number of spanning forests of the graph. Let $G = (V, E)$ be finite graph without self loops. The *graph Laplacian* $L(G)$ is the singular positive semidefinite $|V| \times |V|$ matrix given by

$$L(G)_{x,y} = \begin{cases} \deg_G(x) & \text{if } x = y \\ -m_{x,y} & \text{otherwise,} \end{cases}$$

where $m_{x,y}$ is the multiplicity of the edge $\{x, y\}$ in E. We can equivalently define the graph Laplacian as $L(G) = D - A$ where D is the *degree* matrix and A is the *adjacency* matrix. We observe that if G has c connected components then the rank of $L(G)$ is $|V| - c$. Viewing the graph Laplacian as an abelian group homomorphism $L(G) : \mathbb{Z}^{|V|} \to \mathbb{Z}^{|V|}$ we can define the critical group $K(G)$ as the unique finite abelian group such that the cokernel can be expressed as

$$\mathbb{Z}^{|V|} / \operatorname{im} L(G) \cong \mathbb{Z}^c \oplus K(G). \tag{1}$$

A useful observation is that the critical group of a graph is the product of the critical groups of its connected components, so we can restrict our study to connected graphs without loss of generality. Assume $G = (V, E)$ is a connected graph without self loops. If we consider the reduced graph Laplacian $\overline{L(G)}^{x,y}$ obtained by striking out row x and column y of $L(G)$, then we can express the critical group as

$$K(G) \cong \mathbb{Z}^{|V|-1} / \operatorname{im} \overline{L(G)}^{x,y}. \tag{2}$$

Below we present Kirchhoff's Matrix Tree Theorem which implies the fact stated earlier that $|K(G)| = \kappa(G)$.

Theorem 1 (Kirchhoff's Matrix Tree Theorem, [4])**.** *If $G = (V, E)$ is a connected graph where $|V| = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > \lambda_n = 0$ are the eigenvalues of $L(G)$, then*

$$\kappa(G) = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n}$$

or equivalently for any choice of x and y

$$\kappa(G) = (-1)^{x+y}\overline{L(G)}^{x,y}.$$

2.2. **Theory of lattices.** Here we present a brief overview of some lattices associated with a graph. More can be found in [1]. Let $G = (V, E)$ be a graph with $|E| = m$. We consider the *cycle* space $Z^{\mathbb{R}}$ and its orthogonal complement the *bond* space $B^{\mathbb{R}}$. The cycle space and bond space give us a rational orthogonal decomposition $\mathbb{R}^E \cong \mathbb{R}^m = B^{\mathbb{R}} \oplus Z^{\mathbb{R}}$. We define a rational orthogonal decomposition below in Definition 2. To make the identification $\mathbb{R}^E \cong \mathbb{R}^m$ we must fix an arbitrary orientation of the edges $e \in E$. The basis element corresponding to $\{u, v\} \in E$ oriented from u to v will denoted by $e = (u, v)$ with the convention that $-e = (v, u)$. Recall the cycle space is the subspace of $\mathbb{R}^E$ spanned by the cycles in G after fixing our arbitrary orientation. Also the bond space is the subspace of $\mathbb{R}^E$ spanned by the bonds or *cuts* of G again using the fixed orientation. If $G = (V, E)$ and $V = V_1 \sqcup V_2$ is any partition of our vertex set, we denote the the sign indicent crossing the cut (partition) by $b_G(V_1, V_2)$.

Definition 2. A *rational orthogonal decomposition* is an $\mathbb{R}$-vector space decomposition $\mathbb{R}^m = B^{\mathbb{R}} \oplus Z^{\mathbb{R}}$ in which $B^{\mathbb{R}}$ and $Z^{\mathbb{R}}$ are rational $\mathbb{R}$-subspaces, meaning they are spanned by $\mathbb{Z}^m$.

Given an r-dimensional lattice $\Lambda^{\mathbb{R}} \subset \mathbb{R}^m$, we inherit the inner product $\langle \cdot, \cdot \rangle$. Our space $\Lambda^{\mathbb{R}}$ contains the lattice $\Lambda := \Lambda^{\mathbb{R}} \cap \mathbb{Z}^m$ which is of rank r. The *dual lattice*

$$\Lambda^{\#} := \{x \in \Lambda^{\mathbb{R}} : \langle x, \lambda \rangle \in \mathbb{Z} \text{ for all } \lambda \in \Lambda^{\mathbb{R}}\}$$

is also contained in $\Lambda^{\mathbb{R}}$ and has rank r. Furthermore since $< \Lambda, \Lambda > \subset < \mathbb{Z}^m, \mathbb{Z}^m > = \mathbb{Z}$, it follows that $\Lambda \subset \Lambda^{\#}$. We call their quotient the *determinant group*

$$\det(\Lambda) := \Lambda^{\#}/\Lambda.$$

We can now define the *critical group* of a rational orthogonal decomposition $\mathbb{R}^m = B^{\mathbb{R}} \oplus Z^{\mathbb{R}}$ as

$$K := \mathbb{Z}^m/(B \oplus Z). \tag{3}$$

Furthermore the determinant groups $\det(B)$ and $\det(Z)$ are both isomorphic to the critical group. Taking $Z^{\mathbb{R}}$ and $B^{\mathbb{R}}$ to be the cycle space and bond space of our graph, the expression in (3) gives an alternative presentation of the critical group for a graph G.

2.3. **The line graph and edge subdivision graph.** In addition to studying critical groups a bipartite graph, we will examine the critical groups of the associated *line graph.* Often times it is also useful to consider the *edge subdivision graph.* For a graph G we denote the line graph by $\operatorname{line} G$ and the edge subdivision graph by $\operatorname{sd} G$. Given the graph $G = (V, E)$, $\operatorname{sd} G$ is obtained by placing a new vertex at the midpoint of every edge in G. The line graph is $\operatorname{line} G = (V_{\operatorname{line} G}, E_{\operatorname{line} G})$ where $V_{\operatorname{line} G} = E$ and there is an edge in $E_{\operatorname{line} G}$ corresponding to each pair of edges in E incident on a given vertex $v \in V$.

We will denote $\mathbb{Z}/d\mathbb{Z}$ by $\mathbb{Z}_d$. Assuming G is connected, $\beta(G) = |E|-|V|+1$ is the number of independent cycles of G. The presentation $K(G) = Z^{\#}/Z$ easily shows that the number of generators for $K(G)$ is bounded by $\beta(G)$. This presentation also gives the following simple relationship between $K(G)$ and $K(\operatorname{sd} G)$ in (4) and (5) below, due to Lorenzini [7]:

$$K(G) = \bigoplus_{i=1}^{\beta(G)} \mathbb{Z}_{d_i} \tag{4}$$

$$K(\operatorname{sd} G) = \bigoplus_{i=1}^{\beta(G)} \mathbb{Z}_{2d_i} \tag{5}$$

Also under certain conditions $\beta(G)$ gives a bound on the number of generators for $K(\operatorname{line} G)$.

Theorem 3 ([3])**.** *If a simple graph G is 2-edge-connected, then the critical group $K(\operatorname{line} G)$ an be generated by $\beta(G)$ elements.*

We also have a relationship between the number of spanning trees of G and $\operatorname{line} G$ is the case of regular graphs.

Theorem 4 ([5])**.** *If G is d-regular, then*

$$\begin{aligned} \kappa(\operatorname{line} G) &= d^{\beta(G)-2} 2^{\beta(G)} \kappa(G) \\ &= d^{\beta(G)-2} \kappa(\operatorname{sd} G). \end{aligned}$$

Theorem 4 suggests a nice relationship between $K(G)$ and $K(\operatorname{line} G)$. In [3] Berget, Manion, Maxwell, Potechin, and Reiner proved several useful theorems relating $K(G)$ and $K(\operatorname{line} G)$. They were able to completely classify the relationship for simple, connected, d-regular graphs which are nonbipartite. Since the critical group is a finite abelian group, it is completely determined if we know the *p-Sylow subgroups* for each prime p.

Theorem 5 ([3])**.** *Let $G = (V, E)$ be a connected simple graph with at least one cycle of even length. Let $k(p)$ be the largest power such that $p^{k(p)}$ divides $deg_G(u) + deg_G(v)$ for all $\{u, v\} \in E$. Then for every prime p with $k(p) \geq 1$ we have*

$$K(\operatorname{line} G)/p^{k(p)} K(\operatorname{line} G) \cong \mathbb{Z}_{p^{k(p)}}^{\beta(G)-1} \oplus \mathbb{Z}_{\gcd(p^{k(p)}, |V|)}$$

for G bipartite and

$$K(\operatorname{line} G)/p^{k(p)} K(\operatorname{line} G) \cong \mathbb{Z}_{p^{k(p)}}^{\beta(G)-2} \oplus \begin{cases} 0 & \text{if } p \text{ is odd} \\ \mathbb{Z}_2^2 & \text{if } p = 2 \text{ and } |V| \text{ is even} \\ \mathbb{Z}_4 & \text{if } p = 2 \text{ and } |V| \text{ is odd} \end{cases}$$

for G nonbipartite.

We also have the following exact sequence relating $K(G)$ and $K(\operatorname{line} G)$ through the edge subdivision graph, to be compared with Theorem 4.

Theorem 6 ([3])**.** *For any connected d-regular simple graph G with $d \geq 3$ there is a natural group homomorphism $f : K(\operatorname{line} G) \to K(\operatorname{sd} G)$ whose kernel-cokernel exact sequence takes the form*

$$0 \to \mathbb{Z}_d^{\beta(G)-2} \oplus C \to K(\operatorname{line} G) \xrightarrow{f} K(\operatorname{sd} G) \to C \to 0$$

in which the cokernel C is the following cyclic d-torsion group:

$$C = \begin{cases} 0 & \textit{if } G \textit{ non-bipartite and } d \textit{ is odd} \\ \mathbb{Z}_2 & \textit{if } G \textit{ non-bipartite and } d \textit{ is even} \\ \mathbb{Z}_d & \textit{if } G \textit{ bipartite} \end{cases}$$

Theorem 5 and Theorem 6 combine to completely determine $K(\mathrm{line}\, G)$ from $K(G)$ if G is nonbipartite.

Corollary 7 ([3]). *For G a simple, connected, d-regular graph with $d \geq 3$ which is nonbipartite, after uniquely expressing*

$$K(G) \cong \bigoplus_{i=1}^{\beta(G)} \mathbb{Z}_{d_i}$$

with d_i dividing d_{i+1}, one has

$$K(\mathrm{line}\, G) \cong \bigoplus_{i=1}^{\beta(G)-2} \mathbb{Z}_{2dd_i} \oplus \begin{cases} \mathbb{Z}_{2d_{\beta(G)-1}} \oplus \mathbb{Z}2d_{\beta(G)} & \textit{if } |V| \textit{ even} \\ \mathbb{Z}_{4d_{\beta(G)-1}} \oplus \mathbb{Z}d_{\beta(G)} & \textit{if } |V| \textit{ odd} \end{cases}$$

The relationship between $K(G)$ and $K(\mathrm{line}\, G)$ is not known in general for bipartite graphs. However, we do know some special cases.

Example 8 (The complete bipartite graph). Let $G = K_{n_1,n_2}$, then both $K(G)$ and $K(\mathrm{line}\, G)$ are known [7, 2]:

$$K(G) \cong \mathbb{Z}_{n_1}^{n_2-2} \oplus \mathbb{Z}_{n_2}^{n_1-2} \oplus \mathbb{Z}_{n_1 n_2} \tag{6}$$

$$K(\mathrm{line}\, G) \cong \oplus \mathbb{Z}_{(n_1+n_2)}^{(n_1-2)(n_2-2)+1} \oplus \mathbb{Z}_{n_1(n_1+n_2)}^{n_1-2} \oplus \mathbb{Z}_{n_2(n_1+n_2)}^{n_2-2} \tag{7}$$

In particular when $n_1 = n_2 = n$ so that $K_{n_1,n_2} = K_{n,n}$ is n-regular one has:

$$K(G) \cong \mathbb{Z}_n^{2n-4} \oplus \mathbb{Z}_{n^2} \tag{8}$$

$$K(\mathrm{line}\, G) \cong \mathbb{Z}_{2n}^{(n-2)^2+1} \oplus \mathbb{Z}_{2n^2}^{2n-4} \tag{9}$$

2.4. **Circulant graphs.** We will use $C_n(a_1, a_2, \ldots, a_m)$ to denote the undirected circulant graph with the vertex set $\{0, 1, \ldots, n-1\}$ and edges given by $\{i, i \pm a_j \mod n\}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Thus we will assume without loss of generality that $a_i \leq a_{i+1}$ and $1 \leq a_i \leq \lfloor \frac{n}{2} \rfloor$ for all i.

First we note that the circulant graph $C_n(a_1, a_2, \ldots, a_m)$ is connected if and only if $\gcd(a_1, a_2, \ldots, a_m, n) = 1$. If $\gcd(a_1, a_2, \ldots, a_m, n) = k$ then $C_n(a_1, a_2, \ldots, a_m)$ is isomorphic to k copies of the connected circulant graph $C_{\frac{n}{k}}(\frac{a_1}{k}, \frac{a_2}{k}, \ldots, \frac{a_m}{k})$.

It is clear from the definition that the circulant graph $C_n(a_1, a_2, \ldots, a_m)$ is a regular graph. If we assume each a_i is distinct in $C_n(a_1, a_2, \ldots, a_m)$ then we have a d-regular graph where

$$d = \begin{cases} 2m - 1 & \text{if } a_m = \frac{n}{2} \\ 2m & \text{otherwise} \end{cases}.$$

Also given any $b \in \mathbb{Z}_n^\times$ we have that $C_n(a_1, a_2, \ldots, a_m) \cong C_n(ba_1, ba_2, \ldots, ba_m)$.

Let G be the d-regular circulant graph $C_n(a_1, a_2, \ldots, a_m)$. Then the graph Laplacian is a circulant matrix and takes the form

$$L(G) = \begin{bmatrix} d & x_1 & \dots & x_{n-2} & x_{n-1} \\ x_{n-1} & d & x_1 & & x_{n-2} \\ \vdots & x_{n-1} & d & \ddots & \vdots \\ x_2 & & \ddots & \ddots & x_1 \\ x_1 & x_2 & \dots & x_{n-1} & d \end{bmatrix}$$

where $x_j = -1$ if $\pm j \in \{a_1, a_2, \dots, a_m\}$ and $x_j = 0$ otherwise. The eigenvalues of $L(G)$ are $\lambda_j = d + \sum_{k=1}^{n-1} x_k \omega_j^k$ where the $\omega_j = \left(e^{\frac{2\pi i}{n}}\right)^j$ are nth roots of unity and $i^2 = -1$. From this along with Theorem 1 we obtain

$$\kappa(C_n(a_1, a_2, \dots, a_m)) = \frac{1}{n} \prod_{j=1}^{n-1} \left(d - \sum_{i=1}^{m} \left(\omega_j^{a_i} - \omega_j^{-a_i} \right) \right) \tag{10}$$

as an expression for the number of spanning trees (forests) of a circulant graph.

Lastly we note that if $C_n(a_1, a_2, \dots, a_m)$ is connected, then it is bipartite if and only if n is even and $a_1, a_2, \dots, a_m$ are all odd. This gives us a rich source of bipartite examples to study.

2.5. **Smith normal form and matrices.** Here we will introduce some notational conventions and common matrices that will be used throughout. We will use I_n to denote the $n \times n$ identity matrix. The $n \times n$ matrices of all 1s will be denoted by J_n. Also 0 will be used to denote both the integer constant as well as the integer matrix consisting of all zeros.

Given a graph $G = (V, E)$ we can obtain the critical group $K(G)$ by computing the *Smith normal form* of the graph Laplacian $L(G) \in \mathbb{Z}^{|V| \times |V|}$. When computing the Smith normal form one is allowed to do the following row and column operations:

- permute rows or columns
- scale rows or columns by ± 1
- add an integer multiple of one row (column) to another row (column)

These operations can be performed by left (right) multiplication by an integer matrix in the *general linear group* $GL_n(\mathbb{Z})$ for row (column) operations.

Given two matrices A, $B \in \mathbb{Z}^{n \times n}$ we say A and B are *equivalent* over $\mathbb{Z}$ (denoted $A \sim B$) if $B = PAQ$ for matrices P, $Q \in GL_n(\mathbb{Z})$. Note that $A \sim B$ implies $\operatorname{coker} A \cong \operatorname{coker} B$. We will use $\operatorname{diag}(d_1, d_2, \dots, d_n)$ to denote the $n \times n$ diagonal matrix with $d_1, d_2, \dots, d_n$ along the main diagonal and zeros elsewhere. The Smith normal form of A is the diagonal matrix $D = \operatorname{diag}(d_1, d_2, \dots, d_n)$ where $A \sim D$ and $d_i | d_{i+1}$ for $1 \leq i < n$. We also call two matrices A, $B \in \mathbb{Z}^{n \times n}$ *similar* (denoted $A \approx B$) if $B = PAP^{-1}$ for some $P \in GL_n(\mathbb{Z})$. Note that $A \approx B$ implies $A \sim B$.

3. Matrix reductions

In this chapter we will provide some technical lemmas which will be used later.

Definition 9. If M and T are square matrices of the same size, $H_n(M,T)$ is the $n \times n$ (block) matrix defined by

$$H_n(M,T) := \begin{bmatrix} M-T & -T & \dots & -T \\ -T & M-T & \ddots & \vdots \\ \vdots & \ddots & \ddots & -T \\ -T & \dots & -T & M-T \end{bmatrix}$$

Lemma 10 ([6])**.**

$$H_n(M,T) \approx \begin{bmatrix} M & & & & \\ & \ddots & & & \\ & & M & & \\ & & & M & T \\ & & & 0 & M-nT \end{bmatrix}$$

Proof. We will use the two matrices

$$P = \begin{bmatrix} I_n & 0 & \dots & 0 & -I_n & 0 \\ 0 & I_n & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -I_n & 0 \\ 0 & \dots & 0 & I_n & -I_n & 0 \\ 0 & \dots & 0 & 0 & I_n & 0 \\ -I_n & \dots & -I_n & -I_n & -I_n & -I_n \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} I_n & 0 & \dots & 0 & I_n & 0 \\ 0 & I_n & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & I_n & 0 \\ 0 & \dots & 0 & I_n & I_n & 0 \\ 0 & \dots & 0 & 0 & I_n & 0 \\ -I_n & \dots & -I_n & -I_n & -(n-1)I_n & -I_n \end{bmatrix}.$$

Then $H_n \approx P(H_n(M,T))P^{-1}$ which gives the desired result. $\square$

We will now look at the special case $H_n(1,1) = I_n - J_n$. Starting from Lemma 10 we can completely reduce $I_n - J_n$ to diagonal form by performing one more column operation.

Lemma 11.

$$I_n - J_n \sim \operatorname{diag}(1, 1, \dots, 1, n-1)$$

Proof. We will slightly modify the matrices used in Lemma 10 to obtain the desired result. We will use the $n \times n$ matrices X and Y where

$$X = \begin{bmatrix} 1 & 0 & \dots & 0 & -1 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -1 & 0 \\ 0 & \dots & 0 & 1 & -1 & 0 \\ 0 & \dots & 0 & 0 & 1 & 0 \\ -1 & \dots & -1 & -1 & -1 & -1 \end{bmatrix}$$

$$Y = \begin{bmatrix} 1 & 0 & \dots & 0 & 1 & 1 \\ 0 & 1 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 1 & 1 \\ 0 & \dots & 0 & 1 & 1 & 1 \\ 0 & \dots & 0 & 0 & 1 & 1 \\ -1 & \dots & -1 & -1 & -(n-1) & -(n-2) \end{bmatrix}.$$

Then we have $I_n - J_n \sim X(I_n - J_n)Y = \mathrm{diag}(1, 1, \dots, 1, n-1)$. □

Lemma 12.

$$XY = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 1 & 1 \\ & & & 0 & -1 \end{bmatrix}$$

Proof. Follows from matrix multiplication of X and Y. □

Lemma 13. *Letting* $D := \mathrm{diag}(1, 1, \dots, 1, n-1)$, *one has*

$$((n-1)XY-D)((n-1)XY+D) = \begin{bmatrix} n(n-2) & & & & \\ & \ddots & & & \\ & & n(n-2) & & \\ & & & n(n-2) & (n-1)(n-2) \\ & & & 0 & 0 \end{bmatrix}$$

Proof. Follows from direct computation. Note that $XYXY = I_n$ and clearly $D^2 = \mathrm{diag}(1, 1, \dots, 1, (n-1)^2)$. We also have

$$XYD = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 1 & (n-1) \\ & & & 0 & -(n-1) \end{bmatrix}$$

and

$$DXY = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 1 & 1 \\ & & & 0 & -(n-1) \end{bmatrix}.$$

□

4. Some specific regular bipartite graphs

Here we will look at the critical groups of some specific graphs along with their associated line graphs. The first such graph in closely related to the complete bipartite graph. The later graphs are some examples of circulant bipartite graphs

4.1. **The almost complete bipartite graph.** Here we investigate the critical group of the complete bipartite graph minus a perfect matching. Recall the critical of the complete bipartite graph and its line graph are known and can be found in Equations (6) and (7).

Theorem 14. *Let $G = K_{n,n} - M$ where M is a perfect matching and $n \geq 4$, then*

$$K(G) \cong \mathbb{Z}_{n-2} \oplus \mathbb{Z}_{n(n-2)}^{n-3} \oplus \mathbb{Z}_{n(n-1)(n-2)}$$

$$K(\operatorname{line} G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{2(n-1)}^{n^2-4n+1} \oplus \mathbb{Z}_{2(n-1)(n-2)} \oplus \mathbb{Z}_{2n(n-1)(n-2)}^{n-2}.$$

We will prove Theorem 14 with the following two lemmas.

Lemma 15. *If $G = K_{n,n} - M$ where M is a perfect matching and $n \geq 4$, then*

$$K(G) \cong \mathbb{Z}_{n-2} \oplus \mathbb{Z}_{n(n-2)}^{n-3} \oplus \mathbb{Z}_{n(n-1)(n-2)}.$$

Proof. Let $G = K_{n,n} - M$, using Lemma 11, Lemma 12, and Lemma 13 we reduce the graph Laplacian.

$$\begin{aligned}
L(G) &= \begin{bmatrix} (n-1)I_n & I_n - J_n \\ I_n - J_n & (n-1)I_n \end{bmatrix} \\
&\sim \begin{bmatrix} I_n - J_n & (n-1)I_n \\ (n-1)I_n & I_n - J_n \end{bmatrix} \\
&\sim \begin{bmatrix} X(I_n - J_n)Y & (n-1)XI_nY \\ (n-1)XI_nY & X(I_n - J_n)Y \end{bmatrix} \\
&\sim \begin{bmatrix} D & (n-1)XY \\ (n-1)XY & D \end{bmatrix} \\
&\sim \begin{bmatrix} D & D + (n-1)XY \\ (n-1)XY & D + (n-1)XY \end{bmatrix} \\
&\sim \begin{bmatrix} D & D + (n-1)XY \\ (n-1)XY - D & 0 \end{bmatrix} \\
&\sim \begin{bmatrix} D & 0 \\ (n-1)XY - D & ((n-1)XY - D)((n-1)XY + D) \end{bmatrix} \\
&\sim \begin{bmatrix} D & 0 \\ 0 & ((n-1)XY - D)((n-1)XY + D) \end{bmatrix} \\
&\sim \begin{bmatrix} I_n & 0 \\ 0 & \operatorname{diag}(n-2, n(n-2), \ldots, n(n-2), n(n-1)(n-2), 0) \end{bmatrix}
\end{aligned}$$

□

Lemma 16. *Let $G = K_{n,n} - M$ where M is a perfect matching and $n \geq 4$, then*

$$K(\operatorname{line} G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{2(n-1)}^{n^2-4n+1} \oplus \mathbb{Z}_{2(n-1)(n-2)} \oplus \mathbb{Z}_{2n(n-1)(n-2)}^{n-2}.$$

Proof. Let $G = K_{n,n}$ for $n \geq 4$. Note that G is $(n-1)$-regular and that $\beta(G) = n^2 - 3n + 1$. From Lemma 15 we know $K(G)$ and hence $K(\operatorname{sd} G)$ by the relationship in (4) and (5):

$$K(G) \cong \mathbb{Z}_{n-2} \oplus \mathbb{Z}_{n(n-2)}^{n-3} \oplus \mathbb{Z}_{n(n-1)(n-2)}$$
$$K(\operatorname{sd} G) \cong \mathbb{Z}_2^{n^2-4n2} \oplus \mathbb{Z}_{2(n-2)} \oplus \mathbb{Z}_{2n(n-2)}^{n-3} \oplus \mathbb{Z}_{2n(n-1)(n-2)}$$

From Theorem 6 we know that for any prime p which does not divide the degree $n-1$ we must have $\operatorname{Syl}_p(\operatorname{line} G) \cong \operatorname{Syl}_p(\operatorname{sd} G)$. Since $n \geq 4$ we know that $\gcd(n, n-1) = \gcd(n-2, n-1) = 1$, so we need not consider any prime factors of n or $n-2$. The conditions of Theorem 5 and for a prime p we let $p^{k(p)}$ be the largest power which divides $2(n-1)$.

Let us first consider the case where n is even and our graph G has odd degree. Recalling that $|V| = 2n$ we observe $\gcd(n-1, |V|) = 1$. Invoking Theorem 5 we see that $K(\operatorname{line} G)/(n-1)K(\operatorname{line} G) \cong \mathbb{Z}_{n-1}^{\beta(G)-1}$. Considering the cardinality relationship $|K(\operatorname{line} G)| = (n-1)^{\beta(G)-2}|K(\operatorname{sd} G)|$ our result follows for even n.

Next let us consider the case where n is odd, and therefore our graph has even degree. Noting that $|V| = 2n$ and thus $\gcd(2^k, |V|) = 2$ for any k, we then use Theorem 5 to conclude that $K(\operatorname{line} G)/2^{k(2)}K(\operatorname{line} G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{2^{k(2)}}^{\beta(G)-1}$. Now let us consider the p-Sylow subgroup part of $K(\operatorname{line} G)$ for odd primes p which divide $n-1$. Clearly $\gcd(p^{k(p)}, |V|) = 1$, so $K(\operatorname{line} G)/p^{k(p)} K(\operatorname{line} G) \cong \mathbb{Z}_{p^{k(p)}}^{\beta(G)-1}$. Again because $|K(\operatorname{line} G)| = (n-1)^{\beta(G)-2} |K(\operatorname{sd} G)|$ our result follows for odd n. $\square$

4.2. **Bipartite circulant graphs.** We know the circulant graph $C_n(a_1, a_2, \ldots, a_m)$ is bipartite whenever n is even and all a_i's are odd. Thus circulant graphs can give us many bipartite examples. Conjectures on two specific examples follow.

Conjecture 17. *If $G = C_{2(2l+1)}(1, 2l+1)$ where $2l+1 = 3^k m$ with $\gcd(3, m) = 1$, then we have*

$$K(G) \cong \mathbb{Z}_{3^k} \oplus \mathbb{Z}_{3^k d_1} \oplus \mathbb{Z}_{3^{k+1} d_2}$$

$$K(\operatorname{line} G) \cong \mathbb{Z}_6^{2l-1} \oplus \mathbb{Z}_{2\cdot 3^k} \oplus \mathbb{Z}_{2\cdot 3^{k+1} d_1} \oplus \mathbb{Z}_{2\cdot 3^{k+1} d_2}$$

where 3 does not divide d_1 or d_2.

Conjecture 18. *If $G = C_{2\cdot 2l}(1, 2l-1)$, then we have*

$$K(G) \cong \begin{cases} \mathbb{Z}_4^4 \oplus \mathbb{Z}_8^{2l-4} \oplus \mathbb{Z}_{8l} & \text{if } l \text{ is even} \\ \mathbb{Z}_2^2 \oplus \mathbb{Z}_8^{2l-2} \oplus \mathbb{Z}_{8l} & \text{if } l \text{ is odd} \end{cases}$$

In the case that l is odd we have a conjecture for the line graph.

Conjecture 19. *If $G = C_{2\cdot 2l}(1, 2l-1)$ and l is odd, then we have*

$$K(\operatorname{line} G) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_8^{2l} \oplus \mathbb{Z}_{16}^2 \oplus \mathbb{Z}_{64}^{2l-3} \oplus \mathbb{Z}_{64l}$$

5. A few general results

Here we present results for general regular bipartite graphs. We first present a result on the quotient group $K(G)/dK(G)$ for d-regular bipartite graphs G. Then we give a few results on perfect matchings in regular bipartite graphs.

5.1. **The quotient group.** Recalling Theorem 5 and Theorem 6 we see that to understand the relationship between $K(G)$ and $K(\operatorname{line} G)$ for a d-regular bipartite graph it is important to understand the p-Sylow subgroup of $K(G)$ for primes p which divide d. In this section we look at the quotient group $K(G)/dK(G)$. In the proposition below we show that $K(G)/dK(G)$ is at least $\mathbb{Z}_d$. Recalling the exact sequence in Theorem 6 and the relationship between $K(G)$ and $K(\operatorname{sd} G)$ in (4) and (5), we see that $K(G)$ must surject onto $\mathbb{Z}_d$ for any odd d. Below we show there must be a surjection regardless of the parity of d.

Proposition 20. *If $G = (V, E)$ is a connected simple d-regular bipartite graph, then there exists a surjective homomorphism $f : K(G)/dK(G) \to \mathbb{Z}_d$.*

Proof. We will use the cycle and bond lattice presentation of the critical group given in Equation (3), that is $K(G) \cong \mathbb{Z}^E/(B \oplus Z)$. We then see that

$$K(G)/dK(G) \cong Z^E/(B \oplus Z + d\mathbb{Z}^E).$$

First we choose an orientation of G. Let $V = V_1 \sqcup V_2$ be our bipartition. We orient each edge in E as $e = (x, y)$ for $x \in V_1$ and $y \in V_2$. Next we define the surjective homomorphism $g : \mathbb{Z}^E \to \mathbb{Z}_d$ such that $g(e) = \bar{1}$ for all $e \in E$. Now we claim that $(B \oplus Z + d\mathbb{Z}^E) \subset \ker(g)$. Clearly $d\mathbb{Z}^E \subset \ker(g)$. Since G is bipartite each cycle will be of even length, and with our chosen orientation of edges the sign of the edges in any cycle will alternate. Thus we have $Z \subset \ker(g)$. Since G is d regular any element of B must map to a multiple of d, so $Z \subset \ker(g)$. From knowing $(B \oplus Z + d\mathbb{Z}^E) \subset \ker(g)$ and that g is surjective we get a surjective homomorphism $f : \mathbb{Z}^E/(B \oplus Z + d\mathbb{Z}^E) \to \mathbb{Z}_d$. $\square$

Remark 21. The behavior in Proposition 20 is unique to bipartite graphs. A simple counterexample for nonbipartite graphs is the 3-regular graph $G = C_{10}(2, 5)$ where $K(G) \cong \mathbb{Z}_{19} \oplus \mathbb{Z}_{95}$ and thus $K(G)/3K(G) \cong 0$.

5.2. **Perfect matchings.** To understand the relationship between $K(G)$ and $K(\text{line}\, G)$ in d-regular bipartite graph G it is important to understand the parts of $K(G)$ which have d-torsion. In this section we will attempt to further this understanding by showing that in $K(G)$ the perfect matchings of the vertices in G generate a cyclic subgroup with d-torsion.

Proposition 22. *If $G = (V, E)$ is bipartite, then in $K(G)$*

$$\sum_{e \in E} e = 0.$$

Proof. Recall the presentation $K(G) \cong \mathbb{Z}^E/(B \oplus Z)$ in Equation (3). If $G = (V, E)$ is bipartite summing over all edges in G is equivalent to summing over all edges adjacent to each vertex $x \in V_1$ where $V = V_1 \sqcup V_2$ is a bipartition of the vertices in G. Thus we see

$$\sum_{e \in E} e \in B.$$

$\square$

Proposition 23. *If $G = (V, E)$ a connected bipartite graph and M_1 and M_2 are two perfect matchings, then in $K(G)$*

$$\sum_{e \in M_1} e = \sum_{e \in M_2} e.$$

Proof. Recall the presentation $K(G) \cong \mathbb{Z}^E/(B \oplus Z)$ in Equation (3). We observe

$$\sum_{e \in M_1} e - \sum_{e \in M_2} e \in Z.$$

$\square$

Proposition 24. *Let $G = (V, E)$ be a connected d-regular bipartite graph and M be a perfect matching of the vertices in G, then in $K(G)$*

$$d \sum_{e \in M} e = 0.$$

Proof. If $G = (V, E)$ is a d-regular bipartite graph then we can express the edge set $E = M_1 \sqcup M_2 \sqcup \cdots \sqcup M_d$ as the disjoint union of d prefect matchings. Usings Propositions 22 and 23 we see

$$0 = \sum_{e \in E} e = \sum_{i=1}^{d} \sum_{e \in M_i} e = d \sum_{e \in M} e,$$

where M is any arbitrarily chosen perfect matching. □

6. Looking forward

Here we discuss the various behaviors we have observed. We present some examples exhibiting these behaviors. We discuss trends we observed in the data, and attempt to present counterexamples to any trends we found misleading.

First we point out, as a result of Theorem 6, if we have a d-regular bipartite graph, then for each prime p which does not divide d we know $\mathrm{Syl}_p(K(\mathrm{line}\, G)) \cong \mathrm{Syl}_p(K(\mathrm{sd}\, G))$. So when analyzing data it is only necessary to look at primes p which divide the degree d.

6.1. Odd primes.

We will first consider only odd primes. Odds primes generally exhibit simpler behavior.

Question 25. *What is the simplest behavior observed?*

Given a d-regular bipartite graph G, the simplest behavior we observed was when the exact sequence in Theorem 6 can be broken into two short exact sequences. That is let $p^{k(p)}$ be the largest power that divides d for a prime p where $k(p) \geq 1$, then we have

$$0 \;\to\; \mathbb{Z}_{p^{k(p)}}^{\beta(G)-1} \;\to\; \mathbb{Z}_{p^{k(p)}}^{\beta(G)-1} \;\xrightarrow{f}\; \mathbb{Z}_{p^{k(p)}} \;\to\; \mathbb{Z}_{p^{k(p)}} \;\to\; 0.$$

This behavior is present in the following example.

Example 26 (Simple Behavior)**.** Consider the graph $G = C_6(1,3)$. We note that this graph is 3-regular and $\beta(G) = 6$.

$$K(G) \cong \mathbb{Z}_{11} \oplus \mathbb{Z}_{165}$$
$$K(\mathrm{sd}\, G) \cong \mathbb{Z}_2^4 \oplus \mathbb{Z}_{22} \oplus \mathbb{Z}_{330}$$
$$K(\mathrm{line}\, G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_6^3 \oplus \mathbb{Z}_{66}^2 \oplus \mathbb{Z}_{330}$$

Here the only prime diving the degree is 3, and below we give the exact sequence from Theorem 6 for the prime 3.

$$0 \;\to\; \mathbb{Z}_3^5 \;\to\; \mathbb{Z}_3^5 \;\xrightarrow{f}\; \mathbb{Z}_3 \;\to\; \mathbb{Z}_3 \;\to\; 0$$

Question 27. *Are there any sufficient conditions which imply this simple behavior?*

Using Theorems 4, 5, and 6 we can conclude this simple behavior must occur when $K(G)$ restricted to the primes dividing d is simply the cyclic group $\mathbb{Z}_d$. From the data it appears that this hold when $\gcd(|V|, d) = 1$, however considering the following example we see this is not true.

Example 28. Consider the graph G. With the graph Laplacian matrix

$$L(G) = \begin{bmatrix} 9 & -3 & 0 & -6 \\ -3 & 9 & -6 & 0 \\ 0 & -6 & 9 & -3 \\ -6 & 0 & -3 & 9 \end{bmatrix}.$$

This is a 9-regular bipartite graph on 4 vertices, and $K(G) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{36}$.

We see in the graph Laplacian matrix in Example 28 that each entry is divisible by 3, so all invariant factors will also be divisible by 3. For any composite degree d we can construct a graph with similar properties using multiple edges. However if our degree is prime there is no such construction.

Question 29. *What other behavior is observed with odd primes?*

We were not able to cleanly formulate other classes of behaviors, but they certainly do exist. Here is one such example.

Example 30 (Other behavior)**.** Consider the graph $G = C_6(1,3)$. We note that this graph is 3-regular and $\beta(G) = 4$.

$$K(G) \cong \mathbb{Z}_3^2 \oplus \mathbb{Z}_9$$
$$K(\operatorname{sd} G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_6^2 \oplus \mathbb{Z}_{18}$$
$$K(\operatorname{line} G) \cong \mathbb{Z}_6^2 \oplus \mathbb{Z}_{18}^2$$

Here the only prime diving the degree is 3, and below we consider the exact sequence from Theorem 6 for the prime 3.

$$0 \to \mathbb{Z}_3^3 \to \mathbb{Z}_3^2 \oplus \mathbb{Z}_9^2 \xrightarrow{f} \mathbb{Z}_3^2 \oplus \mathbb{Z}_9 \to \mathbb{Z}_3 \to 0$$

6.2. **The prime 2.** The behavior becomes more complex when we look at graphs with even degree. In the following example the odd prime portion of the degree exhibits the simple behavior seen previously, while the prime 2 exhibits different behavior.

Question 31. *Is there an analogous simple behavior for the prime* 2*?*

There is nothing as simple as what was observed in the odd prime case. The nicest pattern found was the *mirror pattern* of

$$0 \to \mathbb{Z}_{2^k}^{\beta(G)-1} \to \mathbb{Z}_2 \oplus \mathbb{Z}_{2^{k+1}}^{\beta(G)-1} \xrightarrow{f} \mathbb{Z}_2^{\beta(G)-1} \oplus \mathbb{Z}_{2^{k+1}} \to \mathbb{Z}_{2^k} \to 0$$

where G is a d-regular bipartite graph and 2^k is the largest power of 2 dividing d. Below are some examples.

Example 32 (Mirror pattern)**.** Consider the graph $G = C_{10}(1,3)$. We note that this graph is 4-regular and $\beta(G) = 11$. Here the only prime diving the degree is 2. Below we consider the exact sequence from Theorem 6 for the prime 2.

$$0 \to \mathbb{Z}_4^{10} \to \mathbb{Z}_2 \oplus \mathbb{Z}_8^{10} \xrightarrow{f} \mathbb{Z}_2^{10} \oplus \mathbb{Z}_8 \to \mathbb{Z}_4 \to 0$$

Example 33 (Simple behavior and mirror pattern)**.** Consider the graph $G = C_{14}(1,3,5)$. We note that this graph is 6-regular and $\beta(G) = 29$. Here the only prime diving the degree are 2 and 3. Below we consider the exact sequences from Theorem 6 for the primes 2 and 3.

$$0 \to \mathbb{Z}_3^{28} \to \mathbb{Z}_3^{28} \xrightarrow{f} \mathbb{Z}_3 \to \mathbb{Z}_3 \to 0$$
$$0 \to \mathbb{Z}_2^{28} \to \mathbb{Z}_2 \oplus \mathbb{Z}_4^{28} \xrightarrow{f} \mathbb{Z}_2^{28} \oplus \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$$

Question 34. *What else can happen with the prime* 2*?*

The prime 2 by far exhibits the most exotic behavior. Consider the following example of *exponent permutation.*

Example 35 (Exponent permutation)**.** Consider the graph $G = C_{12}(1,3)$. We note that this graph is 4-regular and $\beta(G) = 13$. Here the only prime diving the degree is 2. Below we consider the exact sequence from Theorem 6 for the primes 2.

$$0 \to \mathbb{Z}_4^{12} \to \mathbb{Z}_2 \oplus \mathbb{Z}_8^{10} \oplus \mathbb{Z}_{16}^2 \xrightarrow{f} \mathbb{Z}_2^{10} \oplus \mathbb{Z}_8^2 \oplus \mathbb{Z}_{16} \to \mathbb{Z}_4 \to 0$$

As demonstrated by the following examples behavior can get even less predictable.

Example 36. Consider the graph $G = C_{12}(1,5)$. We note that this graph is 4-regular and $\beta(G) = 13$. Here the only prime diving the degree is 2. Below we consider the exact sequence from Theorem 6 for the primes 2.

$$0 \to \mathbb{Z}_4^{12} \to \mathbb{Z}_2 \oplus \mathbb{Z}_8^6 \oplus \mathbb{Z}_{16}^2 \oplus \mathbb{Z}_{64}^4 \xrightarrow{f} \mathbb{Z}_2^6 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_{16}^5 \to \mathbb{Z}_4 \to 0$$

Below is an example where both the prime 2 and the odd primes dividing the degree exhibit behavior which appears unpredictable.

Example 37. Consider the graph $G = C_{20}(1,3,5)$. We note that this graph is 6-regular and $\beta(G) = 41$. Here the only primes diving the degree are 2 and 3. Below we consider the exact sequences from Theorem 6 for the primes 2 and 3.

$$\begin{array}{ccccccccccc}
0 & \to & \mathbb{Z}_2^{40} & \to & \mathbb{Z}_4^{41} & \xrightarrow{f} & \mathbb{Z}_2^{40} \oplus \mathbb{Z}_8 & \to & \mathbb{Z}_4 & \to & 0 \\
0 & \to & \mathbb{Z}_3^{40} & \to & \mathbb{Z}_3^{38} \oplus \mathbb{Z}_9^2 & \xrightarrow{f} & \mathbb{Z}_3^3 & \to & \mathbb{Z}_3 & \to & 0
\end{array}$$

6.3. **Example exact sequences.** We will close by looking at some potential scenarios for the exact sequence given in Theorem 6. The first two examples come from graphs where we know both $K(G)$ and $K(\operatorname{line} G)$.

Example 38. Let $G = K_{n,n}$, then from (8) and (9) we have:

$$\begin{aligned}
K(G) &\cong \mathbb{Z}_n^{2n-4} \oplus \mathbb{Z}_{n^2} \\
K(\operatorname{sd} G) &\cong \mathbb{Z}_2^{(n-2)^2} \oplus \mathbb{Z}_{2n}^{2n-4} \oplus \mathbb{Z}_{2n^2} \\
K(\operatorname{line} G) &\cong \mathbb{Z}_{2n}^{(n-2)^2+1} \oplus \mathbb{Z}_{2n^2}^{2n-4}
\end{aligned}$$

Note that G is n-regular and $\beta(G) = n^2 - 2n + 1$. We speculate the exact sequence splits:

$$\begin{array}{ccccccccccc}
0 & \to & \mathbb{Z}_n^{n^2-2n} & \to & K(\operatorname{line} G) & \xrightarrow{f} & K(\operatorname{sd} G) & \to & \mathbb{Z}_n & \to & 0 \\
 & & & & & \cong & & & & & \\
0 & \to & \mathbb{Z}_n^{(n-2)^2} & \to & \mathbb{Z}_{2n}^{(n-2)^2} & \to & \mathbb{Z}_2^{(n-2)^2} & \to & 0 & \to & 0 \\
 & & & & & \oplus & & & & & \\
0 & \to & \mathbb{Z}_n^{2n-4} & \to & \mathbb{Z}_{2n^2}^{2n-4} & \to & \mathbb{Z}_{2n}^{2n-4} & \to & 0 & \to & 0 \\
 & & & & & \oplus & & & & & \\
0 & \to & 0 & \to & \mathbb{Z}_{2n} & \to & \mathbb{Z}_{2n^2} & \to & \mathbb{Z}_n & \to & 0
\end{array}$$

Example 39. Let $G = K_{n,n} - M$, then from Theorem 14 we have:

$$\begin{aligned}
K(G) &\cong \mathbb{Z}_{n-2} \oplus \mathbb{Z}_{n(n-2)}^{n-3} \oplus \mathbb{Z}_{n(n-1)(n-2)} \\
K(\operatorname{sd} G) &\cong \mathbb{Z}_2^{n^2-4n+2} \oplus \mathbb{Z}_{2(n-2)} \oplus \mathbb{Z}_{2n(n-2)}^{n-3} \oplus \mathbb{Z}_{2n(n-1)(n-2)} \\
K(\operatorname{line} G) &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_{2(n-1)}^{n^2-4n+1} \oplus \mathbb{Z}_{2(n-1)(n-2)} \oplus \mathbb{Z}_{2n(n-1)(n-2)}^{n-2}
\end{aligned}$$

Note that G is $(n-1)$-regular and $\beta(G) = n^2 - 3n + 1$. We speculate the exact sequence splits:

$$\begin{array}{ccccccccccc}
0 & \to & \mathbb{Z}_{n-1}^{n^3-3n} & \to & K(\operatorname{line} G) & \xrightarrow{f} & K(\operatorname{sd} G) & \to & \mathbb{Z}_{n-1} & \to & 0 \\
 & & & & & \cong & & & & & \\
0 & \to & \mathbb{Z}_{n-1}^{n^2-4n+1} & \to & \mathbb{Z}_{2(n-1)}^{n^2-4n+1} & \to & \mathbb{Z}_2^{n^2-4n+1} & \to & 0 & \to & 0 \\
 & & & & & \oplus & & & & & \\
0 & \to & 0 & \to & \mathbb{Z}_2 & \to & \mathbb{Z}_2 & \to & 0 & \to & 0 \\
 & & & & & \oplus & & & & & \\
0 & \to & \mathbb{Z}_{n-1} & \to & \mathbb{Z}_{2(n-1)(n-2)} & \to & \mathbb{Z}_{2(n-2)} & \to & 0 & \to & 0 \\
 & & & & & \oplus & & & & & \\
0 & \to & \mathbb{Z}_{n-1}^{n-3} & \to & \mathbb{Z}_{2n(n-1)(n-2)}^{n-3} & \to & \mathbb{Z}_{2n(n-2)}^{n-3} & \to & 0 & \to & 0 \\
 & & & & & \oplus & & & & & \\
0 & \to & \mathbb{Z}_{n-1} & \to & \mathbb{Z}_{2n(n-1)(n-2)} & \to & \mathbb{Z}_{2n(n-1)(n-2)} & \to & \mathbb{Z}_{n-1} & \to & 0
\end{array}$$

We now consider potential exact sequences assuming some conjectures hold true.

Example 40. Let $G = C_{2\cdot(2l+1)}(1, 2l-1)$, assuming truth of Conjecture 17 we have:

$$\begin{aligned}
K(G) &\cong \mathbb{Z}_{3^k} \oplus \mathbb{Z}_{3^k d_1} \oplus \mathbb{Z}_{3^{k+1} d_2} \\
K(\operatorname{sd} G) &\cong \mathbb{Z}_2^{2l-1} \mathbb{Z}_{2\cdot 3^k} \oplus \mathbb{Z}_{2\cdot 3^k d_1} \oplus \mathbb{Z}_{2\cdot 3^{k+1} d_2} \\
K(\operatorname{line} G) &\cong \mathbb{Z}_6^{2l-1} \mathbb{Z}_{2\cdot 3^k} \oplus \mathbb{Z}_{2\cdot 3^{k+1} d_1} \oplus \mathbb{Z}_{2\cdot 3^{k+1} d_2}
\end{aligned}$$

Note that G is 3-regular and $\beta(G) = 2l + 2$. We speculate the exact sequence splits in one of the two following ways:

(1)

$$\begin{array}{ccccccccccc}
0 & \to & \mathbb{Z}_3^{2l+1} & \to & K(\operatorname{line} G) & \xrightarrow{f} & K(\operatorname{sd} G) & \to & \mathbb{Z}_3 & \to & 0 \\
 & & & & & \cong & & & & & \\
0 & \to & \mathbb{Z}_3^{2l-1} & \to & \mathbb{Z}_6^{2l-1} & \to & \mathbb{Z}_2^{2l-1} & \to & 0 & \to & 0 \\
 & & & & & \oplus & & & & & \\
0 & \to & 0 & \to & \mathbb{Z}_{2\cdot 3^k} & \to & \mathbb{Z}_{2\cdot 3^k} & \to & 0 & \to & 0 \\
 & & & & & \oplus & & & & & \\
0 & \to & \mathbb{Z}_3 & \to & \mathbb{Z}_{2\cdot 3^{k+1} d_1} & \to & \mathbb{Z}_{2\cdot 3^k d_1} & \to & 0 & \to & 0 \\
 & & & & & \oplus & & & & & \\
0 & \to & \mathbb{Z}_3 & \to & \mathbb{Z}_{2\cdot 3^{k+1} d_2} & \to & \mathbb{Z}_{2\cdot 3^{k+1} d_2} & \to & \mathbb{Z}_3 & \to & 0
\end{array}$$

(2)

$$\begin{array}{ccccccccccc}
0 & \to & \mathbb{Z}_3^{2l+1} & \to & K(\operatorname{line} G) & \xrightarrow{f} & K(\operatorname{sd} G) & \to & \mathbb{Z}_3 & \to & 0 \\
 & & & & & \cong & & & & & \\
0 & \to & \mathbb{Z}_3^{2l-1} & \to & \mathbb{Z}_6^{2l-1} & \to & \mathbb{Z}_2^{2l-1} & \to & 0 & \to & 0 \\
 & & & & & \oplus & & & & & \\
0 & \to & \mathbb{Z}_3 & \to & \mathbb{Z}_{2\cdot 3^k} & \to & \mathbb{Z}_{2\cdot 3^k} & \to & \mathbb{Z}_3 & \to & 0 \\
 & & & & & \oplus & & & & & \\
0 & \to & \mathbb{Z}_3 & \to & \mathbb{Z}_{2\cdot 3^{k+1} d_1} & \to & \mathbb{Z}_{2\cdot 3^k d_1} & \to & 0 & \to & 0 \\
 & & & & & \oplus & & & & & \\
0 & \to & 0 & \to & \mathbb{Z}_{2\cdot 3^{k+1} d_2} & \to & \mathbb{Z}_{2\cdot 3^{k+1} d_2} & \to & 0 & \to & 0
\end{array}$$

Example 41. Let $G = C_{2\cdot 2l}(1, 2l-1)$ for l odd. Assuming truth of Conjectures 18 and 19 we have:

$$K(G) \cong \mathbb{Z}_2^2 \oplus \mathbb{Z}_8^{2l-2} \oplus \mathbb{Z}_{8l}$$
$$K(\operatorname{sd} G) \cong \mathbb{Z}_2^{2l} \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_{16}^{2l-2} \oplus \mathbb{Z}_{16l}$$
$$K(\operatorname{line} G) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_8^{2l} \oplus \mathbb{Z}_{16}^2 \oplus \mathbb{Z}_{64}^{2l-3} \oplus \mathbb{Z}_{64l}$$

Note that G is 4-regular and $\beta(G) = 4l + 1$. We speculate the exact sequence splits as follows:

$$\begin{array}{ccccccccccc}
0 & \to & \mathbb{Z}_4^{4l} & \to & K(\operatorname{line} G) & \xrightarrow{f} & K(\operatorname{sd} G) & \to & \mathbb{Z}_4 & \to & 0 \\
 & & & & & \cong & & & & & \\
0 & \to & \mathbb{Z}_4^{2l} & \to & \mathbb{Z}_8^{2l} & \to & \mathbb{Z}_2^{2l} & \to & 0 & \to & 0 \\
 & & & & & \oplus & & & & & \\
0 & \to & \mathbb{Z}_4^{2} & \to & \mathbb{Z}_{16}^{2} & \to & \mathbb{Z}_4^{2} & \to & 0 & \to & 0 \\
 & & & & & \oplus & & & & & \\
0 & \to & \mathbb{Z}_4^{2l-3} & \to & \mathbb{Z}_{64}^{2l-3} & \to & \mathbb{Z}_{16}^{2l-3} & \to & 0 & \to & 0 \\
 & & & & & \oplus & & & & & \\
0 & \to & 0 & \to & \mathbb{Z}_4 & \to & \mathbb{Z}_{16} & \to & \mathbb{Z}_4 & \to & 0 \\
 & & & & & \oplus & & & & & \\
0 & \to & \mathbb{Z}_4 & \to & \mathbb{Z}_{64l} & \to & \mathbb{Z}_{16l} & \to & 0 & \to & 0
\end{array}$$

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School of Mathematics, University of Minnesota, Minneapolis, MN 55455

E-mail address: `macha052@umn.edu`