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Upper Binomial Posets and Signed Permutation Statistics

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We derive generating functions counting signed permutations by two statistics, using a hyperoctahedral analogue of the binomial poset technique of Stanley [7].

1. Introduction

In two previous papers [5, 6] on signed permutation statistics, we derived generating functions for such statistics using hyperoctahedral analogues of methods of Garsia and Gessel. In this paper we derive more of these generating functions using a variation on the binomial poset technique of Stanley [7]. In our presentation, we have chosen to omit certain proofs that require only routine modifications of analogous proofs in [7].

2. Upper Binomial Posets

An *upper binomial poset* is a partially ordered set P satisfying the following conditions:
(1) P has a greatest element $\hat{1}$, contains arbitrarily long finite chains, and all of its intervals $[x, y]$ are finite and graded (i.e. all maximal chains $x = x_0 < x_1 < \cdots < x_n = y$ have the same length $n = l[x, y]$).
(2) For any interval $[x, y]$ in P, the number of maximal chains in $[x, y]$ depends only on $l[x, y]$ and on whether or not $y = \hat{1}$. If $y \neq \hat{1}$ and $l[x, y] = n$, we say that $[x, y]$ is an *n-interval,* and we denote by $B(P, n)$ (or just $B(n)$) the number of maximal chains in $[x, y]$. If $y = \hat{1}$, and $l[x, y] = n + 1$, we say that $[x, y]$ is an *$\hat{n}$-interval,* and we denote by $B(P, \hat{n})$ (or just $B(\hat{n})$) the number of maximal chains in $[x, y]$.

Example. If $B(\hat{n}) = B(n + 1)$, then P is called a *binomial poset.* See [7, 8] for applications of this concept, particularly to permutation enumeration.

Example. Let $\hat{V}_{n,q}$ be a $2n$-dimensional symplectic space of a finite field $\mathbf{F}_q$ of order q, i.e. $\hat{V}_{n,q}$ is a $2n$-dimensional vector space endowed with a non-degenerate skew-symmetric biliner form $\langle \cdot, \cdot \rangle$. By the linear algebra of skew-symmetric forms, all such spaces $\hat{V}_{n,q}$ are equivalent to one with a basis $\{e_i, f_i\}_{i=1}^n$ in which $\langle e_i, e_j \rangle = 0$ for all i, j, and

$$\langle e_i, f_j \rangle = -\langle f_i, e_i \rangle = \delta_{i,j}.$$

A subspace $W \subseteq \hat{V}_{n,q}$ is *isotropic* if $\langle w, w' \rangle = 0$ for all w, w' in W. Let $L(\hat{V}_{n,q})$ denote the lattice of isotropic subspaces of $\hat{V}_{n,q}$ with a greatest element 1 adjoined. Using the natural inclusion $i: \hat{V}_{n,q} \hookrightarrow \hat{V}_{n+1,q}$, we can embed $L(\hat{V}_{n,q})$ into $L(\hat{V}_{n+1,q})$ via the map $\phi_n: W \mapsto i(W) + \mathbf{F}_q e_{n+1}$, so that the image $\phi_n(L(\hat{V}_{n,q}))$ is the upper interval $[\mathbf{F}_q e_{n+1}, \hat{1}]$. This makes

$$\{\phi_n: L(\hat{V}_{n,q}) \to L(\hat{V}_{n+1,q})\}$$

into a directed system, and we denote by $L(\hat{V}_q)$ its direct limit. This lattice $L(\hat{V}_q)$ is an

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upper binomial poset with $B(n)$, $B(\hat{n})$ given by the following proposition:

PROPOSITION 2.1. *Let* $[n]_q = (q^n - 1)/(q - 1) = 1 + q + q^2 + \cdots + q^{n-1}$. *Then*

$$B(n) = [n]!_q = [n]_q[n-1]_q \cdots [2]_q[1]_q,$$
$$B(\hat{n}) = (2[n])!_q = [2n]_q[2(n-1)]_q \cdots [4]_q \cdots [2]_q.$$

PROOF. We prove both by induction on n, the $n = 0$ cases being trivial. To count $B(n)$, note that if $[W_0, W_n]$ is an n-interval in $L[\hat{V}_q]$, then W_n is isotropic, and hence all of its subspaces are isotropic. So a maximal chain $W_0 \subset \cdots \subset W_n$ in $L(\hat{V}_q)$ is simply a maximal chain of subspaces. We can choose W_1 by choosing a line in W_n/W_0 in $(q^n - 1)/(q - 1) = [n]_q$ ways, and then choose $W_1 \subset \cdots \subset W_n$ in $B(n-1)$ ways. Therefore $B(n) = [n]_q B(n-1)$, and we are done by induction.

Similarly, to count $B(\hat{n})$, note that if $[W_0, \hat{1}]$ is an $\hat{n}$-interval in $L(\hat{V}_q)$, then it is isomorphic to the interval $[\{\underline{0}\}, \hat{1}]$ in $L(\hat{V}_{n,q})$. We can choose a maximal chain $\{\underline{0}\} \subset W_1 \cdots \subset W_n \subset \hat{1}$ by first choosing the line W_1 in $(q^{2n} - 1)/(q - 1) = [\hat{n}]_q$ ways, and then choose a maximal chain

$$W_2/W_1 \subset \cdots W_n/W_1 \subset \hat{1}$$

of isotropic subspaces in $W_1^{\perp}/W_1$. Since $W_1^{\perp}/W_1$ is isomorphic to $\hat{V}_{n-1,q}$ as symplectic spaces, the latter choice can be made in $B(\widehat{n-1})$ ways. So $B(\hat{n}) = [\hat{n}]_q B(\widehat{n-1})]$, and again we are done by induction. □

By convention, we let $L(\hat{V}_1)$ be the lattice of *cofinite signed subsets* of a countable set, i.e. all vectors $(\varepsilon_1, \varepsilon_2, \ldots)$ where $\varepsilon_i = 0$, $+1$ or -1 and only finitely many $\varepsilon_i \neq +1$, ordered componentwise by $0 < +1, 0 < -1$. This is also an upper binomial poset, with $B(n) = n!$, $B(\hat{n}) = 2^n n!$.

EXAMPLE. Given any upper binomial poset P and $r \in \mathbf{P}$, we can form a new upper binomial poset by putting the componentwise partial order on the set

$$P_r = \{(x_1, \ldots, x_r): x_i \in P, l[x_1, \hat{1}] = \cdots = l[x_r, \hat{1}]\}.$$

One can check that

$$B(P_r, n) = B(p, n)^r \qquad \text{and} \qquad B(P_r, \hat{n}) = B(P, \hat{n})^r.$$

If $P = L(\hat{V}_q)$ from the previous example, we denote P_r by $L_r(\hat{V}_q)$.

EXAMPLE. Given any upper binomial poset P and $k \in \mathbf{P}$, the following subposet

$$P^{(k)} = \{x \in P: l[x, \hat{1}] - 1 \text{ is divisible by } k\}$$

is also upper binomial, and one can check that

$$B(P^{(k)}, n) = \frac{B(P, kn)}{B(P, k)^n} \qquad \text{and} \qquad B(P^{(k)}, \hat{n}) = \frac{B(P, \widehat{kn})}{B(P, k)^n}.$$

3. MÖBIUS FUNCTIONS

Recall that if P is a poset the intervals of which are finite, its *incidence algebra* $I(P)$ is the $\mathbf{C}$-vector space of all functions f: $\mathrm{Int}(P) \to \mathbf{C}$ (where $\mathrm{Int}(P)$ denotes the set of all non-empty intervals $[x, y]$ in P), endowed with multiplication $f * g$, defined by

$$(f * g)[x, y] = \sum_{z \in [x, y]} f[x, z]g[z, y].$$

If P is upper binomial, we let $R(P)$ be the vector subspace of $I(P)$ consisting of all functions which satisfy $f[x,y]=f[x',y']=f(n)$ if $[x,y]$, $[x',y']$ are both n-intervals, and $f[x,\hat{1}]=f[x',\hat{1}]=f(\hat{n})$ if $[x,\hat{1}]$, $[x',\hat{1}]$ are both $\hat{n}$-intervals. In the language of [4], $R(P)$ is the *reduced incidence algebra of P* corresponding to the order-compatible equivalence relation which sets all n-intervals equivalent, and all $\hat{n}$-intervals equivalent.

PROPOSITION 3.1. *If P is upper binomial, the map*

$$f \overset{\phi}{\mapsto} \sum_{n\geq 0}\left(\frac{f(n)x^n}{B(n)}+\frac{f(\hat{n})x^n y}{B(\hat{n})}\right)$$

is a **C**-*algebra isomorphism*

$$R(P) \Rightarrow \mathbf{C}\langle\langle x,y\rangle\rangle/(yx,y^2),$$

where $\mathbf{C}\langle\langle x,y\rangle\rangle$ *is the ring of formal power series in two non-commuting variables x, y, and* (yx,y^2) *is the two-sided ideal generated by yx and* y^2.

PROOF. Since $\mathbf{C}\langle\langle x,y\rangle\rangle/(yx,y^2)$ has **C**-basis $\{1,x,x^2,\ldots,y,xy,x^2y,\ldots\}$, the map ϕ is clearly an isomorphism of **C**-vector spaces. It only remains to show that $\phi(f*g)=\phi(f)\cdot\phi(g)$.

Let $[a_n,b_n]$ be a typical n-interval, and $[c_n,\hat{1}]$ a typical $\hat{n}$-interval in P. We have

$$\begin{aligned}\phi(f*g)&=\sum_{n\geq 0}\left(\frac{(f*g)(n)x^n}{B(n)}+\frac{(f*g)(\hat{n})x^n y}{B(\hat{n})}\right)\\&=\sum_{n\geq 0}\left(\sum_{z\in[a_n,b_n]}\frac{f[a_n,z]g[z,b_n]x^n}{B(n)}+\sum_{z\in[c_n,\hat{1}]}\frac{f[c_n,z]g[z,\hat{1}]x^n y}{B(\hat{n})}\right).\end{aligned}$$

There are $B(n)/B(i)B(n-i)$ elements $z\in[a_n,b_n]$ for which $l[a_n,z]=i\leq n$, and $B(\hat{n})/B(i)B(\widehat{n-i})$ elements $z\in[c_n,\hat{1}]$ for which $l[c_n,z]=i\leq n$. This gives us

$$\begin{aligned}\phi(f*g)&=\sum_{n\geq 0}\sum_{i=0}^{n}\frac{f(i)}{B(i)}\frac{g(n-i)}{B(n-i)}x^n+\sum_{n\geq 0}\sum_{i=0}^{n}\frac{f(i)}{B(i)}\frac{g(\widehat{n-i})}{B(\widehat{n-i})}x^n y+\sum_{n\geq 0}\frac{f(\hat{n})g(0)}{B(\hat{n})}x^n y\\&=\sum_{n\geq 0}\frac{f(n)}{B(n)}x^n\sum_{n\geq 0}\frac{g(n)}{B(n)}x^n+\sum_{n\geq 0}\frac{f(n)}{B(n)}x^n\sum_{n\geq 0}\frac{g(\hat{n})}{B(\hat{n})}x^n y+g(0)\sum_{n\geq 0}\frac{f(\hat{n})}{B(\hat{n})}x^n y.\end{aligned}$$

Meanwhile,

$$\phi(f)\cdot\phi(g)=\sum_{n\geq 0}\left(\frac{f(n)x^n}{B(n)}+\frac{f(\hat{n})x^n y}{B(\hat{n})}\right)\sum_{n\geq 0}\left(\frac{g(n)x^n}{B(n)}+\frac{g(\hat{n})x^n y}{B(\hat{n})}\right)$$

which, when multiplied out using the rules $yx=y^2=0$, gives the same result as the preceding line. □

We recall two key elements of $I(P)$: the *zeta function*, defined by $\zeta[x,y]=1$ for all intervals $[x,y]$, and its multiplicative inverse, the *Möbius function* μ. If P is upper binomial, then both ζ and μ are also elements of $R(P)$, and the preceding proposition may be used to calculate μ.

EXAMPLE. Let $P=L(\hat{V}_1)$. Then

$$\phi(\zeta)=\sum_{n\geq 0}\left(\frac{x^n}{n!}+\frac{x^n y}{2^n n!}\right)=e^x+e^{x/2}y.$$

Since $\mu = \zeta^{-1}$ in $I(P)$, we have

$$\phi(\mu) = \phi(\zeta)^{-1} = \mathrm{e}^{-x} - \mathrm{e}^{-x/2}y = \sum_{n \geq 0} \left(\frac{(-1)^n x^n}{n!} + \frac{(-1)^{n+1} x^n y}{2^n n!} \right)$$

and hence $\mu(n) = (-1)^n$, $\mu(n) = (-1)^{n+1}$. More generally, $p(x) + q(x)y$ is invertible in $\mathbf{C}\langle\langle x, y\rangle\rangle/(yx, y^2)$ iff $p(0) \neq 0$, and its inverse is given by $p(x)^{-1}(1 - p(0)^{-1}q(x)y)$.

There are two ways in which to rank-select intervals in an upper binomial poset P. Given $S \subseteq \mathbf{P}$ and an interval $[x, \hat{1}]$ in P, we define two *rank-selected* subposets

$$[x, \hat{1}]_S = \{x, \hat{1}\} \cup \{z \in [x, \hat{1}]: l[x, z] \in S\},$$
$$[x, \hat{1}]_{\hat{S}} = \{x, \hat{1}\} \cup \{z \in [x, \hat{1}]: l[z, \hat{1}] \in S\}.$$

We write $\mu_S(\hat{n}) = \mu_{[x,\hat{1}]_S}[x, \hat{1}]$ and $\mu_{\hat{S}}(\hat{n}) = \mu_{[x,\hat{1}]_{\hat{S}}}[x, \hat{1}]$, where $[x, \hat{1}]$ is any $\hat{n}$-interval.

The next three propositions allow us to calculate $\mu_S(\hat{n})$ and $\mu_{\hat{S}}(\hat{n})$, and are proven analogously to their corresponding results in [7]:

PROPOSITION 3.2 (cf. [7], Theorem 2.2]). *If P is upper binomial and $S \subseteq \mathbf{P}$, then*

$$-\sum_{n \geq 0} \frac{\mu_{\hat{S}}(\hat{n}) x^n}{B(\hat{n})} = \sum_{n \geq 0} \frac{x^n}{B(\hat{n})} + \sum_{n \geq 0} \frac{x^n}{B(n)} \sum_{n+1 \in S} \frac{\mu_{\hat{S}}(\hat{n}) x^n}{B(\hat{n})}.$$

PROPOSITION 3.3 (cf. [7, Corollary 2.4]). *If P is upper binomial and $S = k\mathbf{P}$ for some $k \in \mathbf{P}$, then*

$$-\sum_{n \geq 0} \frac{\mu_{\hat{S}}(\hat{n}) x^n}{B(\hat{n})} = \sum_{n \geq 0} \frac{x^n}{B(\hat{n})} + \sum_{n \geq 0} \frac{x^n}{B(n)} \sum_{n \geq 0} \frac{x^{kn-1}}{N(\widehat{kn-1})} \left(\sum_{i=0}^{n} \frac{x^{kn}}{B(kn)} \right)^{-1}.$$

LEMMA 3.4 (cf. [7, Lemma 2.5]). *Let P be upper binomial, and define three elements f, g, h of $R(P)$ as follows*:

$$\begin{aligned}
&f(0) = 0 && \\
&f(n) = (1+t)^{n-1}, && n \geq 1, \\
&f(\hat{n}) = 0, && n \geq 0, \\
&g(n) = 0, && n \geq 0, \\
&g(\hat{n}) = (1+t)^n, && n \geq 0, \\
&h(n) = 0, && n \geq 0, \\
&h(\hat{n}) = \sum_{S \subseteq \{1,\ldots,n\}} \mu_S(\hat{n}) t^{n-\#S}, && n \geq 0.
\end{aligned}$$

Then $h = -(1+f)^{-1}g$ in $R(P)$.

4. SIGNED PERMUTATIONS

Let B_n denote the group of signed permutations on n elements, i.e. all permutations and sign changes of the co-ordinates in $\mathbf{R}^n$. We may view B_n as a *Coxeter group* with *simple generators* $S = \{s_1, \ldots, s_n\}$ (see [2] for background). Here S_i is the transposition of co-ordinates i and $i+1$ for $1 \leq i \leq n-1$, and s_n is a sign change in the last co-ordinate. The *length* $l(\pi)$ for $\pi \in B_n$ is defined by

$$l(\pi) = \min\{t: \pi = s_{i_1} s_{i_2} \cdots s_{i_t} \text{ for some } s_{i_k} \in S\}$$

and the *descent set* of π is defined by

$$D(\pi)=\{i: l(\pi s_i)<l(\pi)\}.$$

We let $[n]$ denote the set $\{1,\ldots,n\}$. The key relation between upper binomial posets and B_n is given by the following theorem.

THEOREM 4.1 (cf. [7, Theorem 3.1]). *Let $P=L_r(\hat{V}_q)$ and $K\subseteq\{1,\ldots,n\}$. Then*

$$(-1)^{\#K+1}\mu_K(\hat{n})=\sum_{\substack{(\pi_1,\ldots,\pi_r)\in B_n^r\\ \bigcup_i D(\pi_i)=K}} q^{l(\pi_1)+\cdots+l(\pi_r)}.$$

PROOF. Let

$$g_r(K)=(-1)^{\#K+1}\mu_K(\hat{n})$$

and

$$f_r(L)=\sum_{K\subseteq L} g_r(K)$$

for $K, L\subseteq[n]$. We need to show that $g_r(K)$ is equal to the right-hand side of the theorem. When $r=1$, this is exactly the assertion of [1, equation 4.20] (we need the special case in which G is a finite Chevalley group over $\mathbf{F}_q$ of type C_n, and the corresponding building is the flag complex of $L(\hat{V}_{n,q})$; see [2, Section V.6] and [1] for more details).

For $r>1$, by inclusion–exclusion, it would suffice to show that

$$f_r(L)=\sum_{\substack{(\pi_1,\ldots,\pi_r)\in B_n^r\\ \bigcup_i D(\pi_i)\subseteq L}} q^{l(\pi_1)+\cdots+l(\pi_r)},$$

which we now set out to prove. Let $[x,\hat{1}]$ be any $\hat{n}$-interval in $L_r(\hat{V}_q)$. We have

$$f_r(L)=\sum_{K\subseteq L}(-1)^{\#K+1}\mu_K(\hat{n})$$

$$=\sum_{K\subseteq L}(-1)^{\#K+1}\sum_{\text{chains } c\subseteq[x,\hat{1}]_K}(-1)^{\text{length}(c)}$$

by P. Hall's Theorem [8, Proposition 3.8.5]

$$=\sum_{\text{chains } c\subseteq[x,\hat{1}]_L}(-1)^{\text{length}(c)}\sum_{r(c)\subseteq K\subseteq L}(-1)^{\#K+1},$$

where $r(c)=\{l[x,c_i]: c_i\in c\}$

$$=\sum_{\text{chains } c\subseteq[x,\hat{1}]_L}(-1)^{\text{length}(c)}\delta_{r(c),L}$$

$$=\#\{\text{maximal chains in } [x,\hat{1}]_L\}$$

$$=\#\{\text{maximal chains in } [x',\hat{1}]_L\}^r,$$

where $[x',\hat{1}]$ is any $\hat{n}$-interval in $L_1(\hat{V}_q)$

$$=f_1(L)^r$$

by reversing the argument so far

$$=\Bigg(\sum_{\substack{\pi\in B_n\\ D(\pi)\subseteq L}} q^{l(\pi)}\Bigg)^r$$

by the $r=1$ case of the theorem

$$=\sum_{\substack{(\pi_1,\ldots,\pi_r)\in B_n^r\\ \bigcup_i D(\pi_i)\subseteq L}} q^{l(\pi_1)+\cdots+l(\pi_r)},$$

as we wanted. □

Having this interpretation for $\mu_K(\hat{n})$, we can deduce our first result on signed permutation statistics:

THEOREM 4.2 (cf. [7, Corollary 3.3]). *For $k \in \mathbf{P}$, let*

$$f_{krq} = \sum_{\substack{(\pi_1,\ldots,\pi_r)\in B_n^r \\ \bigcup_i D(\pi_i)=(n+1-k\mathbf{P})\cap[n]}} q^{l(\pi_1)+\cdots+l(\pi_r)},$$

where $(n+1-k\mathbf{P})\cap[n]=\{n+1-i\colon i\in k\mathbf{P},\ n+1-i\in[n]\}$. Then

$$\sum_{n\geq 0}\frac{(-1)^{\lfloor n/k\rfloor}f_{krg}x^n}{(2[n])!_q^r}=\sum_{n\geq 0}\frac{x^n}{(2[n])!_q^r}-\sum_{n\geq 1}\frac{x^n}{[n]!_q^r}\sum_{n\geq 1}\frac{x^{kn-1}}{(2[kn-1])!_q^r}\left(\sum_{n\geq 0}\frac{x^{kn}}{[kn]!_q^r}\right)^{-1}.$$

PROOF. If we let $S=k\mathbf{P}$, then

$$\mu_{\hat{S}}(\hat{n})=\mu_{(n+1-k\mathbf{P})\cap[n]}(\hat{n})$$

and $\lfloor n/k\rfloor=\#(n+1-k\mathbf{P}\cap[n])$, so $(-1)^{\lfloor n/k\rfloor}f_{krq}=\mu_{\hat{S}}(\hat{n})$ by the previous theorem. Now apply Proposition 3.3. □

To eliminate the $(-1)^{\lfloor n/k\rfloor}$ factor, we use a lemma proven very similarly to [7, Lemma 3.4]:

LEMMA 4.3. *If $F(x)=\sum_{n\geq 0}(-1)^{\lfloor n/k\rfloor}f(n)x^n$, then*

$$\sum_{n\geq 0}f(n)x^n=\frac{2}{k}\sum_{j=0}^{k-1}\frac{F(\zeta^{2j+1}x)}{1-\zeta^{-(2j+1)}},$$

where $\zeta=\mathrm{e}^{\pi i/k}$.

COROLLARY 4.4. *Let*

$$\mathcal{A}_n=\{\pi\in B_n\colon D(\pi)=\{n-1,n-3,\ldots\}\}$$

be the set of alternating signed permutations. *Then*

$$\sum_{n\geq 0}\frac{(-1)^{\lfloor n/k\rfloor}\sum_{\pi\in\mathcal{A}_n}q^{l(\pi)}x^n}{(2[n])!_q}=\sum_{n\geq 0}\frac{x^n}{(2[n])!_q}-\sum_{n\geq 1}\frac{x^n}{[n]!_q}\sum_{n\geq 1}\frac{x^{2n-1}}{(2[2n-1])!_q}\left(\sum_{n\geq 0}\frac{x^{2n}}{[2n]!_q}\right)^{-1}$$

and

$$\sum_{n\geq 0}\frac{\#\mathcal{A}_nx^n}{2^nn!}=\frac{\cos(x/2)+\sin(x/2)}{\cos(x)}.$$

PROOF. The first equation is Theorem 4.2 with $k=2$ and $r=1$. If we set $q=1$ in the first equation, we obtain

$$\sum_{n\geq 0}\frac{(-1)^{\lfloor n/k\rfloor}\#\mathcal{A}_nx^n}{2^nn!}=\sum_{n\geq 0}\frac{x^n}{2^nn!}-\sum_{n\geq 1}\frac{x^n}{n!}\sum_{n\geq 1}\frac{x^{2n-1}}{2^{2n-1}(2n-1)!}\left(\sum_{n\geq 0}\frac{x^{2n}}{(2n)!}\right)^{-1}$$

$$=\mathrm{e}^{x/2}-(\mathrm{e}^x-1)\frac{\sinh(x/2)}{\cosh(x)}=\frac{e^{x/2}}{\cosh(x)}.$$

Applying the preceding lemma with $k=2$ gives

$$\sum_{n\geq 0}\frac{\#\mathcal{A}_nx^n}{2^nn!}=\frac{\mathrm{e}^{\mathrm{i}x/2}}{(1+\mathrm{i})\cosh(\mathrm{i}x)}+\frac{\mathrm{e}^{-\mathrm{i}x/2}}{(1-\mathrm{i})\cosh(-\mathrm{i}x)}$$

$$=\frac{\cos(x/2)+\sin(x/2)}{\cos(x)}.$$ □

REMARK. A generalization of this last result to the *wreath product* $C_k \wr S_n$ of a cyclic group of order k with the symmetric group S_n is given in [9].

If we set $k=1$ in Theorem 4.2, and replace x by $-x$, we obtain the following:

COROLLARY 4.5.

$$\sum_{n\geq 0} \sum_{\substack{(\pi_1,\dots,\pi_r)\in B_n^r \\ \bigcup_i D(\pi_i)=[n]}} \frac{q^{l(\pi_1)+\cdots+l(\pi_r)}x^n}{(2[n])!_q^r} = \sum_{n\geq 0} \frac{(-x)^n}{(2[n])!_q^r} \left(\sum_{n\geq 0} \frac{(-x)^n}{[n]!_q^r}\right)^{-1}.$$

When $r=2$, this gives the hyperoctahedral q-analogue of a result from [3].

Next we consider generating functions that count descents. For $\pi \in B_n$, define its number of descents to be $d(\pi) = \#D(\pi)$.

THEOREM 4.6. *Let*

$$G_{nkr}(t,q) = \sum_{\substack{(\pi_1,\dots,\pi_r)\in B_n^r \\ \bigcup_i D(\pi_i)\subseteq k[n]}} q^{l(\pi_1)+\cdots+l(\pi_r)} t^{n-\#\bigcup_i D(\pi_i)}$$

and let

$$B_{kr}(t,q,x) = \sum_{n\geq 0} \frac{G_{nkr}(t,q)x^n}{(2[kn])!_q^r}.$$

Then

$$B_{kr}(t,q,x) = \left(1 - \sum_{n\geq 1} \frac{(t-1)^{n-1}x^n}{[kn]!_q^r}\right)^{-1} \sum_{n\geq 0} \frac{(t-1)^n x^n}{2[kn])!_q^r}.$$

PROOF. Let $P = L_r(\hat{V}_q)$, so that $P^{(k)} = L_r(\hat{V}_q)^{(k)}$. We have

$$\begin{aligned}
B_{kr}(t,q,B(P,k)^r x)y &= \sum_{n\geq 0} \frac{G_{nkr}(t,q)B(P,k)^{nr}x^n}{B(P,\widehat{kn})} \\
&= \sum_{n\geq 0} \sum_{S\subseteq k[n]} \sum_{\substack{(\pi_1,\dots,\pi_r)\in B_n^r \\ \bigcup_i D(\pi_i)=S}} \frac{q^{l(\pi_i)+\cdots+l(\pi_r)}t^{n-\#\bigcup_i D(\pi_i)}x^n}{B(P^{(k)},\hat{n})} \\
&= \sum_{n\geq 0} \sum_{S\subseteq k[n]} \frac{(-1)^{\#S+1}\mu_S(\hat{n})t^{n-\#S}x^n y}{B(P^{(k)},\hat{n})}
\end{aligned}$$

by Theorem 4.1, where $\mu_S(\hat{n})$ here refers to an $\hat{n}$-interval in P, not $P^{(k)}$,

$$= \sum_{n\geq 0} (-1)^{n+1} \sum_{S\subseteq[n]} \frac{\mu_S(\hat{n})(-t)^{n-\#S}x^n y}{B(P^{(k)},\hat{n})}$$

where $\mu_S(\hat{n})$ now refers to an $\hat{n}$-interval in $P^{(k)}$, not P!

$$= \sum_{n\geq 0} \frac{h(\hat{n})|_{-t}(-x)^n(-y)}{B(P^{(k)},\hat{n})},$$

where h is the element of $R(P^{(k)})$ defined in Lemma 3.4,

$$= \phi(h)|_{-t,-x,-y}.$$

Therefore

$$B_{kr}(t,q,x)y = T[\phi(h)],$$

where T is the operator which substitutes $-t$ for t, $-x/B(P,k)^r$ for x, and $-y$ for y. Since $h = -(1+f)^{-1}g$ in $R(P^{(k)})$ by Lemma 3.4, one concludes from Proposition 3.1 that

$$\begin{aligned}
B_{kr}(t,q,x)y &= T[-(1+\phi(f))^{-1}\phi(g)] \\
&= T\left[-\left(1 + \sum_{n\geq 1} \frac{(1+t)^{n-1}x^n}{B(P^{(k)},n)}\right)^{-1} \sum_{n\geq 0} \frac{(1+t)^n x^n y}{B(P^{(k)},\hat{n})}\right] \\
&= \left(1 - \sum_{n\geq 1} \frac{(t-1)^{n-1}x^n}{[kn]!_q^r}\right)^{-1} \sum_{n\geq 0} \frac{(t-1)^n x^n}{(2[kn]!_q^r} y,
\end{aligned}$$

which implies the result. □

By setting $k = r = 1$ in this theorem, we recover a special case of a result from [5]; namely, a generating function for a hyperoctahedral q-analogue of the *Eulerian polynomials* (see [8], Section 1.3]):

COROLLARY 4.7.

$$\sum_{n\geq 0} \frac{\sum_{\pi\in B_n} q^{l(\pi)} t^{n-d(\pi)} x^n}{(2[n])!_q} = B_{11}(t, q, x)$$

$$= \left(1 - \sum_{n\geq 1} \frac{(t-1)^{n-1} x^n}{[n]!_q}\right)^{-1} \sum_{n\geq 0} \frac{(t-1)^n x^n}{(2[n])!_q}$$

and hence, setting $q = 1$, we have

$$\sum_{n\geq 0} \frac{\sum_{\pi\in B_n} t^{n-d(\pi)} x^n}{2^n n!} = \frac{(t-1)e^{(t-1)x/2}}{t - e^{(t-1)x}}.$$

Note that, in general, when $r = q = 1$, the expressions in Theorems 4.2 and 4.6 can be written in terms of the exponential functions e^x and $e^{x/2}$, so one expects such functions to occur naturally in signed permutation enumeration problems, similar to the occurrences of e^x in permutation enumeration.

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