Upper Binomial Posets and Signed Permutation Statistics

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We derive generating functions counting signed permutations by two statistics, using a hyperoctahedral analogue of the binomial poset technique of Stanley [7].

1. Introduction

In two previous papers [5, 6] on signed permutation statistics, we derived generating functions for such statistics using hyperoctahedral analogues of methods of Garsia and Gessel. In this paper we derive more of these generating functions using a variation on the binomial poset technique of Stanley [7]. In our presentation, we have chosen to omit certain proofs that require only routine modifications of analogous proofs in [7].

2. Upper Binomial Posets

An upper binomial poset is a partially ordered set $P$ satisfying the following conditions:

1. $P$ has a greatest element $\hat{1}$, contains arbitrarily long finite chains, and all of its intervals $[x, y]$ are finite and graded (i.e. all maximal chains $x = x_0 < x_1 < \cdots < x_n = y$ have the same length $n = l(x, y)$).

2. For any interval $[x, y]$ in $P$, the number of maximal chains in $[x, y]$ depends only on $l(x, y)$ and on whether or not $y = \hat{1}$. If $y \neq \hat{1}$ and $l(x, y) = n$, we say that $[x, y]$ is an $n$-interval, and we denote by $B(P, n)$ (or just $B(n)$) the number of maximal chains in $[x, y]$. If $y = \hat{1}$, and $l(x, y) = n + 1$, we say that $[x, y]$ is an $\bar{n}$-interval, and we denote by $B(P, \bar{n})$ (or just $B(\bar{n})$) the number of maximal chains in $[x, y]$.

Example. If $B(\bar{n}) = B(n + 1)$, then $P$ is called a binomial poset. See [7, 8] for applications of this concept, particularly to permutation enumeration.

Example. Let $\hat{V}_{n,q}$ be a $2n$-dimensional symplectic space of a finite field $\mathbb{F}_q$ of order $q$, i.e. $\hat{V}_{n,q}$ is a $2n$-dimensional vector space endowed with a non-degenerate skew-symmetric biliner form $\langle \cdot, \cdot \rangle$. By the linear algebra of skew-symmetric forms, all such spaces $\hat{V}_{n,q}$ are equivalent to one with a basis $\{e_i, f_i\}_{i=1}^n$ in which $\langle e_i, e_j \rangle = \delta_{i,j}$ and $\langle f_i, f_j \rangle = -\langle f_i, e_j \rangle = \delta_{i,j}$.

A subspace $W \subseteq \hat{V}_{n,q}$ is isotropic if $\langle w, w' \rangle = 0$ for all $w, w' \in W$. Let $L(\hat{V}_{n,q})$ denote the lattice of isotropic subspaces of $\hat{V}_{n,q}$ with a greatest element 1 adjoined. Using the natural inclusion $i: \hat{V}_{n,q} \hookrightarrow \hat{V}_{n+1,q}$, we can embed $L(\hat{V}_{n,q})$ into $L(\hat{V}_{n+1,q})$ via the map $\phi_n: W \mapsto i(W) + \mathbb{F}_qe_{n+1}$, so that the image $\phi_n(L(\hat{V}_{n,q}))$ is the upper interval $[\mathbb{F}_qe_{n+1}, \hat{1}]$. This makes $\{\phi_n: L(\hat{V}_{n,q}) \to L(\hat{V}_{n+1,q})\}$ into a directed system, and we denote by $L(\hat{V}_q)$ its direct limit. This lattice $L(\hat{V}_q)$ is an

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upper binomial poset with \( B(n) \), \( B(\hat{n}) \) given by the following proposition:

**Proposition 2.1.** Let \([n]_q = (q^n - 1)/(q - 1) = 1 + q + q^2 + \cdots + q^{n-1}\). Then

\[
B(n) = [n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q,
\]

\[
B(\hat{n}) = (2[n])_q! = [2n]_q[2n-1]_q \cdots [4]_q \cdots [2]_q.
\]

**Proof.** We prove both by induction on \( n \), the \( n = 0 \) cases being trivial. To count \( B(n) \), note that if \([W_0, W_n] \) is an \( n \)-interval in \( L(\hat{V}_q) \), then \( W_n \) is isotropic, and hence all of its subspaces are isotropic. So a maximal chain \( W_0, \ldots, W_n \) in \( L(\hat{V}_q) \) is simply a maximal chain of subspaces. We can choose \( W_1 \) by choosing a line in \( W_n/W_0 \) in \((q^n - 1)/(q - 1) = [n]_q \) ways, and then choose \( W_1, \ldots, W_n \) in \( B(n-1) \) ways. Therefore \( B(n) = [n]_q B(n-1) \), and we are done by induction.

Similarly, to count \( B(\hat{n}) \), note that if \([W_0, \hat{1}] \) is an \( \hat{n} \)-interval in \( L(\hat{V}_q) \), then it is isomorphic to the interval \([0], \hat{1}\) in \( L(\hat{V}_{n,q}) \). We can choose a maximal chain \( W_0, \ldots, W_n \subset \hat{1} \) by first choosing the line \( W_1 \) in \((q^{2n} - 1)/(q - 1) = [\hat{n}]_q \) ways, and then choose a maximal chain

\[
W_2/W_1 \subset \cdots 
\]

of isotropic subspaces in \( W_2/W_1 \). Since \( W_2/W_1 \) is isomorphic to \( \hat{V}_{n-1,q} \) as symplectic spaces, the latter choice can be made in \( B(n-1) \) ways. So \( B(\hat{n}) = [\hat{n}]_q B(n-1) \), and again we done by induction. \( \square \)

By convention, we let \( L(\hat{V}_q) \) be the lattice of *cofinite signed subsets* of a countable set, i.e. all vectors \( (e_1, e_2, \ldots) \) where \( e_i = 0, +1 \) or \(-1\) and only finitely many \( e_i \neq \pm 1 \), ordered componentwise by \( 0 < +1, 0 < -1 \). This is also an upper binomial poset, with \( B(n) = n! \), \( B(\hat{n}) = 2^n n! \).

**Example.** Given any upper binomial poset \( P \) and \( r \in P \), we can form a new upper binomial poset by putting the componentwise partial order on the set

\[
P_r = \{(x_1, \ldots, x_r): x_i \in P, l[x_1, \hat{1}] = \cdots = l[x_r, \hat{1}] \}.
\]

One can check that

\[
B(P_r, n) = B(P, n)^r \quad \text{and} \quad B(P_r, \hat{n}) = B(P, \hat{n})^r.
\]

If \( P = L(\hat{V}_q) \) from the previous example, we denote \( P_r \) by \( L_r(\hat{V}_q) \).

**Example.** Given any upper binomial poset \( P \) and \( k \in P \), the following subposet

\[
P^{(k)} = \{ x \in P: l[x, \hat{1}] - 1 \text{ is divisible by } k \}
\]

is also upper binomial, and one can check that

\[
B(P^{(k)}, n) = \frac{B(P, kn)}{B(P, k)^n} \quad \text{and} \quad B(P^{(k)}, \hat{n}) = \frac{B(P, \hat{kn})}{B(P, k)^\hat{n}}.
\]

3. **Möbius Functions**

Recall that if \( P \) is a poset the intervals of which are finite, its *incidence algebra* \( I(P) \) is the \( \mathbb{C} \)-vector space of all functions \( f: \text{Int}(P) \to \mathbb{C} \) (where \( \text{Int}(P) \) denotes the set of all non-empty intervals \([x, y]\) in \( P \)), endowed with multiplication \( f \ast g \), defined by

\[
(f \ast g)(x, y) = \sum_{z \in [x, y]} f(x, z)g(z, y).
\]
If $P$ is upper binomial, we let $R(P)$ be the vector subspace of $I(P)$ consisting of all functions which satisfy $f[x, y] = f(x', y') = f(n)$ if $[x, y], [x', y']$ are both $n$-intervals, and $f[x, i] = f(x', i) = f(\hat{n})$ if $[x, i], [x', i]$ are both $\hat{n}$-intervals. In the language of [4], $R(P)$ is the reduced incidence algebra of $P$ corresponding to the order-compatible equivalence relation which sets all $n$-intervals equivalent, and all $\hat{n}$-intervals equivalent.

**Proposition 3.1.** If $P$ is upper binomial, the map

$$f \mapsto \sum_{n \geq 0} \left( \frac{f(n)x^n}{B(n)} + \frac{f(\hat{n})x^n}{B(\hat{n})} \right)$$

is a $\mathbb{C}$-algebra isomorphism

$$R(P) \cong \mathbb{C}\langle \langle x, y \rangle \rangle/(yx, y^2),$$

where $\mathbb{C}\langle \langle x, y \rangle \rangle$ is the ring of formal power series in two non-commuting variables $x, y,$ and $(yx, y^2)$ is the two-sided ideal generated by $yx$ and $y^2$.

**Proof.** Since $\mathbb{C}\langle \langle x, y \rangle \rangle/(yx, y^2)$ has $\mathbb{C}$-basis $\{1, x, x^2, \ldots, y, xy, x^2y, \ldots\}$, the map $\phi$ is clearly an isomorphism of $\mathbb{C}$-vector spaces. It only remains to show that $\phi(f \ast g) = \phi(f) \cdot \phi(g)$.

Let $[a_n, b_n]$ be a typical $n$-interval, and $[c_n, \hat{i}]$ a typical $\hat{n}$-interval in $P$. We have

$$\phi(f \ast g) = \sum_{n \geq 0} \left( \frac{f \ast g(n)x^n}{B(n)} + \frac{f \ast g(\hat{n})x^n}{B(\hat{n})} \right)$$

$$= \sum_{n \geq 0} \left( \sum_{z \in [a_n, b_n]} f[z]x^n + \sum_{z \in [c_n, \hat{i}]} f[z]x^n \right).$$

There are $B(n)/B(i)B(n-i)$ elements $z \in [a_n, b_n]$ for which $l[a_n, z] = i \leq n$, and $B(\hat{n})/B(i)B(n-i)$ elements $z \in [c_n, \hat{i}]$ for which $l[c_n, z] = i \leq n$. This gives us

$$\phi(f \ast g) = \sum_{n \geq 0} \frac{f(i)}{B(i)} \frac{g(n-i)}{B(n-i)} x^n + \sum_{n \geq 0} \frac{f(n)}{B(n)} \frac{g(\hat{n})}{B(\hat{n})} x^n + \sum_{n \geq 0} \frac{f(\hat{n})}{B(\hat{n})} x^n y$$

$$= \sum_{n \geq 0} \frac{f(n)}{B(n)} x^n + \sum_{n \geq 0} \frac{f(\hat{n})}{B(\hat{n})} x^n y.$$}

Meanwhile,

$$\phi(f) \cdot \phi(g) = \sum_{n \geq 0} \left( \frac{f(n)x^n}{B(n)} + \frac{f(\hat{n})x^n}{B(\hat{n})} \right) \sum_{n \geq 0} \left( \frac{g(n)x^n}{B(n)} + \frac{g(\hat{n})x^n}{B(\hat{n})} \right)$$

which, when multiplied out using the rules $yx = y^2 = 0$, gives the same result as the preceding line.

We recall two key elements of $I(P)$: the zeta function, defined by $\zeta[x, y] = 1$ for all intervals $[x, y]$, and its multiplicative inverse, the Möbius function $\mu$. If $P$ is upper binomial, then both $\zeta$ and $\mu$ are also elements of $R(P)$, and the preceding proposition may be used to calculate $\mu$.

**Example.** Let $P = L(\hat{V}_i)$. Then

$$\phi(\zeta) = \sum_{n \geq 0} \left( \frac{x^n}{n!} + \frac{y^n}{2^n n!} \right) = e^x + e^{y/2}.$$
Since \( \mu = \xi^{-1} \) in \( I(P) \), we have
\[
\phi(\mu) = \phi(\xi)^{-1} = e^{-x} - e^{-x/2} = \sum_{n \geq 0} \left( \frac{(-1)^n x^n}{n!} + \frac{(-1)^{n+1} x^n y}{2^n n!} \right)
\]
and hence \( \mu(n) = (-1)^n, \mu(n) = (-1)^{n+1} \). More generally, \( p(x) + q(x)y \) is invertible in \( \mathbb{C}(\langle x, y \rangle)/(yx, y^2) \) iff \( p(0) \neq 0 \), and its inverse is given by \( p(x)^{-1}(1 - p(0)^{-1}q(x)y) \).

There are two ways in which to rank-select intervals in an upper binomial poset \( P \). Given \( S \subseteq \mathcal{P} \) and an interval \([x, \hat{1}]\) in \( P \), we define two rank-selected subposets
\[
[x, \hat{1}]_S = \{ x, \hat{1} \} \cup \{ z \in [x, \hat{1}]: l[x, z] \in S \},
[x, \hat{1}]_S = \{ x, \hat{1} \} \cup \{ z \in [x, \hat{1}]: l[z, \hat{1}] \in S \}.
\]
We write \( \mu_S(n) = \mu_{[x, \hat{1}]_S}[x, \hat{1}] \) and \( \mu_S(n) = \mu_{[x, \hat{1}]_S}[x, \hat{1}] \), where \([x, \hat{1}]\) is any \( \hat{n} \)-interval.

The next three propositions allow us to calculate \( \mu_S(\hat{n}) \) and \( \mu_S(n) \), and are proven analogously to their corresponding results in [7]:

**Proposition 3.2 (cf. [7], Theorem 2.2).** If \( P \) is upper binomial and \( S \subseteq \mathcal{P} \), then
\[
- \sum_{n \geq 0} \frac{\mu_S(\hat{n}) x^n}{B(\hat{n})} = \sum_{n \geq 0} \frac{x^n}{B(n)} + \sum_{n \geq 0} \frac{x^n}{B(n)} \sum_{n+1 \in S} \frac{\mu_S(\hat{n}) x^n}{B(\hat{n})}.
\]

**Proposition 3.3 (cf. [7], Corollary 2.4).** If \( P \) is upper binomial and \( S = k \mathcal{P} \) for some \( k \in \mathcal{P} \), then
\[
- \sum_{n \geq 0} \frac{\mu_S(\hat{n}) x^n}{B(\hat{n})} = \sum_{n \geq 0} \frac{x^n}{B(n)} + \sum_{n \geq 0} \frac{x^n}{B(n)} \sum_{n+1 \in S} \frac{\mu_S(\hat{n}) x^n}{B(\hat{n})}.
\]

**Lemma 3.4 (cf. [7], Lemma 2.5).** Let \( P \) be upper binomial, and define three elements \( f, g, h \) of \( R(P) \) as follows:
\[
f(0) = 0, \\
f(n) = (1 + t)^{n-1}, \quad n \geq 1, \\
f(\hat{n}) = 0, \quad n \geq 0,
\]
\[
g(n) = 0, \quad n \geq 0, \\
g(\hat{n}) = (1 + t)^n, \quad n \geq 0,
\]
\[
h(n) = 0, \quad n \geq 0, \\
h(\hat{n}) = \sum_{s \in \{1, \ldots, n\}} \mu_S(\hat{n}) \epsilon^{n-s}, \quad n \geq 0.
\]

Then \( h = -(1 + f)^{-1}g \) in \( R(P) \).

4. **Signed Permutations**

Let \( B_n \) denote the group of signed permutations on \( n \) elements, i.e. all permutations and sign changes of the co-ordinates in \( \mathbb{R}^n \). We may view \( B_n \) as a Coxeter group with simple generators \( S = \{ s_1, \ldots, s_n \} \) (see [2] for background). Here \( S_i \) is the transposition of co-ordinates \( i \) and \( i + 1 \) for \( 1 \leq i \leq n - 1 \), and \( s_n \) is a sign change in the last co-ordinate. The length \( l(\pi) \) for \( \pi \in B_n \) is defined by
\[
l(\pi) = \min \{ t : \pi = s_{i_1}s_{i_2} \cdots s_{i_t} \text{ for some } s_{i_k} \in S \}.
\]
and the descent set of $\pi$ is defined by

$$D(\pi) = \{i : l(\pi s_i) < l(\pi)\}.$$ 

We let $[n]$ denote the set $\{1, \ldots, n\}$. The key relation between upper binomial posets and $B_n$ is given by the following theorem.

**Theorem 4.1 (cf. [7, Theorem 3.1]).** Let $P = L_r(\hat{V}_q)$ and $K \subseteq \{1, \ldots, n\}$. Then

$$(-1)^{\#K} \mu_K(\hat{n}) = \sum_{(\pi_1, \ldots, \pi_r) \in B_n^r \cap \bigcup_{\pi \in \Delta(D(\pi)) \subseteq K} q^{l(\pi_1) + \cdots + l(\pi_r)}.$$

**Proof.** Let

$$g_r(K) = (-1)^{\#K} \mu_K(\hat{n})$$

and

$$f_r(L) = \sum_{K \subseteq L} g_r(K)$$

for $K, L \subseteq [n]$. We need to show that $g_r(K)$ is equal to the right-hand side of the theorem. When $r = 1$, this is exactly the assertion of [1, equation 4.20] (we need the special case in which $G$ is a finite Chevalley group over $F_q$ of type $C_n$, and the corresponding building is the flag complex of $L(\hat{V}_{n,q})$; see [2, Section V.6] and [1] for more details).

For $r > 1$, by inclusion–exclusion, it would suffice to show that

$$f_r(L) = \sum_{(\pi_1, \ldots, \pi_r) \in B_n^r \cap \bigcup_{\pi \in \Delta(D(\pi)) \subseteq L} q^{l(\pi_1) + \cdots + l(\pi_r)},$$

which we now set out to prove. Let $[x, \hat{1}]$ be any $\hat{n}$-interval in $L_r(\hat{V}_q)$. We have

$$f_r(L) = \sum_{K \subseteq L} (-1)^{\#K} \mu_K(\hat{n})$$

$$= \sum_{K \subseteq L} (-1)^{\#K} \sum_{\text{chains } c \subseteq [x, \hat{1}]} (-1)^{\text{length}(c)}$$

by P. Hall's Theorem [8, Proposition 3.8.5]

$$= \sum_{\text{chains } c \subseteq [x, \hat{1}]} (-1)^{\text{length}(c)} \sum_{r(c) \subseteq K \subseteq L} (-1)^{\#K}.$$ 

where $r(c) = \{l[x, c_i] : c_i \in c\}$

$$= \sum_{\text{chains } c \subseteq [x, \hat{1}]} (-1)^{\text{length}(c)} \delta_{r(c), L}$$

$$= \#\{\text{maximal chains in } [x, \hat{1}]\}$$

$$= \#\{\text{maximal chains in } [x', \hat{1}]\},$$

where $[x', \hat{1}]$ is any $\hat{n}$-interval in $L_{r-1}(\hat{V}_q)$

$$= f_r(L)^r$$

by reversing the argument so far

$$= \left( \sum_{\pi \in B_n \cap \Delta(D(\pi)) \subseteq L} q^{l(\pi)} \right)^r$$

by the $r = 1$ case of the theorem

$$= \sum_{(\pi_1, \ldots, \pi_r) \in B_n^r \cap \bigcup_{\pi \in \Delta(D(\pi)) \subseteq L} q^{l(\pi_1) + \cdots + l(\pi_r)},$$

as we wanted.

$\square$
Having this interpretation for $\mu_k(\bar{n})$, we can deduce our first result on signed permutation statistics:

**Theorem 4.2 (cf. [7, Corollary 3.3]).** For $k \in \mathbb{P}$, let

$$f_{kn} = \sum_{(\pi_1, \ldots, \pi_r) \in B_r(n+1-k \mathbb{P}) \cap [n]} q^l(\pi_1) \cdots l(\pi_r),$$

where $(n+1-k \mathbb{P}) \cap [n] = \{n+1-i : i \in k \mathbb{P}, n+1-i \in [n]\}$. Then

$$\sum_{n \geq 0} \frac{(-1)^{|n|/k} f_{kn} x^n}{(2[n])!} = \sum_{n \geq 0} \frac{x^n}{(2[n])!} \sum_{r 
\geq 1} \frac{x^n}{[n]^r} \sum_{n \geq 1} \frac{x^{kn-1}}{(2[kn-1])!} \left( \sum_{n \geq 0} \frac{x^{kn}}{[kn]^r} \right)^{-1}.$$

**Proof.** If we let $S = k \mathbb{P}$, then

$$\mu_S(\bar{n}) = \mu_{(n+1-k \mathbb{P}) \cap [n]}(\bar{n})$$

and $|n/k| = \#(n+1-k \mathbb{P}) \cap [n])$, so $(-1)^{|n|/k} f_{kn} = \mu_{S}(\bar{n})$ by the previous theorem. Now apply Proposition 3.3.

To eliminate the $(-1)^{|n|/k}$ factor, we use a lemma proven very similarly to [7, Lemma 3.4]:

**Lemma 4.3.** If $F(x) = \sum_{n \geq 0} (-1)^{|n|/k} f(n) x^n$, then

$$\sum_{n \geq 0} f(n) x^n = \frac{k-1}{k} \sum_{j=0}^{k-1} \frac{F(\xi^{j+1}) x^j}{1 - \xi^{-(j+1)}},$$

where $\xi = e^{\pi i 2k}$.

**Corollary 4.4.** Let

$$\mathcal{A}_n = \{ \pi \in B_n : D(\pi) = \{n-1, n-3, \ldots\} \}$$

be the set of alternating signed permutations. Then

$$\sum_{n \geq 0} \frac{(-1)^{|n|/k} \# \mathcal{A}_n x^n}{(2[n])!} = \sum_{n \geq 0} \frac{x^n}{(2[n])!} \sum_{r 
\geq 1} \frac{x^n}{[n]^r} \sum_{n \geq 1} \frac{x^{kn-1}}{(2[2kn-1])!} \left( \sum_{n \geq 0} \frac{x^{kn}}{[kn]^r} \right)^{-1}$$

and

$$\sum_{n \geq 0} \frac{\# \mathcal{A}_n x^n}{2^n n!} = \frac{\cos(x/2) + \sin(x/2)}{\cos(x)}.$$

**Proof.** The first equation is Theorem 4.2 with $k = 2$ and $r = 1$. If we set $q = 1$ in the first equation, we obtain

$$\sum_{n \geq 0} \frac{(-1)^{|n|/2} \# \mathcal{A}_n x^n}{2^n n!} = \sum_{n \geq 0} \frac{x^n}{2^n n!} \sum_{r 
\geq 1} \frac{x^n}{[n]^r} \sum_{n \geq 1} \frac{x^{2n-1}}{(2[n]!)} \left( \sum_{n \geq 0} \frac{x^{2n}}{[2n]^r} \right)^{-1}$$

$$= e^{x/2} - (e^x - 1) \frac{\sinh(x/2)}{\cosh(x)} = \frac{e^{x/2} - e^{-x/2}}{\cosh(x)}.$$

Applying the preceding lemma with $k = 2$ gives

$$\sum_{n \geq 0} \frac{\# \mathcal{A}_n x^n}{2^n n!} = \frac{e^{ix/2}}{(1+i) \cosh(ix)} + \frac{e^{-ix/2}}{(1-i) \cosh(-ix)}$$

$$= \frac{\cos(x/2) + \sin(x/2)}{\cos(x)}. \quad \Box$$
REMARK. A generalization of this last result to the wreath product \( C_k \wr S_n \) of a cyclic group of order \( k \) with the symmetric group \( S_n \) is given in [9].

If we set \( k = 1 \) in Theorem 4.2, and replace \( x \) by \(-x\), we obtain the following:

**Corollary 4.5.**

\[
\sum_{n \geq 0} \frac{\sum_{(\pi_1, \ldots, \pi_r) \in B_n} q^{l(\pi_1) + \cdots + l(\pi_r)} x^n}{(2[n])!^r_q} = \sum_{n \geq 0} \frac{(-x)^n}{(2[n])!^r_q} \left( \sum_{n \geq 0} \frac{(-x)^n}{[n]!^r_q} \right)^{-1}.
\]

When \( r = 2 \), this gives the hyperoctahedral \( q \)-analogue of a result from [3].

Next we consider generating functions that count descents. For \( \pi \in B_n \), define its number of descents to be \( d(\pi) = \# D(\pi) \).

**Theorem 4.6.** Let

\[
G_{nk}(t, q) = \sum_{(\pi_1, \ldots, \pi_r) \in B_n} q^{l(\pi_1) + \cdots + l(\pi_r)} x^n - \# \cup \cup D(\pi)
\]

and let

\[
B_{kt}(t, q, x) = \sum_{n \geq 0} \frac{G_{nk}(t, q) x^n}{(2[kn])!^r_q}.
\]

Then

\[
B_{kt}(t, q, x) = \left( 1 - \sum_{n \geq 1} \frac{(t-1)^{n-1} x^n}{[kn]!^r_q} \right) \sum_{n \geq 0} \frac{(t-1)^n x^n}{2[kn])!^r_q}.
\]

**Proof.** Let \( P = L_r(\hat{V}_q) \), so that \( P^{(k)} = L_r(\hat{V}_q)^{(k)} \). We have

\[
B_{kt}(t, q, B(P, k)y) = \sum_{n \geq 0} \frac{G_{nk}(t, q) B(P, k) y^n}{B(P, kn)}
\]

\[
= \sum_{n \geq 0} \sum_{S \subseteq [k]} \sum_{(\pi_1, \ldots, \pi_r) \in B_n} q^{l(\pi_1) + \cdots + l(\pi_r)} x^n - \# \cup \cup D(\pi)
\]

\[
\sum_{n \geq 0} \sum_{S \subseteq [k]} \frac{(-1)^{n+1} \mu_S(\hat{n})}{B(P^{(k)}, \hat{n})} \left( -t \right)^n x^n y^n
\]

by Theorem 4.1, where \( \mu_S(\hat{n}) \) here refers to an \( \hat{n} \)-interval in \( P \), not \( P^{(k)} \),

\[
= \sum_{n \geq 0} (-1)^n \sum_{S \subseteq [k]} \mu_S(\hat{n})(-t)^n \# g_S x^n y^n
\]

where \( \mu_S(\hat{n}) \) now refers to an \( \hat{n} \)-interval in \( P^{(k)} \), not \( P \),

\[
= \sum_{n \geq 0} \frac{h(\hat{n})}{B(P^{(k)}, \hat{n})} (-t)^n \frac{(-y)^n}{B(P^{(k)}, \hat{n})},
\]

where \( h \) is the element of \( R(P^{(k)}) \) defined in Lemma 3.4,

\[
= \phi(h)|_{-t, -x, y}.
\]

Therefore

\[
B_{kt}(t, q, x) y = T[\phi(h)],
\]

where \( T \) is the operator which substitutes \(-t\) for \( t \), \(-x/B(P, k)\) for \( x \), and \(-y\) for \( y \). Since \( h = -1 + f^{-1}g \) in \( R(P^{(k)}) \) by Lemma 3.4, one concludes from Proposition 3.1 that

\[
B_{kt}(t, q, x) y = T[1 + \frac{\phi(f)^{-1}}{B(P^{(k)}, n)} \sum_{n \geq 0} \frac{(1 + t)^n x^n y}{B(P^{(k)}, \hat{n})}]
\]

\[
= \left( 1 - \sum_{n \geq 1} \frac{(t-1)^{n-1} x^n}{[kn]!^r_q} \right) \sum_{n \geq 0} \frac{(t-1)^n x^n y}{2[2kn])!^r_q},
\]

which implies the result. \( \square \)
By setting \( k = r = 1 \) in this theorem, we recover a special case of a result from [5]; namely, a generating function for a hyperoctahedral \( q \)-analogue of the Eulerian polynomials (see [8], Section 1.3):

**Corollary 4.7.**

\[
\sum_{n \geq 0} \frac{\sum_{x \in B_n} t^{n - d(x)} x^n}{(2[n])!_q} = B_{11}(t, q, x) \\
= \left( 1 - \sum_{n \geq 1} \frac{(t - 1)^{n-1} x^n}{[n]_q} \right)^{-1} \sum_{n \geq 0} \frac{(t - 1)^n x^n}{(2[n])!_q}
\]

and hence, setting \( q = 1 \), we have

\[
\sum_{n \geq 0} \frac{\sum_{x \in B_n} t^{n - d(x)} x^n}{2^n n!} = \frac{(t - 1)e^{(t-1)x/2}}{1 - e^{(t-1)x}}.
\]

Note that, in general, when \( r = q = 1 \), the expressions in Theorems 4.2 and 4.6 can be written in terms of the exponential functions \( e^x \) and \( e^{x/2} \), so one expects such functions to occur naturally in signed permutation enumeration problems, similar to the occurrences of \( e^x \) in permutation enumeration.

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**References**


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