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Representation Stability for Cohomology of Configuration Spaces in $\mathbb{R}^d$

Patricia Hersh[1,*] and Victor Reiner[2]

[1]Department of Mathematics, North Carolina State University, Raleigh, NC 27695 and [2]School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

*Correspondence to be sent to: e-mail: plhersh@ncsu.edu

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This paper studies representation stability in the sense of Church and Farb for representations of the symmetric group S_n on the cohomology of the configuration space of n ordered points in $\mathbb{R}^d$. This cohomology is known to vanish outside of dimensions divisible by $d-1$; it is shown here that the S_n-representation on the $i(d-1)$st cohomology stabilizes sharply at $n = 3i$ (resp. $n = 3i+1$) when d is odd (resp. even). The result comes from analyzing S_n-representations known to control the cohomology: the Whitney homology of set partition lattices for d even, and the higher Lie representations for d odd. A similar analysis shows that the homology of any rank-selected subposet in the partition lattice stabilizes by $n \geq 4i$, where i is the maximum rank selected. Further properties of the Whitney homology and more refined stability statements for S_n-isotypic components are also proven, including conjectures of J. Wiltshire-Gordon.

1 Introduction

Much has been written recently on *representation stability*, in papers of Church, Ellenberg, Farb, and others [3–7, 14, 27, 32, 38, 47, 50], particularly, for sequences of (complex, finite-dimensional) S_n-representations $\{V_n\}$. Recall that the irreducible representations V^λ of S_n are indexed by integer partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell)$ of $n = |\lambda| := \sum_i \lambda_i$. Say that

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$\{V_n\}$ *stabilizes beyond* $n = n_0$ if the unique S_n-irreducible decomposition

$$V_{n_0} = \bigoplus_{\lambda:|\lambda|=n_0} \left(V^{\lambda}\right)^{\oplus c_\lambda}$$

determines V_n for every $n \geq n_0$ as follows:

$$V_n = \bigoplus_{\lambda:|\lambda|=n_0} \left(V^{(\lambda_1+(n-n_0),\lambda_2,\ldots,\lambda_\ell)}\right)^{\oplus c_\lambda}.$$

If n_0 is minimal with the above property, say that $\{V_n\}$ *stabilizes sharply at* n_0.

Our starting point was the following celebrated result of T. Church on the nth *(ordered) configuration space* of a topological space X

$$\mathrm{Conf}(n, X) := \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j \text{ for } 1 \leq i < j \leq n\}.$$

The S_n-action permuting the coordinates in X^n restricts to $\mathrm{Conf}(n, X)$, giving rise to S_n-representations on the cohomology $H^*(\mathrm{Conf}(n, X)) := H^*(\mathrm{Conf}(n, X), \mathbb{Q})$ with rational co-efficients (All cohomology groups in this paper will be taken with coefficients in $\mathbb{Q}$.).

Theorem. ([3, Theorem 1]) Fix $i \geq 1$. For a connected, orientable d-manifold X with $H^*(X)$ finite-dimensional, the sequence of S_n-representations $\{H^i(\mathrm{Conf}(n, X))\}$ vanishes unless $d-1$ divides i, in which case it stabilizes beyond

$$\begin{cases} n = 2i & \text{for } d \geq 3, \\ n = 4i & \text{for } d = 2. \end{cases}$$

□

Our first main result improves this when $X = \mathbb{R}^d$, giving the sharp onset of stabilization.

Theorem 1.1. Fix integers $d \geq 2$ and $i \geq 1$. Then $H^i(\mathrm{Conf}(n, \mathbb{R}^d))$ vanishes unless $d-1$ divides i, in which case it stabilizes sharply at

$$\begin{cases} n = 3\frac{i}{d-1} & \text{for } d \geq 3 \text{ odd}, \\ n = 3\frac{i}{d-1} + 1 & \text{for } d \geq 2 \text{ even}. \end{cases}$$

In particular, $H^i(\mathrm{Conf}(n, \mathbb{R}^2))$ stabilizes sharply at $n = 3i + 1$. □

In fact, one has finer information about the onset of stabilization for individual S_n-irreducible multiplicities in $H^i(\mathrm{Conf}(n, \mathbb{R}^d))$; see Theorem 5.1.

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There are several motivations to focus on the manifolds $X = \mathbb{R}^d$ in Church's result.

- His analysis for more general manifolds relies on the stability properties for the case of $X = \mathbb{R}^d$ as a key input, via a result of Totaro [46]; see Section 11.1.
- One can identify $\mathrm{Conf}(n, \mathbb{R}^2)$ with the complement in $\mathbb{C}^n$ of the reflection arrangement of type A_{n-1}, an Eilenberg–MacLane $K(PB_n, 1)$ space for the pure braid group PB_n on n strands. Thus $H^*(\mathrm{Conf}(n, \mathbb{R}^2))$ computes the group cohomology $H^*(PB_n; \mathbb{Q})$.
- The S_n-representation on $H^*(\mathrm{Conf}(n, \mathbb{R}^2))$ plays a role in counting *polynomial statistics* on squarefree monic polynomials in $\mathbb{F}_q[x]$, the focus of further work of Church *et al.* in [4], as well as work of Matchett-Wood and Vakil in [47]. In fact, our results will give an improvement on the stable range of $H^*(\mathrm{Conf}(n, \mathbb{R}^2))$ that leads to a better *power-saving bound* (see [4, Section 1.1]) for the convergence rate of these counts—see Remark 3.5.

The proof of Theorem 1.1 starts with the descriptions of the S_n-representations on $H^i(\mathrm{Conf}(n, \mathbb{R}^d))$. For $d = 2$, this is known from work of Arnol'd [1] and of Lehrer and Solomon [23]. For arbitrary $d \geq 2$, such descriptions go back to work of Cohen [8, Chapter III]; see also Cohen and Taylor [9]. We will use a formulation for $d \geq 2$ closer to that of Sundaram and Welker [43]. The descriptions are in terms of *higher Lie characters* Lie_λ when d is odd, and the *Whitney homology* of the lattice Π_n of set partitions of $\{1, 2, \ldots, n\}$ when d is even; see Sections 2.3, 2.5, and 2.6 for definitions. The key to stability is recasting the descriptions in the following form (This was pointed out in the case $d = 2$ by Church and Farb [7, Section 4.1] using different language.):

$$H^{i(d-1)}(\mathrm{Conf}(n, \mathbb{R}^d)) \cong \begin{cases} \bigoplus_{m=i+1}^{2i} M_n(\widehat{\mathrm{Lie}}^i_m) & \text{for } d \text{ odd} \\ \bigoplus_{m=i+1}^{2i} M_n(\widehat{\mathrm{W}}^i_m) & \text{for } d \text{ even.} \end{cases}$$

Here $\widehat{\mathrm{Lie}}^i_m, \widehat{\mathrm{W}}^i_m$ are certain subrepresentations (These are the subrepresentations carried by *FI-generators* of the FI-modules $\{H^i(\mathrm{Conf}(n, \mathbb{R}^d))\}$ as in [6].) of higher Lie characters and Whitney homology which are defined just after Proposition 2.8, and $\chi \mapsto M_n(\chi)$ is this operation taking S_m-representations to S_n-representations:

$$M_n(\chi) = \begin{cases} \left(\chi \otimes \mathbf{1}_{S_{n-m}}\right) \uparrow^{S_n}_{S_m \times S_{n-m}} & \text{if } m \leq n, \\ 0 & \text{if } m < n. \end{cases}$$

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Sequences of S_n-representations of the form $\{M_n(\chi)\}$ were shown already by Church [3] to exhibit representation stability. We will show in Lemma 2.3 that the onset of stability is controlled by bounds on $|\lambda| + \lambda_1$ for λ arising in the irreducible expansion $\chi = \sum_\lambda c_\lambda \chi^\lambda$. The crux of our analysis is to bound the irreducible expansions of the characters $\widehat{\mathrm{Lie}}_n^i, \widehat{\mathrm{W}}_n^i$; this is achieved in Section 4, utilizing symmetric functions and *plethysm* (reviewed in Section 2.1).

It remains an open question (see Question 11.1) to give explicit irreducible decompositions for $\widehat{\mathrm{Lie}}_n^i, \widehat{\mathrm{W}}_n^i$ in general (Some data on their decompositions are given in Tables A1 and A2 of Appendix 2.). However, in Theorem 1.3 below we *do* give explicit irreducible decompositions for the sums $\widehat{\mathrm{Lie}}_n := \sum_i \widehat{\mathrm{Lie}}_n^i$ and $\widehat{\mathrm{W}}_n := \sum_i \widehat{\mathrm{W}}_n^i$. It is here that one discovers a close connection to *derangements*, that is, fixed-point free permutations. It turns out (see Remark 2.9) that these S_n-representations have the following properties:

- $\widehat{\mathrm{Lie}}_n, \widehat{\mathrm{W}}_n$ have degree equal to the number d_n of all derangements in S_n.
- $\widehat{\mathrm{Lie}}_n^i, \widehat{\mathrm{W}}_n^i$ have degree d_n^{n-i}, the number of derangements in S_n with $n-i$ cycles.

After writing down product generating functions for the Frobenius characters of $\widehat{\mathrm{Lie}}_n^i, \widehat{\mathrm{W}}_n^i$ in terms of power sum symmetric functions (Corollary 2.18), we use the generating functions in Section 6 to prove representation-theoretic lifts of a well-known derangement recurrence

$$d_n = nd_{n-1} + (-1)^n \text{ for } n \geq 2.$$

Theorem 1.2. Letting $\widehat{\mathrm{Lie}}_0 := \widehat{\mathrm{W}}_0 := \mathbf{1}_{S_0}, \widehat{\mathrm{Lie}}_1 := \widehat{\mathrm{W}}_1 := 0$ by convention, then for $n \geq 1$,

$$\widehat{\mathrm{Lie}}_n = \widehat{\mathrm{Lie}}_{n-1} \uparrow_{S_{n-1}}^{S_n} + (-1)^n \epsilon_n, \tag{1}$$

$$\widehat{\mathrm{W}}_n = \widehat{\mathrm{W}}_{n-1} \uparrow_{S_{n-1}}^{S_n} + (-1)^n \tau_n, \tag{2}$$

where ϵ_n is the sign character of S_n, and τ_n is this **virtual** S_n-character of degree 1:

$$\tau_n := \begin{cases} \mathbf{1}_{S_n} & \text{for } n = 0, 1, 2, 3, \\ \chi^{(3,1^{n-3})} - \chi^{(2,2,1^{n-4})} & \text{for } n \geq 4. \end{cases} \qquad \square$$

While (2) appears to be new, the recurrence (1) appears implicitly in work of Désarménien and Wachs [10], who studied the symmetric function which is the Frobenius image of $\widehat{\mathrm{Lie}}_n$. Recurrence (1) is also equivalent, upon tensoring with ϵ_n, to a recurrence of Reiner

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and Webb [28, Proposition 2.2] for the S_n-representation on the homology $H_n(M)$ of the *complex of injective words*.

Theorem 1.2 also leads to the next result, giving irreducible decompositions for $\widehat{\mathrm{Lie}}_n, \widehat{\mathrm{W}}_n$.

Theorem 1.3. One has the following irreducible decompositions

$$\widehat{\mathrm{Lie}}_n = \sum_{Q} \chi^{\mathrm{shape}(Q)} \tag{3}$$

$$\widehat{\mathrm{W}}_n = \sum_{Q} \chi^{\mathrm{shape}(Q)} \tag{4}$$

in which the sums in (3), (4), respectively, range over the set of desarrangement tableaux, Whitney-generating tableaux Q of size n (defined in Section 7). □

In this paper, we also address two other conjectures on the structure of $\widehat{\mathrm{W}}_n^i$, due to John Wiltshire-Gordon, that were mentioned in [6, Section 3.1, p. 37]. One of his conjectures is (6) below, an analogue of another derangement recurrence

$$d_n^k = n(d_{n-1}^k + d_{n-2}^{k-1}),$$

and will be proven in Section 8 as part of the following theorem.

Theorem 1.4. For $n \geq 2$ and $i \geq 1$, one has an isomorphism of S_{n-1}-representations

$$\widehat{\mathrm{Lie}}_n^i \downarrow \cong \left(\widehat{\mathrm{Lie}}_{n-1}^{i-1} \downarrow \quad \oplus \quad \widehat{\mathrm{Lie}}_{n-2}^{i-1}\right) \uparrow, \tag{5}$$

$$\widehat{\mathrm{W}}_n^i \downarrow \cong \left(\widehat{\mathrm{W}}_{n-1}^{i-1} \downarrow \quad \oplus \quad \widehat{\mathrm{W}}_{n-2}^{i-1}\right) \uparrow, \tag{6}$$

where $\uparrow$ and $\downarrow$ are induction $(-)\ \uparrow_{S_n}^{S_{n+1}}$ and restriction $(-)\ \downarrow_{S_{n-1}}^{S_n}$ applied to S_n-representations. □

He also made a second conjecture

Conjecture 1.5. (J. Wiltshire-Gordon) Fixing $n \geq 2$, the S_n-representations $\{W_n^\bullet\}$ admit a cochain complex structure with cohomology only in degree $n-1$, affording character $\chi^{(2,1^{n-2})}$. □

The following more precise version of this conjecture is discussed in Section 11.4, and is proven in Appendix 1, joint with Steven Sam. The particular cochain complex

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$F_n(A^{\bullet})$ arising in Theorem 1.6 is related to the Orlik–Solomon algebra, as explained in Section 11.4.

Theorem 1.6. When $n \geq 2$, the S_n-cochain complex $F_n(A^{\bullet})$ has nonvanishing cohomology only in degree $n-1$, affording the character $\chi^{(2,1^{n-2})}$, thus affirmatively answering Conjecture 1.5. □

We will also show in Section 9 that Conjecture 1.5 predicts the correct *Euler characteristic*:

Theorem 1.7. As virtual characters, for $n \geq 2$ one has

$$\sum_{i \geq 0} (-1)^i \widehat{\mathrm{W}}_n^i = (-1)^{n-1} \chi^{(2,1^{n-2})}. \qquad \square$$

While Theorem 1.7 already follows from Theorem 1.6, Section 9 is included nonetheless because it gives a self-contained proof of Theorem 1.7, in contrast to the proof of Theorem 1.6 in Appendix 1 which relies on techniques not appearing elsewhere in this paper.

The above results on stability of the Whitney homology of Π_n suggest other questions, for instance, the question of representation stability more generally for the so-called rank-selected homology of Π_n, described next.

Sundaram [39, Proposition 1.9] related the ith Whitney homology $WH_i(P)$ of a *Cohen–Macaulay poset* P with G-action to the *rank-selected homology* representations $\beta_S(P)$, extensively studied in combinatorics; see Section 2.4 for the definition of Cohen–Macaulay posets and $\beta_S(P)$. She observed that one has a G-module isomorphism

$$WH_i(P) \cong \beta_{\{1,2,\ldots,i-1\}}(P) \oplus \beta_{\{1,2,\ldots,i\}}(P).$$

Combining this with Theorem 1.1 implies that for fixed i, the S_n-representations $\beta_{\{1,2,\ldots,i\}}(\Pi_n)$ also stabilize sharply at $n = 3i+1$; see Corollary 5.4. More generally, for any rank set S, we prove the following in Section 10.

Theorem 1.8. For a subset S of positive integers with $\max(S) = i$, the sequence $\beta_S(\Pi_n)$ stabilizes beyond $n = 4i$. Furthermore, when $S = \{i\}$, it stabilizes sharply at $n = 4i$. □

Section 11 collects further questions and remarks, including Conjecture 11.3 on the sharp stabilization onset for $\beta_S(\Pi_n)$ given any fixed rank subset S.

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2 Review

2.1 Symmetric functions and S_n-representations

Throughout we will make free use of the identification of (complex, finite-dimensional) representations of a finite group G with their characters, and the fact that when G is the symmetric group S_n, all such representations can be defined over $\mathbb{Q}$. We will extensively use the dictionary between characters of symmetric groups and symmetric functions. This is realized by the *Frobenius isomorphism* $R \xrightarrow{\text{ch}} \Lambda$ of graded rings and (Hopf) algebras. Here

$$R = \bigoplus_{n \geq 0} R_n$$

in which R_n is the $\mathbb{Z}$-lattice of (virtual) complex characters of the symmetric group S_n, and

$$\Lambda = \bigoplus_{n \geq 0} \Lambda_n$$

is the ring of symmetric functions (the symmetric power series of bounded degree in an infinite variable set $x_1, x_2, \ldots$) with $\mathbb{Z}$ coefficients, in which Λ_n is the set of homogeneous degree n symmetric functions. See [15, Section 7.3], [24, Section I.7], [31, Section 4.7], and [36, Section 7.18] for many of the properties of this isomorphism, some of which are reviewed here.

The isomorphism $R \xrightarrow{\text{ch}} \Lambda$ can be defined in each degree $\text{ch} : R_n \to \Lambda_n$. One first defines the symmetric functions

$$p_\lambda := p_{\lambda_1} \cdots p_{\lambda_\ell},$$

for partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of n, where $p_d := x_1^d + x_2^d + \cdots$ is the *power sum* symmetric function. Regarding a virtual complex character in R_n as a $\mathbb{C}$-valued class function f on S_n,

$$\text{ch}(f) := \frac{1}{n!} \sum_{w \in S_n} f(w)\, p_{\lambda(w)} = \sum_{\lambda : |\lambda| = n} f(\lambda)\, \frac{p_\lambda}{z_\lambda}, \tag{7}$$

where here

- $\lambda(w)$ is the cycle type partition of w,
- $f(\lambda)$ is the value of f on any permutation of cycle type λ, and

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- if $\lambda = 1^{m_1}2^{m_2}\cdots$ has m_j parts of size j, then $z_\lambda := 1^{m_1}(m_1!)2^{m_2}(m_2)!\cdots$ is the size of the S_n-centralizer subgroup $Z_{S_n}(w)$ for any permutation of cycle type λ.

This map ch sends $\mathbb{C}$-valued class functions on S_n to symmetric functions with $\mathbb{C}$ coefficients that are homogeneous of degree n. It turns out to restrict to an isomorphism $R_n \to \Lambda_n$ between virtual S_n-characters and degree n symmetric functions with $\mathbb{Z}$ coefficients.

One has a distinguished $\mathbb{Z}$-basis of R_n given by the *irreducible characters* $\{\chi^\lambda\}$ indexed by the set of integer partitions $\lambda = (\lambda_1 \geq \ldots \geq \lambda_\ell)$ of $n = |\lambda| := \sum_{i=1}^{\ell} \lambda_i$. If $\lambda_\ell > 0$, then the *length* $\ell(\lambda) := \ell$. The isomorphism ch sends χ^λ from R_n to the *Schur function* s_λ lying in Λ_n. The *induction product* on characters

$$\begin{array}{ccc} R_{n_1} \times R_{n_2} & \longrightarrow & R_{n_1+n_2} \\ (\chi_1, \chi_2) & \longmapsto & \chi_1 * \chi_2 := (\chi_1 \otimes \chi_2) \uparrow_{S_{n_1} \times S_{n_2}}^{S_{n_1+n_2}} \end{array}$$

is sent by ch to the usual product in Λ, that is, $\mathrm{ch}(\chi_1 * \chi_2) = \mathrm{ch}(\chi_1)\mathrm{ch}(\chi_2)$. In particular, because each *parabolic* or *Young subgroup* $S_\lambda := S_{\lambda_1} \times \cdots \times S_{\lambda_\ell} \subset S_n$ has a tensor product description for its *trivial* and *sign* characters as

$$\mathbf{1}_{S_\lambda} \cong \mathbf{1}_{S_{\lambda_1}} \otimes \cdots \otimes \mathbf{1}_{S_{\lambda_\ell}},$$
$$\epsilon_{S_\lambda} \cong \epsilon_{S_{\lambda_1}} \otimes \cdots \otimes \epsilon_{S_{\lambda_\ell}},$$

the map ch sends the induced representations $\mathbf{1}_{S_\lambda} \uparrow_{S_\lambda}^{S_n}$ and $\epsilon_{S_\lambda} \uparrow_{S_\lambda}^{S_n}$ to the products

$$h_\lambda := h_{\lambda_1} \cdots h_{\lambda_\ell},$$
$$e_\lambda := e_{\lambda_1} \cdots e_{\lambda_\ell},$$

where h_λ, e_λ, respectively, are the *complete homogeneous* and *elementary* symmetric functions indexed by λ and are defined as products of h_d (resp. e_d) where

$$h_d = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_d} x_{i_1} x_{i_2} \cdots x_{i_d},$$
$$e_d = \sum_{1 \leq i_1 < i_2 < \cdots < i_d} x_{i_1} x_{i_2} \cdots x_{i_d}.$$

In other words, h_d is the sum of all monomials of degree d while e_d is the sum of all squarefree monomials of degree d.

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It is worth remarking that as λ runs through the set of partitions of n, the sets $\{h_\lambda\}, \{e_\lambda\}$ and the set $\{s_\lambda\}$ to be defined shortly all give $\mathbb{Z}$-bases for the free $\mathbb{Z}$-module Λ_n, while $\{p_\lambda\}$ gives a $\mathbb{Q}$-basis for the extended $\mathbb{C}$-vector space of all class functions on S_n. We also record two standard identities [24, Chapter I, Section 2] for later use, with conventions $h_0 = e_0 = 1$:

$$H(u) := \sum_{d=0}^{\infty} h_d u^d = \prod_{i=1}^{\infty} (1 - x_i u)^{-1} = \exp\left(\sum_{m\geq 1} \frac{p_m u^m}{m}\right), \tag{8}$$

$$E(u) = H(-u)^{-1} := \sum_{d=0}^{\infty} e_d u^d = \prod_{i=1}^{\infty} (1 + x_i u) = \exp\left(-\sum_{m\geq 1} \frac{p_m (-u)^m}{m}\right). \tag{9}$$

We will also use the well-known identity

$$h_n = \sum_{\lambda : |\lambda| = n} \frac{p_\lambda}{z_\lambda} \tag{10}$$

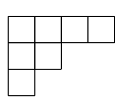
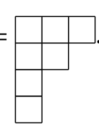
that follows either from (8) or the fact that $h_n = \mathrm{ch}(\mathbf{1}_{S_n})$.

There are many ways to define the *Schur function* (See [36, Section 7.10] for the combinatorial definition via column-strict tableaux.) s_λ. One way is through either of the *Jacobi–Trudi* or *Nägelsbach–Kostka* determinants that express s_λ in terms of h_n or e_n:

$$s_\lambda = \det(h_{\lambda_i - i + j})_{i,j=1,2,\ldots,\lambda_1^t}, \tag{11}$$

$$s_\lambda = \det(e_{\lambda_i^t - i + j})_{i,j=1,2,\ldots,\lambda_1}. \tag{12}$$

Here λ^t is the *conjugate* of λ, obtained by swapping rows and columns of the *Ferrers diagram*:

$\lambda = (4, 2, 1) =$  $\quad \lambda^t = (3, 2, 1, 1) =$ .

The involution on R that sends an S_n-character χ to the tensor product $\epsilon_{S_n} \otimes \chi$ corresponds to the fundamental involution $\Lambda \xrightarrow{\omega} \Lambda$ that swaps $h_n \leftrightarrow e_n$ and $p_n \leftrightarrow (-1)^{n-1} p_n$ for each n, along with swapping $s_\lambda \leftrightarrow s_{\lambda^t}$.

Branching and induction for $S_{n-1} \subset S_n \subset S_{n+1}$ have a well-known symmetric function interpretation [24, Examples I.5.3(c), I.8.26]: for an S_n-character χ with

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$\mathrm{ch}(\chi) = f(p_1, p_2, \ldots)$ one has

$$\mathrm{ch}\left(\chi \downarrow^{S_n}_{S_{n-1}}\right) = \frac{\partial}{\partial p_1}\mathrm{ch}(\chi),$$
$$\mathrm{ch}\left(\chi \uparrow^{S_{n+1}}_{S_n}\right) = p_1 \cdot \mathrm{ch}(\chi). \tag{13}$$

The *Pieri Rule* expresses

$$s_\mu h_r = \sum_\lambda s_\lambda, \tag{14}$$

where the sum is over all partitions λ for which $\lambda_{i+1} \leq \mu_i \leq \lambda_i$ for all i.

The description of the S_n-representations on the cohomology of configuration spaces in $\mathbb{R}^d$, found in Section 2.3 makes crucial use of the *plethysm* operation on characters and symmetric functions $R_{n_1} \times R_{n_2} \longrightarrow R_{n_1 n_2}$, which we will denote $(\chi_1, \chi_2) \longmapsto \chi_1[\chi_2]$. One way to describe it [24, Section I.8] is for genuine characters χ_i with $i = 1, 2$ of S_{n_i}-representations on vector spaces U_i for $i = 1, 2$. Then, their plethysm $\chi_1[\chi_2]$ is the character of an $S_{n_1 n_2}$-representation induced up from the representation of the *wreath product* $S_{n_1}[S_{n_2}]$ which is the normalizer subgroup within $S_{n_1 n_2}$ of the product $(S_{n_2})^{n_1}$. The representation to be induced is the one in which $S_{n_1}[S_{n_2}]$ acts on

$$U_1 \otimes \left(U_2^{\otimes n_1}\right) = U_1 \otimes \underbrace{U_2 \otimes \cdots \otimes U_2}_{n_1 \text{ factors}}$$

by letting

- $(S_{n_2})^{n_1}$ act componentwise on the tensor factors in $U_2^{\otimes n_1}$, and
- S_{n_1} simultaneously acts on U_1, while permuting the tensor positions in $U_2^{\otimes n_1}$.

In terms of the symmetric functions f and g associated with χ_1 and χ_2 by the characteristic map *ch*, the plethysm $f[g]$ is the symmetric function obtained by writing $g = \sum_{i=1}^\infty \mathbf{x}^{\alpha^{(i)}}$ as a sum of monomials $\mathbf{x}^{\alpha^{(i)}} = x_1^{\alpha_1^{(i)}} \cdot x_2^{\alpha_2^{(i)}} \cdots$, each with coefficient 1, and then

$$f[g] := f(x_1, x_2, \ldots)|_{x_i \mapsto \mathbf{x}^{\alpha^{(i)}}}.$$

In particular, $f = f[h_1] = h_1[f]$. We will later use a few plethysm facts; see, for example [24, Section I.8]:

$$(f_1 f_2)[g] = (f_1[g])(f_2[g]) \tag{15}$$

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$$\omega(f[g]) = \omega^n(f)[\omega(g)] \text{ if } g \in \Lambda_n \tag{16}$$

$$s_\lambda[g_1 + g_2] = \sum_{\mu \subseteq \lambda} s_\mu[g_1] s_{\lambda/\mu}[g_2]. \tag{17}$$

In particular, since (11) and (12) show that $h_n = s_{(n)}$ and $e_n = s_{(1^n)}$, one deduces from (17) that

$$h_n[g_1 + g_2] = \sum_{i=0}^{n} h_i[g_1] h_{n-i}[g_2], \tag{18}$$

$$e_n[g_1 + g_2] = \sum_{i=0}^{n} e_i[g_1] e_{n-i}[g_2]. \tag{19}$$

2.2 Representation stability

We start by rephrasing the definition from the introduction.

Definition 2.1. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $m \geq 0$, let $\lambda^{+(m)} := (\lambda_1 + m, \lambda_2, \ldots, \lambda_\ell)$. For example,

$$\lambda = (4, 2, 1) = \ydiagram{4,2,1}$$

will have

$$\lambda^{(+1)} = (5, 2, 1) = \ydiagram{5,2,1}, \qquad \lambda^{(+2)} = (6, 2, 1) = \ydiagram{6,2,1}, \text{ etc.}$$

For virtual S_n-characters χ in R_n with $\chi = \sum_{\lambda : |\lambda| = n} c_\lambda \chi^\lambda$, define $\chi^{(+m)}$ in R_{n+m} via the expansion

$$\chi^{(+m)} = \sum_{\lambda : |\lambda| = n} c_\lambda \chi^{\lambda + (m)}.$$

The operation $\chi \mapsto \chi^{(+m)}$ is simply the mth iterate of the operation $\chi \mapsto \chi^{(+1)}$.

Say that a sequence of S_n-characters $\{\chi_n\}$ *stabilizes beyond* n_0 if $\chi_n = \chi_{n_0}^{(+(n-n_0))}$ for $n \geq n_0$, and that $\{\chi_n\}$ *stabilizes sharply at* n_0 if n_0 is the smallest integer with the above property. □

The following basic stability lemma is a variant of Hemmer's [19, Lemma 2.3, Theorem 2.4]. To state it, for a character χ in R_{n_0}, define a sequence of characters

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$M(\chi) := \{M_n(\chi)\}$ via

$$M_n(\chi) = \begin{cases} \chi * \mathbf{1}_{S_{n-n_0}} & \text{if } n \geq n_0, \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

Equivalently, if $\mathrm{ch}(\chi) = f$, then

$$\mathrm{ch}\,(M_n(\chi)) = \begin{cases} f \cdot h_{n-n_0} & \text{if } n \geq n_0, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Lemma 2.2. For any partition μ, the sequence $M(\chi^\mu)$ obeys this inequality

$$M_{n+1}(\chi^\mu) \quad \geq \quad M_n(\chi^\mu)^{(+1)}, \tag{22}$$

for $n \geq |\mu|$, with equality if and only if $n \geq |\mu| + \mu_1$. Consequently

- $M(\chi^\mu)$ stabilizes sharply at $n_0 = |\mu| + \mu_1$, and more generally,
- any genuine character $\chi = \sum_\mu c_\mu \chi^\mu \geq 0$ has $M(\chi)$ stabilizing sharply at

$$n_0 = \max\{|\mu| + \mu_1 : c_\mu \neq 0\}. \qquad \square$$

Proof. After proving the assertions in the first sentence, the rest follow easily.

The *Pieri rule* (14) says that for $n \geq |\mu|$ one has $M_n(\chi^\mu) = \sum_\lambda \chi^\lambda$ in which λ runs through the set, which we will denote here by $L(n)$, of all partitions of n for which λ/μ is a *horizontal strip* of size $n - |\mu|$, that is, λ/μ is a skew shape whose cells lie in different columns. For example, if $\mu = (7, 6, 3)$ then $\lambda = (10, 6, 5, 1)$ shown below lies in $L(n)$ for $n = |\lambda| = 22$, and squares of the horizontal strip λ/μ are indicated with $\times$ (below the first row) and $\otimes$ (in the first row):

$$\lambda = \begin{ytableau} \ & \ & \ & \ & \ & \ & \ & \otimes & \otimes & \otimes \\ \ & \ & \ & \ & \ & \ \\ \ & \ & \ & \times & \times \\ \times \end{ytableau} \tag{23}$$

The map $\lambda \mapsto \lambda^{(+1)}$ that adds an extra $\otimes$ to the first row shown above gives an injection $L(n) \hookrightarrow L(n+1)$ which shows the inequality (22). The case of equality follows because the elements λ of $L(n+1)$ not in the image of this injection are those where the horizontal strip λ/μ (of size $n + 1 - |\mu|$) is confined within the first μ_1 columns. Such λ exist if and only if $n + 1 - |\mu| \leq \mu_1$, or equivalently, $n < |\mu| + \mu_1$. $\blacksquare$

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We need a refinement of Lemma 2.2 for stabilization of individual irreducible multiplicities.

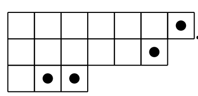
Lemma 2.3. For ν, μ partitions and $n \geq |\mu|$, one has

$$\langle \chi^{(n-|\nu|,\nu)}, M_n(\chi^{\mu}) \rangle_{S_n} = \begin{cases} 1 & \text{if } \nu \subseteq \mu, \text{ with } \mu/\nu \text{ a horizontal strip, and } n \geq |\nu| + \mu_1, \\ 0 & \text{otherwise.} \end{cases}$$

□

Proof. This is just another restatement of the Pieri rule as in the previous proof: the partitions λ in $L(n)$ in that proof biject with the $\nu \subseteq \mu$ for which μ/ν is a horizontal strip and $n \geq |\nu| + \mu_1$, via the inverse bijections $\nu \mapsto \lambda = (n - |\nu|, \nu)$, and $\lambda \mapsto \nu = (\lambda_2, \lambda_3, \ldots)$. The horizontal strip μ/ν occupies the columns of λ *complementary* to those occupied by the horizontal strip λ/μ. One needs $n \geq |\nu| + \mu_1$, or $n - |\nu| \geq \mu_1$, so that the first row of $\lambda = (n - |\nu|, \nu)$ contains the first row of μ. ■

Example 2.4. To illustrate the bijection in this proof, in (23) with $\mu = (7, 6, 3), \lambda = (10, 6, 5, 1)$, one has $\nu = (6, 5, 1)$, so that $\mu/\nu = (7, 6, 3)/(6, 5, 1)$ is the horizontal strip filled with $\boxed{\bullet}$'s in this picture:



□

2.3 Cohomology of configurations of points in $\mathbb{R}^d$

The combinatorial description of the cohomology of $\mathrm{Conf}(n, \mathbb{R}^d)$ as an S_n-representation is known. For $d = 2$, it follows from work of Arnol'd [1] and of Lehrer and Solomon [23]. For arbitrary $d \geq 2$, following on work of Cohen [8, Chapter III] and Cohen and Taylor [9], Sundaram and Welker [43] proved an equivariant version [43, Theorem 2.5] of the *Goresky–MacPherson formula* [18, III.1.3 Theorem A], and used this to show that the cohomology $H^*(\mathrm{Conf}(n, \mathbb{R}^d))$ affords

- for d even, the *Whitney homology* of the set partition lattice (see Section 2.5), and
- for d odd, the closely related *higher Lie characters* (see Section 2.6).

To state their result more precisely, we introduce a few definitions.

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Definition 2.5. A partition $\lambda = (\lambda_1, \ldots, \lambda_\ell) = 1^{m_1} 2^{m_2} \cdots$ with m_j parts of size j has *rank*

$$\mathrm{rank}(\lambda) := \sum_{k \geq 1} (\lambda_k - 1) = \sum_{j \geq 1} (j-1) m_j. \qquad \square$$

Definition 2.6. Let C_n be the subgroup $\langle c \rangle$ generated by an n-cycle c in S_n. The *Lie character* of S_n is the induction of any linear character $C_n \xrightarrow{\chi_\zeta} \mathbb{C}^\times$ that sends c to a primitive nth root of unity:

$$\mathrm{Lie}_{(n)} := \chi_\zeta \uparrow_{C_n}^{S_n} . \tag{24}$$

Denote by ℓ_n the symmetric function which is the Frobenius image of $\mathrm{Lie}_{(n)}$, and by π_n the image of its twist by the sign:

$$\begin{aligned} \ell_n &:= \mathrm{ch}(\mathrm{Lie}_{(n)}), \\ \pi_n &:= \mathrm{ch}(\epsilon_{S_n} \otimes \mathrm{Lie}_{(n)}) \;=\; \omega(\ell_n). \end{aligned}$$

For a partition $\lambda = 1^{m_1} 2^{m_2} \cdots$ of n having m_j parts of size j, define S_n-characters $\mathrm{W}_\lambda, \mathrm{Lie}_\lambda$ as those having as Frobenius images the following symmetric functions:

$$\mathrm{ch}(\mathrm{Lie}_\lambda) = \prod_{j \geq 1} h_{m_j}[\ell_j], \tag{25}$$

$$\mathrm{ch}(\mathrm{W}_\lambda) = \prod_{\text{odd } j \geq 1} h_{m_j}[\pi_j] \prod_{\text{even } j \geq 2} e_{m_j}[\pi_j]. \tag{26}$$

$\square$

Theorem 2.7. [43, Theorem 4.4(iii)] Fix $d \geq 2$ and $i \geq 0$. Then $H^i(\mathrm{Conf}(n, \mathbb{R}^d))$

- vanishes unless i is divisible by $d-1$, say $i = j(d-1)$,
- in which case, as S_n-representations,

$$H^{j(d-1)}(\mathrm{Conf}(n, \mathbb{R}^d)) \cong \begin{cases} \mathrm{Lie}_n^j := \oplus_\lambda \mathrm{Lie}_\lambda & \text{for } d \text{ odd}, \\ WH_j(\Pi_n) := \oplus_\lambda \mathrm{W}_\lambda & \text{for } d \text{ even}, \end{cases}$$

where both direct sums above run over all partitions λ of n having $\mathrm{rank}(\lambda) = j$.

$\square$

We wish to reformulate this result in terms of the construction $M(-)$ from Definition (20). Given a partition $\lambda = 1^{m_1} 2^{m_2} 3^{m_3} \cdots$ of n, let $\hat{\lambda} := 2^{m_2} 3^{m_3} \cdots$ denote the partition

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of $n - m_1$ obtained by removing all of its parts of size 1. Also define

$$\widehat{\mathrm{Lie}}^i := \bigoplus_\lambda \mathrm{Lie}_\lambda,$$
$$\widehat{\mathrm{W}}^i := \bigoplus_\lambda \mathrm{W}_\lambda,$$

with both sums running over all partitions λ having $\mathrm{rank}(\lambda) = i$ and no parts of size 1. Although, these look potentially like infinite sums, they are finite due to the following fact.

Proposition 2.8. A partition λ with no parts of size 1 and $\mathrm{rank}(\lambda) = i$ has $i + 1 \leq |\lambda| \leq 2i$. □

Proof. Note that $\lambda = 2^{m_2}3^{m_3}\cdots$ has $\ell(\lambda) = \sum_{j\geq 2} m_j \leq \sum_{j\geq 2} m_j(j-1) = \mathrm{rank}(\lambda) = i$. Thus $\ell(\lambda)$ lies in the range $[1, i]$, and hence $|\lambda| = i + \ell(\lambda)$ lies in the range $[i+1, 2i]$. ■

Thus one has finer decompositions of $\widehat{\mathrm{Lie}}^i, \widehat{\mathrm{W}}^i$, illustrated in Tables A1 and A2 of Appendix 2:

$$\begin{aligned} \widehat{\mathrm{Lie}}^i &:= \bigoplus_{m=i+1}^{2i} \widehat{\mathrm{Lie}}^i_m, \quad \text{where} \quad \widehat{\mathrm{Lie}}^i_m := \bigoplus_\lambda \mathrm{Lie}_\lambda \\ \widehat{\mathrm{W}}^i &:= \bigoplus_{m=i+1}^{2i} \widehat{\mathrm{W}}^i_m, \quad \text{where} \quad \widehat{\mathrm{W}}^i_m := \bigoplus_\lambda \mathrm{W}_\lambda \end{aligned} \tag{27}$$

with the rightmost sums running over λ with $|\lambda| = m$, no parts of size 1, and $\mathrm{rank}(\lambda) = i$.

Remark 2.9. It is not hard to show using the definition of plethysm that for a partition λ of n, both Lie_λ and W_λ are representations induced up to S_n from one-dimensional characters of the S_n-centralizer Z_λ for a permutation w_λ having cycle type λ; see Lehrer and Solomon [23] for a discussion in the case of W_λ. Consequently, Lie_λ and W_λ both have degree equal to the index $[S_n : Z_\lambda]$, which is the number of permutations in S_n of cycle type λ.

This now allows us to justify some assertions from the introduction about derangements. A permutation w in S_n with cycle type λ is a derangement if and only if λ has no parts of size 1. Also, $\mathrm{rank}(\lambda) = n - \ell(\lambda)$ where $\ell(\lambda)$ is the number of cycles of w. Thus

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- $\widehat{\mathrm{Lie}}^i, \widehat{\mathrm{W}}^i$ have degree d_n, the number of derangements in S_n, and
- $\widehat{\mathrm{Lie}}^i_n, \widehat{\mathrm{W}}^i_n$ have degree d_n^{n-i}, the number of derangements in S_n with $n-i$ cycles, since
- $\mathrm{Lie}_\lambda, \mathrm{W}_\lambda$ have degree $[S_n : Z_\lambda]$, the number of permutations of cycle type λ. □

As mentioned in the introduction, one way to show representation stability is via the construction $\chi \mapsto M_n(\chi)$.

Corollary 2.10. For any partition $\lambda = 1^{m_1}2^{m_2}3^{m_3}\cdots$ of n, with $\hat{\lambda} := 2^{m_2}3^{m_3}\cdots$, one has

$$\mathrm{Lie}_\lambda = M_n\left(\mathrm{Lie}_{\hat{\lambda}}\right),$$
$$\mathrm{W}_\lambda = M_n\left(\mathrm{W}_{\hat{\lambda}}\right),$$

and consequently

$$H^{i(d-1)}(\mathrm{Conf}(n, \mathbb{R}^d)) \cong \begin{cases} M_n\left(\widehat{\mathrm{Lie}}^i\right) & \text{for } d \text{ odd,} \\ M_n\left(\widehat{\mathrm{W}}^i\right) & \text{for } d \text{ even.} \end{cases} \qquad \square$$

Proof. Compare (25), (26) with (21), noting $\ell_1 = \pi_n = h_1$, so $h_{m_1}[\ell_1] = h_{m_1}[\pi_1] = h_{m_1}$. ■

Example 2.11. As special cases of $\widehat{\mathrm{W}}^i_n, \widehat{\mathrm{Lie}}^i_n$ from the introduction, one has

$$\widehat{\mathrm{Lie}}^{n-1}_n = \mathrm{Lie}_{(n)},$$
$$\widehat{\mathrm{W}}^{n-1}_n = \epsilon_{S_n} \otimes \mathrm{Lie}_{(n)}.$$

Thus for $n \leq 5$, their irreducible expansions appear as the $(i, n) = (n-1, n)$ diagonal in Tables A1 and A2, respectively. Multiplicities larger than 1 first appear in the decomposition of $\mathrm{Lie}_{(n)}$ at $n = 6$:

$$\mathrm{Lie}_{(6)} = \chi^{(5,1)} + \chi^{(4,2)} + 2\chi^{(4,1,1)} + \chi^{(3,3)} + 3\chi^{(3,2,1)} + \chi^{(3,1,1,1)} + 2\chi^{(2,2,1,1)} + \chi^{(2,1,1,1,1)}. \qquad \square$$

2.4 Posets, Whitney homology, and rank-selection

Good references for much of this material include Stanley [37], Sundaram [39], and Wachs [48]. Given a finite partially ordered set (*poset*) P, the *order complex* of P, denoted $\Delta(P)$, is the simplicial complex whose i-faces are $(i+1)$-chains $p_0 < p_1 < \cdots < p_i$ of comparable poset elements. We often consider the (reduced) simplicial homology $\tilde{H}_*(\Delta(P)) = \tilde{H}_*(\Delta(P), \mathbb{Q})$ with coefficients in $\mathbb{Q}$, regarded as a representation for any group G of poset

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automorphisms. Throughout we will make free use of the identification of (complex, finite-dimensional) representations of a finite group G with their characters, and the fact that when G is the symmetric group S_n, all such representations can be defined over $\mathbb{Q}$. In particular, we will use that the inner product $\langle \chi_1, \chi_2 \rangle_G := \frac{1}{|G|} \sum_{g \in G} \chi_1(g^{-1})\chi_2(g)$ of two characters χ_1, χ_2 of G-representations V_1, V_2 gives the dimension of their intertwiner space $\mathrm{Hom}_G(V_1, V_2)$, so that when V_1 is irreducible, $\langle \chi_1, \chi_2 \rangle_G$ is the multiplicity of V_1 within V_2.

Say that a finite poset P is *graded* if all of its *maximal chains* (namely its totally ordered subsets which are maximal under inclusion) have the same length. For P a finite graded poset having unique minimum element $\hat{0}$, let $\mathrm{rank}(x)$ be the length ℓ of all the maximal chains $\hat{0} = x_0 < x_1 < \cdots < x_\ell = x$ from $\hat{0}$ to x. Denote by $\Delta(x, y)$ the order complex of the open interval $(x, y) := \{z \in P : x < z < y\}$, so that $\dim \Delta(x, y) = \mathrm{rank}(y) - \mathrm{rank}(x) - 2$. When we speak of the simplicial homology of a poset P or of an interval (u, v) in a poset P, we are always referring to the simplicial homology of its order complex. Say that P is *Cohen–Macaulay* (over $\mathbb{Q}$) if every interval (x, y) in P has

$$\tilde{H}_i(x, y) = 0 \text{ for } i < \mathrm{rank}(y) - \mathrm{rank}(x) - 2.$$

It is known that the Cohen–Macaulay property is inherited when passing to the rank-selected subposets $P^S := \{p \in P : \mathrm{rank}(p) \in S\}$ of a graded Cohen–Macaulay poset P for any subset S of possible ranks. For a group G of automorphisms of P, let $\alpha_S(P)$ denote the G-representation that is the permutation representation on the maximal chains in P^S induced by the G-action on P. Let $\beta_S(P)$ denote the virtual representation defined by

$$\beta_S(P) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_T(P). \tag{28}$$

The Hopf trace formula implies that $\beta_S(P)$ is also the virtual representation that is the alternating sum of G-representations on the homology groups of P^S. When P is a Cohen–Macaulay poset, this second interpretation for $\beta_S(P)$ implies that $\beta_S(P)$ is the actual G-representation on the top homology $\tilde{H}_{|S|-1}(P^S)$ since all the other terms comprising the virtual representation are 0. By inclusion–exclusion, one also has

$$\alpha_S(P) = \sum_{T \subseteq S} \beta_T(P). \tag{29}$$

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Sundaram observed the following relation between the rank-selected homologies $\beta_S(P)$ for initial subsets $S = \{1, 2, \ldots\}$ of ranks, and the *Whitney homology*, defined as follows:

$$WH_i(P) := \bigoplus_{\substack{x \in P: \\ \mathrm{rank}(x) = i}} \tilde{H}_*(\hat{0}, x).$$

Proposition 2.12. (Sundaram [39, Proposition 1.9]) For P any finite Cohen–Macaulay graded poset with a bottom element, one has

$$WH_i(P) = \beta_{\{1,2,\ldots,i\}}(P) + \beta_{\{1,2,\ldots,i-1\}}(P). \qquad \square$$

2.5 The lattice of set partitions

A *set partition* $\pi = \{B_1, \ldots, B_\ell\}$ of $\{1, 2, \ldots, n\}$ is a disjoint decomposition $\{1, 2, \ldots, n\} = \bigsqcup_{i=1}^{\ell} B_i$ into sets B_i called the *blocks* of the partition. The set Π_n of all such partitions is ordered by refinement: $\pi \leq \sigma$ if every block of σ is a union of blocks of π. This partial order gives a well-studied ranked lattice, in which the unique minimum and maximum elements $\hat{0}, \hat{1}$ of Π_n are the partitions with n blocks and 1 block, respectively. The rank of a set partition π turns out to be the same as the rank of the number partition $\lambda = 1^{m_1} 2^{m_2} \cdots$ of $n = |\lambda|$ giving its block sizes, that is, m_i is the number of blocks of size i:

$$\mathrm{rank}(\pi) = \mathrm{rank}(\lambda) = \sum_{i \geq 1} m_i(i-1) = |\lambda| - \ell(\lambda). \tag{30}$$

It is well-known that Π_n is Cohen–Macaulay of rank $n-1$, and therefore the open interval $(\hat{0}, \hat{1})$ has only top homology $\tilde{H}_{n-3}(\Pi_n)$ non zero. Stanley [36] described its S_n-representation.

Theorem 2.13. [36, Theorem 7.3] For $n \geq 1$, the homology S_n-representation $\tilde{H}_{n-3}(\Pi_n)$ is

$$\epsilon_{S_n} \otimes \mathrm{Lie}_{(n)} = \epsilon_{S_n} \otimes \chi_\zeta \uparrow_{C_n}^{S_n}. \qquad \square$$

More generally, Lehrer and Solomon [23] described its Whitney homology of Π_n, as follows; see also Sundaram [39, Theorem 1.8].

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Theorem 2.14. For a partition $\lambda = 1^{m_1}2^{m_2}\cdots$ of n, the S_n-representation

$$\bigoplus_{\substack{\pi\in\Pi_n \text{ with}\\ \text{block sizes } \lambda}} \tilde{H}_*(\hat{0},\pi)$$

is isomorphic to W_λ described earlier, with Frobenius characteristic given by (26). □

2.6 Higher Lie characters

The homology S_n-representation π_n for the partition lattice is well-known (from work by Witt, by Brandt, by Thrall, and by Klyachko) to have a close relation with the theory of *free Lie algebras*, and higher Lie characters. We review this connection here, drawing on expositions of Gessel and Reutenauer [16], Reutenauer [30, Chapter 8], Sundaram [39, Introduction], Schocker [34], and Stanley [36, Exercise 7.89].

For $V = \mathbb{C}^n$, the *tensor algebra* $T(V) := \oplus_{d\geq 0} T^d(V)$ where $T^d(V) := V^{\otimes d}$ may be considered the *free associative algebra* on n generators $e_1,\ldots,e_n$ forming a $\mathbb{C}$-basis for V. It is also the *universal enveloping algebra* $T(V) = \mathcal{U}(\mathcal{L}(V))$ for the *free Lie algebra* $\mathcal{L}(V)$, which is the Lie subalgebra of $T(V)$ generated by $T^1(V) = V$, using the usual Lie bracket operation $[x,y] := xy - yx$ for elements x, y in $T(V)$. The $GL(V)$-action on V extends to an action on $T(V)$, preserving $\mathcal{L}(V)$, and respecting the graded $\mathbb{C}$-vector space decomposition $\mathcal{L}(V) = \bigoplus_{j\geq 0} \mathcal{L}^j(V)$ in which

$$\mathcal{L}^0(V) := \mathbb{C}, \quad \mathcal{L}^1(V) := V, \quad \mathcal{L}^2(V) := [V,V],$$

and $\mathcal{L}^j(V)$ is the $\mathbb{C}$-span of all Lie monomials bracketing j elements of V. Denote by $S(U)$ the *symmetric algebra* of a $\mathbb{C}$-vector space U, that is, $S(U) := \oplus_{d\geq 0} S^d(U)$, where $S^d(U)$ is the dth symmetric power. Then, the *Poincaré–Birkhoff–Witt* vector space isomorphism $\mathcal{U}(L) \cong S(L)$ for a Lie algebra L here provides a $GL(V)$-equivariant isomorphism and decomposition

$$T(V) = \mathcal{U}(\mathcal{L}(V)) \cong S(\mathcal{L}(V)) = S\left(\bigoplus_{j\geq 0} \mathcal{L}^j(V)\right) \cong \bigoplus_{(m_1,m_2,\ldots)\geq 0} \underbrace{S^{m_1}\mathcal{L}^1(V) \otimes S^{m_2}\mathcal{L}^2(V) \otimes \cdots}_{\mathcal{L}_\lambda(V):=}$$

Definition 2.15. The $GL(V)$-representation $\mathcal{L}_\lambda(V)$ is the *higher Lie representation* for λ. □

Theorem 2.16. Letting $n := |\lambda|$, so that $V = \mathbb{C}^n$, the higher Lie representation $\mathcal{L}_\lambda(V)$ is *Schur–Weyl dual* to the S_n-representation Lie_λ from Definition 2.6: Lie_λ is S_n-isomorphic

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to the *multilinear component*, or 1^n-*weight space* in $\mathcal{L}_\lambda(V)$, the subspace on which a matrix in $GL(V)$ having eigenvalues $x_1, \ldots, x_n$ acts via the scalar $x_1 \cdots x_n$. □

Equivalently, the trace of this same diagonal matrix acting on $\mathcal{L}_\lambda(V)$ can be obtained from the symmetric function $\mathrm{ch}(\mathrm{Lie}_\lambda)$ in $x_1, x_2, \ldots$ by setting $x_{n+1} = x_{n+2} = \cdots = 0$.

2.7 Product generating functions

The formulas (25) and (26) have the following product generating function reformulations that we will find useful. They appear in work of Sundaram [39, p. 249], [40, Lemma 3.12], of Hanlon (There are small sign typos which need to be corrected in [22, Equation (8.1)] to accord with (33).) [22, Equation (8.1)], and of Calderbank *et al.* [2, Corollary 4.4] (see also Getzler [17, Theorem 4.5] for subsequent results in greater generality). To state them, we first introduce for $\ell \leq 1$ the Möbius function sum

$$a_\ell(u) := \frac{1}{\ell} \sum_{d \mid \ell} \mu(d) u^{\frac{\ell}{d}}. \tag{31}$$

Theorem 2.17. In $\Lambda[[u]]$, one has the product formulas

$$L(u) := \sum_{\lambda} \mathrm{ch}(\mathrm{Lie}_\lambda) u^{\ell(\lambda)} = \sum_{n,i \geq 0} \mathrm{ch}(\mathrm{Lie}_n^i) u^{n-i} = \prod_{\ell \geq 1} (1 - p_\ell)^{-a_\ell(u)} \tag{32}$$

$$W(u) := \sum_{\lambda} \mathrm{ch}(\mathrm{W}_\lambda) u^{\ell(\lambda)} = \sum_{n,i \geq 0} \mathrm{ch}(\mathrm{W}_i(\Pi_n)) u^{n-i} = \prod_{\ell \geq 1} \left(1 + (-1)^\ell p_\ell\right)^{a_\ell(-u)}. \tag{33}$$

□

We also introduce "hatted" versions $\widehat{\mathrm{W}}(u), \widehat{L}(u)$ of the generating functions $W(u), L(u)$:

$$\widehat{L}(u) := \sum_{\lambda \text{ with no parts } 1} \mathrm{ch}(\mathrm{Lie}_\lambda) u^{\ell(\lambda)} = \sum_{n,i \geq 0} \mathrm{ch}(\widehat{\mathrm{W}}_n^i) u^{n-i} \tag{34}$$

$$\widehat{\mathrm{W}}(u) := \sum_{\lambda \text{ with no parts } 1} \mathrm{ch}(\mathrm{W}_\lambda) u^{\ell(\lambda)} = \sum_{n,i \geq 0} \mathrm{ch}(\widehat{\mathrm{W}}_n^i) u^{n-i}. \tag{35}$$

Corollary 2.18. In $\Lambda[[u]]$, one also has the product formulas

$$\widehat{L}(u) = \frac{L(u)}{H(u)} = \exp\left(- \sum_{m \geq 1} \frac{p_m u^m}{m} \right) \prod_{\ell \geq 1} (1 - p_\ell)^{-a_\ell(u)}, \tag{36}$$

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$$\widehat{\mathrm{W}}(u) = \frac{W(u)}{H(u)} = \exp\left(-\sum_{m\geq 1}\frac{p_m u^m}{m}\right)\prod_{\ell\geq 1}\left(1+(-1)^{\ell}p_{\ell}\right)^{a_{\ell}(-u)}. \tag{37}$$

□

Proof. Comparing Corollary 2.10 and (21) with the definition in (8) of $H(u)$ gives

$$\begin{aligned}
\sum_{n,i\geq 0}\mathrm{ch}(\mathrm{Lie}_n^i)u^{n-i} &= \left(\sum_{n\geq 0}h_n u^n\right) \cdot \left(\sum_{n,i\geq 0}\mathrm{ch}(\widehat{\mathrm{Lie}}_n^i)u^{n-i}\right)\\
L(u) &= H(u) \cdot \widehat{L}(u).\\
\sum_{n,i\geq 0}\mathrm{ch}(\mathrm{W}_i(\Pi_n))u^{n-i} &= \left(\sum_{n\geq 0}h_n u^n\right) \cdot \left(\sum_{n,i\geq 0}\mathrm{ch}(\widehat{\mathrm{W}}_n^i)u^{n-i}\right)\\
W(u) &= H(u) \cdot \widehat{\mathrm{W}}(u),
\end{aligned}$$

giving the first equalities in (36) and (37). Theorem 2.17 and (9) give the second equalities. ■

Remark 2.19. Corollary 2.18 and its proof are modeled on an argument of Hanlon and Hersh [21, pp. 118–119]. They give a product formula for the generating function $\sum_i \mathrm{ch}(H_n^{(i)}(M))u^i$ recording the S_n-representations on the *Hodge components* $H_n^{(i)}(M)$ in the homology $H_n(M)$ of the *complex of injective words*, discussed in the introduction and further in Remark 6.2. □

3 New Tools for Polynomial Characters

The goal of this section is Theorem 3.4 below, refining the discussion of *polynomial characters* from Church *et al.* [6, Section 3.3], [4, Section 3.4]. We begin by reviewing this notion.

Definition 3.1. A polynomial $P = P(x_1, x_2, \ldots)$ in $\mathbb{Q}[x_1, x_2, \ldots]$ gives rise to a class function χ_P called a *polynomial character* on S_n for each n, by setting

$$\chi_P(w) := P(m_1, m_2, \ldots)$$

if w has cycle type $\lambda = 1^{m_1}2^{m_2}\cdots$, that is, w has m_j cycles of size j. Define the *degree* $\deg(P)$ by letting the variable x_j have $\deg(x_j) = j$. □

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As pointed out in [4, Section 3.4], when working with polynomial characters, there is a particularly convenient $\mathbb{Q}$-basis for $\mathbb{Q}[x_1, x_2, \ldots]$. Specifically, since $\mathbb{Q}[x]$ has $\mathbb{Q}$-basis $\{\binom{x}{\ell}\}_{\ell \geq 0}$, one has for $\mathbb{Q}[x_1, x_2, \ldots]$ a $\mathbb{Q}$-basis $\{\binom{X}{\lambda}\}$ given by

$$\binom{X}{\lambda} := \binom{x_1}{m_1}\binom{x_2}{m_2}\cdots,$$

as $\lambda = 1^{m_1}2^{m_2}\cdots$ runs through all number partitions. Additionally, the subset $\{\binom{X}{\lambda} : |\lambda| \leq d\}$ gives a $\mathbb{Q}$-basis for the subspace $\{P \in \mathbb{Q}[x_1, x_2, \ldots] : \deg(P) \leq d\}$.

The next result uses this basis to give a dictionary between polynomial characters and symmetric functions. It will also be used to further analyze the stability of χ_P.

Proposition 3.2. For a partition λ, consider the polynomial character χ_P of degree $|\lambda|$ corresponding to the $\mathbb{Q}$-basis element $P = \binom{X}{\lambda}$ of $\mathbb{Q}[x_1, x_2, \ldots]$ as a class function on S_n. Then one has

$$\mathrm{ch}(\chi_P) = \begin{cases} \dfrac{p_\lambda}{z_\lambda} h_{n-|\lambda|} & \text{for } n \geq |\lambda|, \\ 0 & \text{for } n < |\lambda|. \end{cases} \tag{38}$$

□

Proof. One calculates as follows:

$$\mathrm{ch}\,(\chi_P) = \sum_{\substack{\mu: \\ |\mu|=n}} \chi_P(\mu)\frac{p_\mu}{z_\mu} = \sum_{\substack{\mu=1^{n_1}2^{n_2}\cdots: \\ |\mu|=n, \\ n_j \geq m_j}} p_\mu \cdot \prod_{j\geq 1} \frac{\binom{n_j}{m_j}}{j^{n_j}(n_j!)}.$$

When $n < |\lambda|$, the sum is empty and hence $\mathrm{ch}(\chi_P)$ vanishes. On the other hand, if $n \geq |\lambda|$, one can reindex the sum over $\mu = 1^{m_1}2^{m_2}\cdots$ via $\hat{\mu} := 1^{n_1-m_1}2^{n_2-m_2}\cdots$, to obtain

$$\mathrm{ch}\,(\chi_P) = \sum_{\substack{\hat{\mu}: \\ |\hat{\mu}|=n-|\lambda|}} \frac{p_\lambda p_{\hat{\mu}}}{z_\lambda z_{\hat{\mu}}} = \frac{p_\lambda}{z_\lambda} \sum_{\substack{\hat{\mu}: \\ |\hat{\mu}|=n-|\lambda|}} \frac{p_{\hat{\mu}}}{z_{\hat{\mu}}} = \frac{p_\lambda}{z_\lambda} h_{n-|\lambda|}$$

using (10) in the very last equality. ■

Corollary 3.3. For any polynomial P in $\mathbb{Q}[x_1, x_2, \ldots]$, one can express its polynomial character as $\chi_P = M\left(\sum_\mu c_\mu \chi^\mu\right)$ with $c_\mu \in \mathbb{Q}$ and each μ satisfying $|\mu| \leq \deg(P)$. □

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Proof. It suffices to show this assertion for the $\mathbb{Q}$-basis elements $P = \binom{X}{\lambda}$. In this case, Proposition 3.2 showed that $\chi_P = M(\chi)$ where $\chi = \mathrm{ch}^{-1}\left(\frac{p_\lambda}{z_\lambda}\right)$ is a class function on S_n for $n = |\lambda| = \deg(P)$. Hence $\chi = \sum_{\mu:|\mu|=\deg(P)} c_\mu \chi^\mu$, as desired. ■

This has an important consequence for the stability of polynomial characters, allowing one to sometimes improve on the bound given in [4, Proposition 3.9].

Theorem 3.4. Fix P in $\mathbb{Q}[x_1, x_2, \ldots]$.

(i) The polynomial character χ_P on S_n can be expressed as

$$\chi_P = \sum_\nu d_\nu \chi^{(n-|\nu|,\nu)} \text{ for } n \geq 2\deg(P),$$

with each ν having $|\nu| \leq \deg(P)$, and some d_μ in $\mathbb{Q}$.

(ii) If $\chi = \sum_\mu c_\mu \chi^\mu$ in which each μ has $\mu_1 \leq b$, then

$$\langle\, \chi_P \,,\, M_n(\chi) \,\rangle_{S_n}$$

becomes a constant function of n for $n \geq \max\{2\deg(P), \deg(P) + b\}$. □

Proof. For assertion (i), note that by Lemma 2.2, $M_n(\chi^\mu)$ has such an expansion of the form $\sum_\nu d_\nu \chi^{(n-|\nu|,\nu)}$ in which each ν has $|\nu| \leq |\mu|$, once $n \geq |\mu|+\mu_1$. But then Proposition 3.3 expresses χ_P as a sum of χ^μ with $|\mu| \leq \deg(P)$, so that $|\mu| + \mu_1 \leq 2|\mu| \leq 2\deg(P)$. Thus, once $n \geq 2\deg(P)$, the assertion follows.

For assertion (ii), write $\chi_P = \sum_\nu \chi^{(n-|\nu|,\nu)}$ with $|\nu| \leq \deg(P)$ as in assertion (i). Then, Lemma 2.3 says that each term $\left\langle \chi^{(n-|\nu|,\nu)} \,,\, M_n(\chi) \right\rangle_{S_n}$ is constant in n once $n \geq |\nu|+b$. Hence all of them are constant once $n \geq \deg(P) + b$. ■

Remark 3.5. We explain here how this can be used to sharpen results of Church *et al.* [4] on polynomial statistics over the set

$$C_n(\mathbb{F}_q) := \{\text{monic squarefree } f(T) \text{ of degree } n \text{ in } \mathbb{F}_q[T]\}.$$

A fixed polynomial P in $\mathbb{Q}[x_1, x_2, \ldots]$ gives rise to a statistic on $C_n(\mathbb{F}_q)$ defined by $P(f) := [P(x_1, x_2, \ldots)]_{x_i = m_i}$ if $f(T)$ has m_i irreducible factors in $\mathbb{F}_q[T]$ of degree i. Church *et al.* discuss the following result at the end of [4, Section 1.1]:

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Theorem 3.6. This limit exists:

$$L := \lim_{n\to\infty} \sum_{i=0}^{n} (-q)^{-i} \left\langle \chi_P \, , \, H^i(\mathrm{Conf}_n(\mathbb{C})) \right\rangle_{S_n}.$$

Furthermore, given constants K, C such that

$$\left\langle \chi_P \, , \, H^i(\mathrm{Conf}_n(\mathbb{C})) \right\rangle_{S_n} \text{ is constant when } n \geq Ki + C,$$

then the above limit L also estimates the average of the statistic P as follows:

$$q^{-n} \sum_{f \in C_n(\mathbb{F}_q)} P(f) = L + O(q^{-\frac{n}{K}}). \qquad \square$$

They showed that $\langle \chi_P, H^i(\mathrm{Conf}_n(\mathbb{C}))\rangle_{S_n}$ stabilizes for $n \geq 2i + \deg(P)$, which is of the form $n \geq Ki + C$ where $K = 2$. We explain here why it stabilizes for $n \geq i + (2\deg(P) + 1)$, replacing the $K = 2$ with $K = 1$.

Start by taking $d = 2$ in Corollary 2.10 and (27) to see that $H^i(\mathrm{Conf}_n(\mathbb{C})) \cong M_n(\chi)$ where χ is the sum of characters W_λ where λ is a partition having no parts of size 1 and $\mathrm{rank}(\lambda) = i$. Theorem 4.4(d) below then implies that $\chi = \sum_\mu \chi^\mu$ having $\mu_1 \leq i + 1$ for all μ in the sum. Finally, taking $b = i + 1$ in Theorem 3.4 above shows that

$$\left\langle \chi_P \, , \, H^i(\mathrm{Conf}_n(\mathbb{C})) \right\rangle_{S_n} \text{ is constant for } n \geq \max\{2\deg(P), \deg(P) + i + 1\}. \tag{39}$$

Thus it is constant when $n \geq Ki + C$ for the constants $K := 1$ and $C := 2\deg(P) + 1$. $\square$

4 Bounding the Higher Lie and Whitney Homology Characters

Theorem 2.7 expressed $H^i(\mathrm{Conf}(n, \mathbb{R}^d))$ in the form of $\{M_n(\chi)\}$ for certain representations χ. To apply Lemma 2.2 in determining the onset of stability $\{M_n(\chi)\}$, one needs bounds on the shapes λ appearing in the irreducible expansion $\chi = \sum_\lambda c_\lambda \chi^\lambda$.

We start by developing some simple tools for finding such bounds. For example, the following standard partial order lets one compare characters or symmetric functions.

Definition 4.1. Partially order R_n by decreeing $\chi_1 \leq \chi_2$ when $\chi_2 - \chi_1$ is the character of a genuine, not virtual, S_n-representation, that is, the unique expansion $\chi_2 - \chi_1 = \sum_\lambda c_\lambda \chi^\lambda$ has $c_\lambda \geq 0$ for all partitions λ of n. In particular, $0 \leq \chi_1 \leq \chi_2$ means that χ_1 and χ_2 are characters of genuine representations, with χ_1 a subrepresentation of χ_2. Analogously

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partially order Λ_n by decreeing $f_1 \leq f_2$ if $f_2 - f_1$ is *Schur-positive*, that is, $f_2 - f_1 = \sum_\lambda c_\lambda s_\lambda$ with $c_\lambda \geq 0$. Thus $\chi_1 \leq \chi_2$ if and only if $\mathrm{ch}(\chi_1) \leq \mathrm{ch}(\chi_2)$. □

Definition 4.2. Say that a virtual S_n-character χ is *bounded by N* if the unique expansion $\chi = \sum_\lambda c_\lambda \chi^\lambda$ has the property that $\lambda_1 \leq N$ whenever $c_\lambda \neq 0$. Analogously, say that a symmetric function f is *bounded by N* if its Schur function expansion $f = \sum_\lambda c_\lambda s_\lambda$ has $\lambda_1 \leq N$ whenever $c_\lambda \neq 0$.

When N is smallest with the above property, say that χ or f is *sharply* bounded by N.

Alternatively, a sharp bound for a symmetric function f is the largest power d_1 on the variable x_1 occurring among all monomials $x_1^{d_1} x_2^{d_2} \cdots$ appearing in f. □

Proposition 4.3. Boundedness in Λ enjoys these inheritance properties.

(a) If f_1, f_2 are bounded by N, then so is $f_1 + f_2$.
(b) If $f \geq g \geq 0$ and f is bounded by N, then so is g.
(c) If f_1, f_2 are bounded by N_1, N_2, then $f_1 f_2$ is bounded by $N_1 + N_2$.
(d) If $g \geq 0$ is bounded by N, and if f lies in Λ_n, then $f[g]$ is bounded by nN. □

Proof. Assertions (a) and (b) are straightforward exercises in the definition of boundedness.

Assertion (c) arises either from the characterization of boundedness by highest powers of x_1 in symmetric functions, or from various versions of the *Littlewood–Richardson rule* (e.g., [24, Section I.9], [36, Theorem A1.3.3]) for $s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda$ which show that $c^\lambda_{\mu,\nu} \neq 0$ forces $\lambda_1 \leq \mu_1 + \nu_1$.

For assertion (d), note that it will follow by property (a) if we can show it in the special case where f is any of the $\mathbb{Z}$-basis elements $\{h_\lambda\}_\lambda$ of Λ_n. Furthermore, note that the special case where $f = h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$ follows using (15) and part (c), if we can show it in the special case where $f = h_n$. To show it when $f = h_n$, start with the assumption $g \geq 0$ and write $g = \mathrm{ch}(\chi)$ where χ is the character of a genuine S_m-representation V. Then note that

$$f[g] = h_n[g] = \mathrm{ch}\left(\chi^{\otimes n} \uparrow_{S_n[S_m]}^{S_{nm}}\right),$$

$$g^n = \mathrm{ch}\left(\chi^{\otimes n} \uparrow_{(S_m)^n}^{S_{nm}}\right),$$

and hence $g^n \geq f[g] (\geq 0)$ via the surjection of the corresponding $\mathbb{C}S_{nm}$-modules

$$\mathbb{C}S_{nm} \otimes_{\mathbb{C}(S_m)^n} V^{\otimes n} \twoheadrightarrow \mathbb{C}S_{nm} \otimes_{\mathbb{C}S_n[S_m]} V^{\otimes n}$$

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sending $1 \otimes v \mapsto 1 \otimes v$. Thus g^n is bounded by nN via part (c), so $f[g]$ is also via part (b). ■

Proposition 4.3 helps us bound the factors appearing in the Definition (2.6) of $\mathrm{Lie}_\lambda, \mathrm{W}_\lambda$.

Theorem 4.4. For $m \geq 1$, one has the following column bounds.

(a) All of $h_m[\ell_n], h_m[\pi_n], e_m[\ell_n], e_m[\pi_n]$, are bounded by $m(n-1)$ if $n \geq 3$.
(b) $h_m[\ell_2]$ is sharply bounded by m.
(c) $e_m[\pi_2]$ is sharply bounded by $m+1$.
(d) $\mathrm{Lie}_\lambda, \mathrm{W}_\lambda$ are bounded by $i, i+1$, resp. when λ has no parts of size 1, and $\mathrm{rank}(\lambda) = i$.
(e) Writing $\widehat{\mathrm{Lie}}^i, \widehat{\mathrm{W}}^i$ as $\sum_\mu c_\mu \chi^\mu$, one has $n_0 = \max\{|\mu| + \mu_1 : c_\mu \neq 0\} = 3i, 3i+1$, resp. □

Proof. Part (a) reduces, via Proposition 4.3(d), to the case $m = 1$, that is, showing ℓ_n, π_n are both bounded by $n-1$. To see this, note that χ_ζ is the trivial character of C_n only for $n = 1$, and the sign character of C_n only for $n = 2$. Thus for $n \geq 3$, one has

$$\left\langle \chi_\zeta\, , \mathbf{1}_{S_n} \downarrow^{S_n}_{C_n} \right\rangle = 0 = \left\langle \chi_\zeta\, , \epsilon_{S_n} \downarrow^{S_n}_{C_n} \right\rangle.$$

Frobenius reciprocity then shows that $\mathrm{Lie}_{(n)} = \chi_\zeta \uparrow^{S_n}_{C_n}$ has both $\ell_n = \mathrm{ch}(\mathrm{Lie}_{(n)})$ and $\pi_n = \mathrm{ch}(\epsilon_{S_n} \otimes \mathrm{Lie}_{(n)})$ bounded by $n-1$.

Parts (b) and (c) follow from two identities of Littlewood [36, Exercise 7.28(c), 7.29(b)]:

$$h_m[\ell_2] = h_m[e_2] = \sum_\lambda s_\lambda, \tag{40}$$

$$e_m[\pi_2] = e_m[h_2] = \sum_\lambda s_\lambda, \tag{41}$$

where both sums are over partitions λ of $2m$, but the first sum is over those having only even column sizes, and the second sum over those having Frobenius notation of the form $\lambda = (\alpha_1 + 1 \cdots \alpha_r + 1 | \alpha_1 \cdots \alpha_r)$. The first sum is bounded by m, and sharply so because $s_{(m,m)}$ occurs within it; the second is bounded by $m+1$, sharply because $s_{(m+1,1^m)}$ occurs within it.

Part (d) for $\lambda = 2^{m_2} 3^{m_3} \cdots$ reduces to the case where $\lambda = i^{m_i}$ has only one part size, using the definitions (25) and (26) of $\mathrm{Lie}_\lambda, \mathrm{W}_\lambda$, together with Proposition 4.3(c), and

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the additivity $\mathrm{rank}(\lambda) = \sum_i m_i(i-1) = \sum_i \mathrm{rank}(i^{m_i})$. When $\lambda = i^{m_i}$, the assertions follow from part (a) for $i \geq 3$, and parts (b) and (c) for $i = 2$.

For part (e), note that $\widehat{\mathrm{Lie}}^i, \widehat{\mathrm{W}}^i$ are the sums of $\mathrm{Lie}_\lambda, \mathrm{W}_\lambda$ over all partitions λ of rank i with no parts of size 1. Proposition 2.8 showed that all such λ have $|\lambda| \leq 2i$. Thus, the irreducibles χ^μ that can occur within the expansions of these $\mathrm{Lie}_\lambda, \mathrm{W}_\lambda$ have $|\mu| = |\lambda| \leq 2i$. They also have $\mu_1 \leq i, i+1$, respectively, by part (d). Hence they satisfy $|\mu| + \mu_1 \leq 3i, 3i+1$. The sharpness comes from parts (b) and (c), as $\lambda = (2^i)$ is a partition of rank i, and

$$\begin{aligned} \mathrm{ch}(\mathrm{Lie}_{(2^i)}) &= h_i[\ell_2] \quad \text{has} \quad n_0 = 2i + i = 3i, \\ \mathrm{ch}(\mathrm{W}_{(2^i)}) &= e_i[\pi_2] \quad \text{has} \quad n_0 = 2i + (i+1) = 3i+1. \end{aligned}$$ ■

5 Proof of Theorem 1.1

Recall the statement of the theorem.

Theorem 1.1. Fix integers $d \geq 2$ and $i \geq 1$. Then $H^i(\mathrm{Conf}(n, \mathbb{R}^d))$ vanishes unless $d-1$ divides i, in which case, it stabilizes sharply at

$$\begin{cases} n = 3\frac{i}{d-1} & \text{for } d \text{ odd,} \\ n = 3\frac{i}{d-1} + 1 & \text{for } d \text{ even.} \end{cases}$$

In particular, $H^i(\mathrm{Conf}(n, \mathbb{R}^2))$ stabilizes sharply at $n = 3i+1$. □

Proof. The vanishing assertion is part of Theorem 2.7. Using Corollary 2.10 to recast the cohomology $H^{i(d-1)}(\mathrm{Conf}(n, \mathbb{R}^d))$ as $M_n(\widehat{\mathrm{Lie}}^i), M_n(\widehat{\mathrm{W}}^i)$ when d is odd, even, it remains to show that the latter S_n-representations stabilize sharply at $3i, 3i+1$, respectively. But this follows from Lemma 2.2 applied to $\widehat{\mathrm{Lie}}^i, \widehat{\mathrm{W}}^i$ using Theorem 4.4(e). ■

Theorem 1.1 can also be deduced from the following more precise result on the stabilization as a function of n of individual irreducible multiplicities:

$$f_{i,\nu}(n) := \left\langle \chi^{(n-|\nu|,\nu)}, H^{i(d-1)}(\mathrm{Conf}(n, \mathbb{R}^d)) \right\rangle_{S_n}.$$

Theorem 5.1. Fix $i \geq 0$. Then $f_{i,\nu}(n)$ vanishes unless $|\nu| \leq 2i$ and becomes constant when

$$n \geq n_0 := \begin{cases} |\nu| + i & \text{for d odd,} \\ |\nu| + i + 1 & \text{for d even.} \end{cases}$$ □

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Proof. Let $\sum_\mu c_\mu \chi^\mu$ be the irreducible expansion of $\widehat{\mathrm{Lie}}^i, \widehat{\mathrm{W}}^i$ for d odd, even, respectively. Then Corollary 2.10 shows that

$$f_{i,\nu}(n) = \sum_\mu c_\mu \left\langle \chi^{(n-|\nu|,\nu)}, M_n(\chi^\mu) \right\rangle = \sum_\mu c_\mu,$$

where Lemma 2.3 tells us that the last sum runs over all partitions μ with

- $c_\mu > 0$,
- $\nu \subseteq \mu$,
- μ/ν a horizontal strip,
- $n \geq |\nu| + \mu_1$.

For the vanishing, note $c_\mu > 0$ and Proposition 2.8 show $|\mu| \leq 2i$, hence $\nu \subseteq \mu$ forces $|\nu| \leq 2i$.

For the second assertion, note that as the c_μ are nonnegative, the last sum becomes constant as a function of n once n reaches the maximum of all $|\nu| + \mu_1$ among those μ having $c_\mu \neq 0$ with μ/ν a horizontal strip. Theorem 4.4(d) implies $c_\mu = 0$ unless $\mu_1 \leq i$ for d odd, or $\mu_1 \leq i+1$ for d even. Thus the sum is constant for $n \geq |\nu| + i$ when d is odd, and for $n \geq |\nu| + i + 1$ when d is even. ■

Remark 5.2. Stabilization for the multiplicity of $\chi^{(n)}, \chi^{(n-1,1)}$ within the Whitney homology of Π_n (relevant for d even) was noted already by Sundaram [39, Proposition 1.9, Corollary 2.3(i)], who observed that $\langle \chi^{(n)}, WH_i(\Pi_n) \rangle = 0$ for $n \geq 2$, and $\langle \chi^{(n-1,1)}, WH_i(\Pi_n) \rangle = 2$ for $n \geq 3$. □

Along similar lines, we next obtain an improvement of the stable range in [4, Theorem 1], where Church, *et al.* showed $\left\langle \chi_P, H^i(\mathrm{Conf}(n,\mathbb{R}^2)) \right\rangle_{S_n}$ is constant for $n \geq \deg(P) + 2i$.

Theorem 5.3. Fix $P = P(x_1, x_2, \ldots)$ in $\mathbb{Q}[x_1, x_2, \ldots]$. Then the polynomial character χ_P on S_n has $\left\langle \chi_P, H^{i(d-1)}(\mathrm{Conf}(n,\mathbb{R}^d)) \right\rangle_{S_n}$ constant for

$$n \geq \begin{cases} \max\{2\deg(P), \deg(P) + i\} & \text{if } d \text{ is odd,} \\ \max\{2\deg(P), \deg(P) + i + 1\} & \text{if } d \text{ is even.} \end{cases}$$ □

Proof. Since Corollary 2.10 expresses $H^{i(d-1)}(\mathrm{Conf}(n,\mathbb{R}^d)) = M_n(\chi)$, with $\chi = \widehat{\mathrm{Lie}}^i, \widehat{\mathrm{W}}^i$ for d odd, even, and Theorem 4.4(d) shows χ is bounded by $i, i+1$ for d odd, even, the result then follows directly from Theorem 3.4. ■

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We close this section by observing the following consequence of Theorem 1.1.

Corollary 5.4. The rank-selected homology $\beta_{\{1,\ldots,i\}}(\Pi_n)$ stabilizes sharply at $n = 3i+1$. □

Proof. Induct on i, with trivial base cases $i = 0, 1$. Proposition 2.12 gives the expression

$$\beta_{\{1,\ldots,i\}}(\Pi_n) = WH_i(\Pi_n) - \beta_{\{1,\ldots,i-1\}}(\Pi_n).$$

As $WH_i(\Pi_n)$ stabilizes sharply at $n = 3i+1$ (Theorem 1.1) and $\beta_{\{1,\ldots,i-1\}}(\Pi_n)$ stabilizes beyond $n \geq 3(i-1)+1 = 3i-2$ by induction, $\beta_{\{1,\ldots,i\}}(\Pi_n)$ stabilizes sharply at $n = 3i+1$. ■

6 Proof of Theorem 1.2

Recall the statement of the theorem.

Theorem 1.2. Letting $\widehat{\mathrm{Lie}}_0 := \widehat{\mathrm{W}}_0 := \mathbf{1}_{S_0}, \widehat{\mathrm{Lie}}_1 := \widehat{\mathrm{W}}_1 := 0$ by convention, then for $n \geq 1$,

$$\widehat{\mathrm{Lie}}_n = \widehat{\mathrm{Lie}}_{n-1} \uparrow_{S_{n-1}}^{S_n} + (-1)^n \epsilon_n,$$
$$\widehat{\mathrm{W}}_n = \widehat{\mathrm{W}}_{n-1} \uparrow_{S_{n-1}}^{S_n} + (-1)^n \tau_n,$$

where ϵ_n is the sign character of S_n, and τ_n is this virtual S_n-character of degree 1:

$$\tau_n := \begin{cases} \mathbf{1}_{S_n} & \text{for } n = 0,1,2,3, \\ \chi^{(3,1^{n-3})} - \chi^{(2,2,1^{n-4})} & \text{for } n \geq 4. \end{cases}$$ □

Proof. We will work instead with the symmetric functions

$$\begin{aligned} \kappa_n &:= \mathrm{ch}(\widehat{\mathrm{Lie}}_n) = \textstyle\sum_i \mathrm{ch}(\widehat{\mathrm{Lie}}^i_n) \qquad & \nu_n &:= \mathrm{ch}(\widehat{\mathrm{W}}_n) = \textstyle\sum_i \mathrm{ch}(\widehat{\mathrm{W}}^i_n), \\ \kappa &:= \kappa_0 + \kappa_1 + \kappa_2 + \cdots & \nu &:= \nu_0 + \nu_1 + \nu_2 + \cdots . \end{aligned} \tag{42}$$

Abusing notation, let τ_n also denote the Frobenius image $\mathrm{ch}(\tau_n)$ in Λ_n, that is, $\tau_n := h_n = s_{(n)}$ for $0 \leq n \leq 3$ and $\tau_n = s_{(3,1^{n-3})} - s_{(2,2,1^{n-4})}$ for $n \geq 4$. The theorem then asserts

$$\begin{aligned} \kappa &= p_1\kappa + 1 - e_1 + e_2 - e_3 + \cdots , \\ \nu &= p_1\nu + 1 - \tau_1 + \tau_2 - \tau_3 + \cdots . \end{aligned} \tag{43}$$

To show this, start by setting $u = 1$ in (37), giving

$$\kappa = \widehat{\mathrm{L}}(1) \qquad = \exp\left(-\sum_{m\geq 1} \frac{p_m}{m}\right) \prod_{\ell \geq 1} (1-p_\ell)^{-a_\ell(1)} \qquad = \frac{1}{1-p_1} \sum_{k\geq 0} (-1)^k e_k, \tag{44}$$

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$$\nu = \widehat{W}(1) \qquad = \exp\left(-\sum_{m\geq 1}\frac{p_m}{m}\right)\prod_{\ell\geq 1}\left(1+(-1)^{\ell}p_\ell\right)^{a_\ell(-1)} \qquad = \frac{1+p_2}{1-p_1}\sum_{k\geq 0}(-1)^k e_k. \tag{45}$$

The last equality on each line applied the following consequence of (9) at $u=-1$

$$\exp\left(-\sum_{m\geq 1}\frac{p_m}{m}\right) = 1 - e_1 + e_2 - e_3 + \cdots = \sum_{k\geq 0}(-1)^k e_k,$$

along with these Möbius function calculations:

$$a_\ell(1) = \frac{1}{\ell}\sum_{d|\ell}\mu(d) = \begin{cases} +1 & \text{if } \ell = 1, \\ 0 & \text{if } \ell \geq 2. \end{cases}$$

$$a_\ell(-1) = \frac{1}{\ell}\sum_{d|\ell}\mu(d)(-1)^{\frac{\ell}{d}} = \frac{1}{\ell}\left(\sum_{\substack{d|\ell \\ \frac{\ell}{d}\text{ even}}}\mu(d) - \sum_{\substack{d|\ell \\ \frac{\ell}{d}\text{ odd}}}\mu(d)\right) = \begin{cases} -1 & \text{if } \ell = 1, \\ +1 & \text{if } \ell = 2, \\ 0 & \text{if } \ell \geq 3. \end{cases}$$

Then (44) can be rewritten

$$(1-p_1)\kappa = 1 - e_1 + e_2 - e_3 + \cdots,$$

which is equivalent to the first equation in (43). Meanwhile (45) can be rewritten

$$(1-p_1)\nu = (1 - e_1 + e_2 - e_3 + \cdots)(1+p_2) = 1 - e_1 + \sum_{n\geq 2}(-1)^n(e_n + p_2 e_{n-2}). \tag{46}$$

The identity (12) lets one identify the far right terms $e_n + p_2 e_{n-2}$ as τ_n for $n \geq 4$:

$$\begin{aligned}\tau_n = s_{(3,1^{n-3})} - s_{(2,2,1^{n-4})} &= \det\begin{bmatrix} e_{n-2} & e_{n-1} & e_n \\ 1 & e_1 & e_2 \\ 0 & 1 & e_1 \end{bmatrix} - \det\begin{bmatrix} e_{n-2} & e_{n-1} \\ e_1 & e_2 \end{bmatrix} \\ &= e_n + (e_1^2 - 2e_2)e_{n-2} = e_n + p_2 e_{n-2}.\end{aligned} \tag{47}$$

But one also has $\tau_2 = h_2 = e_2 + p_2$ and $\tau_1 = h_1 = e_1$, so (46) becomes the following identity, equivalent to the second equation in (43):

$$(1-p_1)\nu = 1 - \tau_1 + \tau_2 - \tau_3 + \tau_4 - \cdots. \qquad \blacksquare$$

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Remark 6.1. The authors thank S. Sam for pointing out the following more uniform rephrasing of the definition for the symmetric function τ_n. One has

$$\tau_n = \omega\left(s_{n-2,1,1} - s_{n-2,2}\right),$$

for *all* $n \geq 0$, not just $n \geq 4$, if one broadens the definition of the Schur function s_α to α in $\mathbb{Z}^\ell$ in a standard way via the Jacobi–Trudi determinant:

$$s_\alpha := \det\left(h_{\alpha_i - i + j}\right)_{i,j=1}^{\ell}, \quad \text{where } h_0 := 1 \text{ and } h_i := 0 \text{ for } i < 0.$$

See, for example Tamvakis [44, Sections 2.2 and 3.5]. This convention is consistent with Bott's vanishing theorem for cohomology of line bundles on flag manifolds (see, e.g., Weyman [49, Corollay 4.1.7]): setting $\rho := (\ell-1, \ell-2, \ldots, 1, 0)$, then $s_\alpha = 0$ unless there is a partition λ and (unique) w in S_ℓ with $\alpha + \rho = w(\lambda + \rho)$, in which case $s_\alpha = \epsilon(w) s_\lambda$. □

Remark 6.2. As mentioned in the introduction, Désarménien and Wachs [10] first studied the symmetric function denoted κ_n which appears in the above proof. It was later noted by Reiner and Webb [28, Theorem 2.4] that $\omega(\kappa_n)$ is the Frobenius characteristic of the S_n-representation on the homology $H_n(M)$ of the *complex of injective words*. They noted [28, Proposition 2.2] that it satisfies the following recurrence equivalent to (1):

$$\mathrm{ch}(H_n(M)) = p_1 \mathrm{ch}(H_{n-1}(M)) + (-1)^n h_n.$$

Hanlon and Hersh [21, Theorem 2.3] used the *Eulerian idempotents* in $\mathbb{Q}S_n$ to further decompose the homology $H_n(M)$ of the complex of injective words into a so-called *Hodge decomposition* $H_n(M) = \bigoplus_{i=1}^n H_n^{(i)}(M)$. The summands $H_n^{(i)}(M)$ are S_n-representations having degree equal to the number of derangements in S_n with i cycles. In fact, one can prove an isomorphism $H_n^{(n-i)}(M) \cong \epsilon_n \otimes \widehat{\mathrm{Lie}}_n^{n-i}$ by comparing their formula [21, bottom of p. 118]

$$\sum_i \mathrm{ch}(H_n^{(i)}(M)) u^i = \exp\left(\sum_{m \geq 1} \frac{p_m (-u)^m}{m}\right) \prod_{\ell \geq 1} (1 + (-1)^\ell p_\ell)^{-a_\ell(u)}$$

with the product formula (37), and using $\omega(p_m) = (-1)^{m-1} p_m$. □

Remark 6.3. To further tighten the analogy between recurrences (1) and (2), note that the sequence of symmetric functions $\{\tau_n\}$ in Theorem 1.2 shares the following property with $\{e_n\}$ (or $\{h_n\}$): one has $\frac{\partial}{\partial p_1} \tau_n = \tau_{n-1}$, using, for example, the expression $\tau_n = e_n + p_2 e_{n-2}$

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appearing in (47). In particular, their corresponding virtual S_n-representations $T_n :=$ $\mathrm{ch}^{-1}(\tau_n)$ satisfy $T_n \downarrow^{S_n}_{S_{n-1}} = T_{n-1}$, and they all have (virtual) degree 1. □

7 Proof of Theorem 1.3

We next use Theorem 1.2 to derive an explicit irreducible expansion for $\widehat{\mathrm{W}}_n$. An analogous expansion is already known for the Désarménien–Wachs derangement symmetric function κ_n and the homology $H_n(M)$ of the complex of injective words discussed in Remark 6.2. These expansions involve the notions of tableaux and ascents, which we now recall.

Definition 7.1. A *standard Young tableau* Q of shape λ with $|\lambda| = n$ is a filling of the cells of the Ferrers diagram of λ with $\{1, 2, \ldots, n\}$ bijectively, increasing left-to-right in rows, and top-to-bottom in columns. Call i an *ascent* of Q if $i+1$ lies in a weakly higher row than i in Q, or if $i = n$ the size (For this convention, it helps to imagine Q extended by entries $n+1, n+2, \ldots$ at the end of its first row.) of Q. □

Example 7.2.

$$Q = \begin{array}{|c|c|c|c|}\hline 1&3&6&8\\\hline 2&4&7\\\cline{1-3} 5\\\cline{1-1}\end{array}$$

is a standard Young tableau of shape $\lambda = (4, 3, 1)$ having ascents $\{2, 5, 7, 8\}$ □

Definition 7.3. A *desarrangement tableau* is a standard tableau Q with even first ascent (Wachs dubbed the permutations w having even first ascent *desarrangements*. These are the permutations whose Robinson–Schensted recording tableau Q is a desarrangement tableau as defined here.). □

Definition 7.4. A *Whitney-generating tableau* is a standard tableau Q that either has size $n \leq 3$ and one of the following forms

$$Q = \varnothing, \quad Q = \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \quad Q = \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array},$$

or has the restriction $Q|_{\{1,2,3,4\}}$ to its first four values taking one of the following forms

$$\left\{ T_1 = \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}, \quad T_2 = \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}, \quad T_3 = \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}, \quad T_4 = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array} \right\}$$

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with the following further restrictions in the cases $Q|_{\{1,2,3,4\}} = T_3, T_4$:

(a) If $Q|_{\{1,2,3,4\}} = T_3$ then the first ascent (Recall from Definition 7.1 that n is always an ascent of Q, so this first ascent exists.) $k \geq 4$ is odd, that is, Q contains the entries shown below for some odd $k \geq 5$:

$$\begin{array}{cccc} 1 & 2 & \cdots & \\ 3 & 4 & \cdots & \\ 5 & \cdots & k+1 & \\ 6 & & & \\ \vdots & & & \\ k-1 & & & \\ k & & & \end{array}$$

In particular, $Q \neq T_3$ itself.

(b) If $Q|_{\{1,2,3,4\}} = T_4$ then the first ascent (As in the previous footnote, this first ascent exists.) $k \geq 4$ is even, that is, Q contains the entries shown below for some even $k \geq 4$:

$$\begin{array}{ccccc} 1 & 2 & 3 & \cdots & \\ 4 & \cdots & & & \\ 5 & \cdots & k+1 & & \\ \vdots & \ddots & & & \\ k-1 & & & & \\ k & & & & \end{array}$$

$\square$

Theorem 1.3. One has the following irreducible decompositions

$$\widehat{\mathrm{Lie}}_n = \sum_{Q} \chi^{\mathrm{shape}(Q)},$$

$$\widehat{\mathrm{W}}_n = \sum_{Q} \chi^{\mathrm{shape}(Q)}$$

in which the sums in (3), (4), respectively, range over the set of desarrangement tableaux, Whitney-generating tableaux Q of size n. $\square$

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That is, the desarrangement (resp. Whitney-generating) tableaux predicts the sum across each row of Table A1 (resp. Table A2). Here are both kinds of tableaux up to size $n = 5$, for comparison to Tables A1 and A2:

n	Desarrangement tableaux of size n	Whitney-generating tableaux of size n
0	$\varnothing$	$\varnothing$
1		
2	$\begin{ytableau}1\\2\end{ytableau}$	$\begin{ytableau}1&2\end{ytableau}$
3	$\begin{ytableau}1&3\\2\end{ytableau}$	$\begin{ytableau}1&2\\3\end{ytableau}$
4	$\begin{ytableau}1&3\\2&4\end{ytableau}$ $\begin{ytableau}1\\2\\3\\4\end{ytableau}$ $\begin{ytableau}1&3&4\\2\end{ytableau}$ $\begin{ytableau}1&3\\2\\4\end{ytableau}$	$\begin{ytableau}1&2\\3\\4\end{ytableau}$ $\begin{ytableau}1&2&3\\4\end{ytableau}$ $\begin{ytableau}1&2&4\\3\end{ytableau}$
5	$\begin{ytableau}1&3\\2\\4\\5\end{ytableau}$ $\begin{ytableau}1&3\\2&4\\5\end{ytableau}$ $\begin{ytableau}1&3&4\\2\\5\end{ytableau}$ $\begin{ytableau}1&3&4\\2&5\end{ytableau}$ $\begin{ytableau}1&3&4&5\\2\end{ytableau}$ $\begin{ytableau}1&3&5\\2&4\end{ytableau}$ $\begin{ytableau}1&3&5\\2\\4\end{ytableau}$ $\begin{ytableau}1&3\\2&5\\4\end{ytableau}$ $\begin{ytableau}1&5\\2\\3\\4\end{ytableau}$	$\begin{ytableau}1&2\\3\\4\\5\end{ytableau}$ $\begin{ytableau}1&2\\3&5\\4\end{ytableau}$ $\begin{ytableau}1&2&5\\3\\4\end{ytableau}$ $\begin{ytableau}1&2\\3&4\\5\end{ytableau}$ $\begin{ytableau}1&2&3\\4&5\end{ytableau}$ $\begin{ytableau}1&2&3&5\\4\end{ytableau}$ $\begin{ytableau}1&2&4\\3\\5\end{ytableau}$ $\begin{ytableau}1&2&4\\3&5\end{ytableau}$ $\begin{ytableau}1&2&4&5\\3\end{ytableau}$

Proof of Theorem 1.3. The theorem is equivalent to the following expansions for the symmetric functions κ_n, ν_n defined in (42):

$$\kappa_n = \sum_Q s_{\mathrm{shape}(Q)}, \tag{48}$$

$$\nu_n = \sum_Q s_{\mathrm{shape}(Q)} \tag{49}$$

with the sums ranging over the desarrangement and Whitney-generating tableaux Q of size n, respectively.

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It was shown by Désarménien and Wachs [10] and by Reiner and Webb [28, Proposition 2.3] that

$$\begin{aligned} \kappa_n &= \sum_{Q} s_{\mathrm{shape}(Q)}, \text{ or equivalently,} \\ \mathrm{ch}(H_n(M)) &= \sum_{Q} s_{\mathrm{shape}(Q)^t}, \end{aligned} \tag{50}$$

where Q runs over all standard Young tableaux of size n whose first ascent is even. Thus, it only remains to prove the analogous expansion for $\widehat{\mathrm{W}}_n$.

Let $\tilde{\nu}_n$ be the sum on the right in (49). We will check that $\tilde{\nu}_n = \nu_n$ by induction on n. The base cases where $n \leq 4$ are easily checked. In the inductive step for $n \geq 5$, one need only check that $\tilde{\nu}_n$ satisfies the recurrence from Theorem 1.2, that is

$$\tilde{\nu}_n = p_1 \tilde{\nu}_{n-1} + (-1)^n \tau_n. \tag{51}$$

By the special case of the *Pieri rule* (14) for multiplying a Schur function s_λ by $p_1 (= s_{(1)})$, one wants to show that if one adds a new entry n to all the Whitney-generating tableaux of size $n-1$, in all possible corner cell locations, one obtains a set of tableaux (call it $\mathcal{T}_n$) that *almost* contains exactly one copy of each Whitney-generating tableaux of size n. The exceptions come from considering these two families of tableaux, $A(n)$ for $n \geq 3$, and $B(n)$ for $n \geq 4$:

$$A(n) := \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} \vdots \\ \cline{1-1} n-1 \\ \cline{1-1} n \\ \cline{1-1} \end{array} \qquad B(n) := \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1} \vdots \\ \cline{1-1} n-1 \\ \cline{1-1} n \\ \cline{1-1} \end{array}.$$

Note that $A(n)|_{\{1,2,3,4\}} = T_4$, that $B(n)|_{\{1,2,3,4\}} = T_3$, and that $s_{\mathrm{shape}(A(n))} - s_{\mathrm{shape}(B(n))} = \tau_n$. We explain here why the $(-1)^n \tau_n$ term in the theorem exactly accounts for the discrepancy resulting from these exceptions.

First assume n is even and at least 4. Then $B(n-1)$ is Whitney-generating, but adding n to the bottom of its first column produces $B(n)$ which is not Whitney-generating. However, removing $B(n)$ from the set $\mathcal{T}_n$ and replacing it with $A(n)$ produce

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a set $\mathcal{T}_n \setminus \{B(n)\} \cup \{A(n)\}$ that has each Whitney-generating tableau of size n exactly once. This replacement models adding τ_n.

Next assume n is odd and at least 5. Then $A(n-1)$ is Whitney-generating, but adding n to the bottom of the first column of $A(n-1)$ produces $A(n)$ which is not Whitney-generating. Similarly to the previous case, removing $A(n)$ from the set $\mathcal{T}_n$ and replacing it with $B(n)$ produce a set $\mathcal{T}_n \setminus \{A(n)\} \cup \{B(n)\}$ that has each Whitney-generating tableau of size n exactly once. This replacement models subtracting τ_n.

This shows that $\tilde{\nu}_n$ satisfies the recurrence (51), completing the proof of the theorem. ∎

8 Proof of Theorem 1.4

Recall the statement of the theorem.

Theorem 1.4. For $n \geq 2$ and $i \geq 1$, one has an isomorphism of S_{n-1}-representations

$$\widehat{\mathrm{Lie}}_n^i \downarrow \cong \left(\widehat{\mathrm{Lie}}_{n-1}^{i-1} \downarrow \quad \oplus \quad \widehat{\mathrm{Lie}}_{n-2}^{i-1}\right) \uparrow,$$

$$\widehat{\mathrm{W}}_n^i \downarrow \cong \left(\widehat{\mathrm{W}}_{n-1}^{i-1} \downarrow \quad \oplus \quad \widehat{\mathrm{W}}_{n-2}^{i-1}\right) \uparrow,$$

where $\uparrow$ and $\downarrow$ are induction $(-)\uparrow_{S_n}^{S_{n+1}}$, restriction $(-)\downarrow_{S_{n-1}}^{S_n}$ applied to S_n-representations. □

Recall from (13) that $\downarrow, \uparrow$ correspond via the Frobenius map ch to the operations of $\frac{\partial}{\partial p_1}$ and multiplying by p_1 on symmetric functions. We will prove Theorem 1.4 therefore, by applying $\frac{\partial}{\partial p_1}$ to (36), (37). To this end, extend $\frac{\partial}{\partial p_1}$ as an operator on Λ to one on $\Lambda[[u]]$ via

$$\frac{\partial}{\partial p_1} \sum_n f_n u^n := \sum_n \left(\frac{\partial}{\partial p_1} f_n\right) u^n.$$

Proof of Theorem 1.4. We give the proof for the second recurrence in the theorem by applying $\frac{\partial}{\partial p_1}$ to $\widehat{\mathrm{W}}(u)$; the proof of the first recurrence is exactly the same using $\widehat{\mathrm{Lie}}(u)$ instead.

Recall that (37) factors $\widehat{\mathrm{W}}(u) = H(u)^{-1} W(u)$ where

$$\begin{aligned} H(u)^{-1} &= \exp\left(-\textstyle\sum_{m\geq 1} \frac{p_m u^m}{m}\right) &&= \exp(-p_1 u) \cdot \exp\left(-\textstyle\sum_{m\geq 2} \frac{p_m u^m}{m}\right) \\ W(u) &= \textstyle\prod_{\ell\geq 1} \left(1 + (-1)^\ell p_\ell\right)^{a_\ell(-u)} &&= (1-p_1)^{-u} \cdot \textstyle\prod_{\ell \geq 2} \left(1+(-1)^\ell p_\ell\right)^{a_\ell(-u)}. \end{aligned}$$

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These expressions show that

$$\frac{\partial H(u)^{-1}}{\partial p_1} = -u \cdot H(u)^{-1},$$
$$\frac{\partial W(u)}{\partial p_1} = \frac{u}{1-p_1} \cdot W(u),$$

and hence by the Leibniz rule applied to $\widehat{\mathrm{W}}(u) = H(u)^{-1}W(u)$ one has

$$\begin{aligned}\frac{\partial \widehat{\mathrm{W}}(u)}{\partial p_1} &= H(u)^{-1}\frac{\partial W(u)}{\partial p_1} + \frac{\partial H(u)^{-1}}{\partial p_1} W(u)\\ &= \frac{u}{1-p_1} \cdot H(u)^{-1}W(u) - u \cdot H(u)^{-1}W(u)\\ &= \frac{up_1}{1-p_1}\widehat{\mathrm{W}}(u).\end{aligned}$$

From here, an easy algebraic manipulation reformulates this as follows:

$$\frac{\partial \widehat{\mathrm{W}}(u)}{\partial p_1} = p_1 \frac{\partial \widehat{\mathrm{W}}(u)}{\partial p_1} + up_1\widehat{\mathrm{W}}(u). \tag{52}$$

This is an identity in $\Lambda[[u]]$. Extracting terms of appropriate degree from (52), that is, taking the Λ_n homogeneous component within the coefficient of u^{n+1-i}, yields

$$\frac{\partial}{\partial p_1}\mathrm{ch}(\widehat{\mathrm{W}}^i_{n+1}) = p_1 \cdot \frac{\partial}{\partial p_1}\mathrm{ch}(\widehat{\mathrm{W}}^{i-1}_n) + p_1 \cdot \mathrm{ch}(\widehat{\mathrm{W}}^{i-1}_{n-1}), \tag{53}$$

which is equivalent to the assertion of the theorem via (13). ■

9 Proof of Theorem 1.7

Recall the statement of the theorem.

Theorem 1.7. As virtual characters, for $n \geq 2$ one has

$$\sum_{i \geq 0} (-1)^i \widehat{\mathrm{W}}^i_n = (-1)^{n-1}\chi^{(2,1^{n-2})}.$$ □

Proof. Setting $u = -1$ in Corollary 37, and noting $a_\ell(1) = \frac{1}{\ell}\sum_{d|\ell}\mu(d) = 0$ for $\ell \geq 2$ gives

$$\begin{aligned}-\widehat{\mathrm{W}}(-1) = \sum_{n\geq 0}\left(\sum_{k\geq 0}\mathrm{ch}(\widehat{\mathrm{W}}^k_n)(-1)^{n-1-k}\right) &= -\exp\left(-\sum_{m\geq 1}\frac{p_m(-1)^m}{m}\right)\prod_{\ell\geq 1}\left(1+(-1)^\ell p_\ell\right)^{a_\ell(1)}\\ &= -\exp\left(\sum_{m\geq 1}\frac{(-1)^{m-1}p_m}{m}\right)(1-p_1).\end{aligned}$$

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Applying (9) at $u = 1$, and noting that $p_1 = e_1$, this last expression equals

$$(1 + e_1 + e_2 + \cdots)(e_1 - 1) = -1 + (e_1 - e_1) + (e_1 e_1 - e_2) + (e_1 e_2 - e_3) + (e_1 e_3 - e_4) + \cdots$$
$$= -1 + \sum_{n \geq 2} s_{(2,1^{n-2})},$$

where the last step applied (12) to rewrite $e_1 e_{n-1} - e_n = s_{(2,1^{n-2})}$ for $n \geq 2$. ■

In addition to Theorem 1.7, we point out a simple fact about the S_n-characters $\{\widehat{\mathrm{W}}^i_n\}_{i=1}^{n-1}$ closely related to Conjecture 1.5; it follows, for example, from Sundaram [39, Corollary 2.3(ii)].

Proposition 9.1. For $n \geq 2$ one has

$$\langle \chi^{(2,1^{n-2})}, \widehat{\mathrm{W}}^i_n \rangle = \begin{cases} 0 & \text{for } 0 \leq i \leq n-2, \\ 1 & \text{for } i = n-1. \end{cases}$$

Thus, any cochain complex $(\widehat{\mathrm{W}}^\bullet_n, d)$ would have one copy of $\chi^{(2,1^{n-2})}$ in the homology $H^{n-1}(\widehat{\mathrm{W}}^\bullet_n)$. □

This unique copy of $\chi^{(2,1^{n-2})}$ inside π_n predicted by Proposition 9.1 is distinguished in at least two ways. On one hand it is the top filtration factor in Reutenauer's derived series for the free Lie algebra, as discussed in Reutenauer [29] and Sundaram and Wachs [42, p. 951].

On the other hand, Lehrer and Solomon [23] model $WH_i(\Pi_n)$ via the *Orlik–Solomon algebra* of type A_{n-1}, that is, the quotient $A(n) = E/I$ of an exterior algebra E on generators $\{a_{ij}\}_{1 \leq i < j \leq n}$, by the ideal I having generators

$$a_{ij} a_{ik} - a_{ij} a_{jk} + a_{ik} a_{jk} = 0 \quad \text{for } 1 \leq i < j < k \leq n.$$

This gives a skew-commutative graded algebra $A = \bigoplus_{i=0}^{n-1} A^i$, carrying an S_n-representation defined by $w(e_{ij}) = e_{w(i),w(j)}$, and for which $WH_i(\Pi_n) \cong A^i$. In particular, $A^{n-1} \cong \pi_n$. It is then not hard to show that the images of these n monomials

$$m^{(i)} := a_{1,i} a_{2,i} \cdots a_{i-1,i} a_{i,i+1}, a_{i,i+2} \cdots a_{i,n-1} a_{i,n} \text{ for } 1 \leq i \leq n, \tag{54}$$

satisfy a single relation $\sum_{i=1}^n (-1)^i m^{(i)} = 0$, and span an $(n-1)$-dimensional S_n-stable subspace of A^{n-1}, carrying the unique copy of $\chi^{(2,1^{n-2})}$ predicted by Proposition 9.1.

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10 Proof of Theorem 1.8

Recall the statement of the theorem.

Theorem 1.8. For a subset S of positive integers with $\max(S) = i$, the sequence $\beta_S(\Pi_n)$ stabilizes beyond $n = 4i$. Furthermore, when $S = \{i\}$, it stabilizes sharply at $n = 4i$. $\square$

We break this into two statements, Theorem 10.1 and Proposition 10.2 below, addressing α_S, β_S simultaneously.

Theorem 10.1. For $S \subset \{1, 2, \ldots, n-2\}$ with $i = \max(S)$, both $\{\alpha_S(\Pi_n)\}, \{\beta_S(\Pi_n)\}$ stabilize beyond $n = 4i$. $\square$

See Sundaram [41, Section 5], as well as Hanlon–Hersh [20, Theorem 2.5], Stanley [37, p. 152], and Sundaram [39, Remark 4.10.2], for some related stability results on $\alpha_S(\Pi_n), \beta_S(\Pi_n)$.

Proof of Theorem 10.1. Since (28) expresses $\beta_S(\Pi_n)$ as an alternating sum of $\alpha_T(\Pi_n)$ with $\max(T) \leq \max(S)$, it suffices to prove the desired stability bound for $\alpha_S(\Pi_n)$ for each S. Since $\alpha_S(\Pi_n)$ is the S_n-permutation representation on the S_n-orbits of chains c passing through the rank set S, we are further reduced to understanding each of the transitive coset representations $\mathbf{1}_G \uparrow_G^{S_n}$ where $G := \mathrm{Stab}_{S_n}(c)$, and showing that they stabilize beyond $n = 4i$.

To this end, choose a representative chain c within each S_n-orbit so that the top element π at rank i in c has as the union of its nonsingleton blocks some initial segment of n_0 elements $\{1, 2, \ldots, n_0\}$, along with singleton blocks $\{n_0+1\}, \{n_0+2\}, \ldots, \{n-1\}, \{n\}$. It follows from Proposition 2.8 that $n_0 \leq 2i$ because $\mathrm{rank}(\pi) = i$. Restricting π to $\{1, 2, \ldots, n_0\}$ gives an element π_0 within the subposet Π_{n_0}, where here we consider the partition lattices as a tower $\Pi_1 \subset \Pi_2 \subset \Pi_3 \subset \cdots$, with Π_n included within Π_{n+1} as the subset of partitions having $\{n+1\}$ as a singleton block. Then the entire chain c in Π_n similarly restricts to a chain c_0 in Π_{n_0}, visiting the same rank set S, for which $G = \mathrm{Stab}_{S_n}(c) = G_0 \times S_{n-n_0}$ where $G_0 := \mathrm{Stab}_{S_{n_0}}(c_0)$. Hence

$$\mathbf{1}_G \uparrow_G^{S_n} \cong M_n(\chi) \text{ where } \chi := \mathbf{1}_{G_0} \uparrow_{G_0}^{S_{n_0}} .$$

As an S_{n_0}-character, χ is trivially bounded by n_0, and expands into irreducibles χ^λ with $|\lambda| = n_0$. Hence, Lemma 2.2 shows $M_n(\chi)$ stabilizes beyond $n = n_0 + n_0 \leq 2 \cdot 2i = 4i$. $\blacksquare$

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The next result shows that, in the worst case for S, the bound of Theorem 1.8 is tight.

Proposition 10.2. For $1 \le i \le n-2$, both $\alpha_{\{i\}}(\Pi_n), \beta_{\{i\}}(\Pi_n)$ stabilize sharply at $n = 4i$. □

Proof. Since (29) shows $\alpha_{\{i\}}(\Pi_n) = \beta_{\{i\}}(\Pi_n) + \chi^{(n)}$, the two representations will stabilize sharply at the same value of n, and we need only prove the assertion for $\alpha_{\{i\}}(\Pi_n)$. Similar to the analysis in the previous proof, $\alpha_{\{i\}}(\Pi_n)$ is a sum of representations $\mathbf{1} \uparrow_{\mathrm{Stab}_{S_n}(\pi)}^{S_n}$ for S_n-orbits of set partitions π in Π_n having rank i. In light of Theorem 10.1 we need only find one such set partition π_0 for which $\mathbf{1} \uparrow_G^{S_n}$, where $G := \mathrm{Stab}_{S_n}(\pi_0)$, stabilizes sharply at $n = 4i$.

We claim that any π_0 whose block size number partition is $(2^i, 1^{n-2i})$ will do the trick. To see this, note that, by the definition of plethysm given in Section 2.1, any such π_0 has

$$\mathbf{1}_G \uparrow_G^{S_n} \cong \mathbf{1}_{S_i}[\mathbf{1}_{S_2}] * \mathbf{1}_{S_{n-2i}} = M_n(\mathbf{1}_{S_i}[\mathbf{1}_{S_2}]).$$

On the other hand, applying ω to (40) and using (16) give the expansion

$$\mathrm{ch}(\mathbf{1}_{S_i}[\mathbf{1}_{S_2}]) = h_i[h_2] = \sum_\lambda s_\lambda$$

as λ runs through all partitions of $2i$ with all even parts. This means that it is bounded by $2i$, and sharply so because the single row $\lambda = (2i)$ occupies $2i$ columns. Thus, Lemma 2.2 shows that $\mathbf{1} \uparrow_G^{S_n}$ stabilizes sharply at $n = 4i$. ■

Conjecture 11.3 below suggests for each S the sharp onset of stabilization for $\beta_S(\Pi_n)$.

Remark 10.3. Theorem 1.8 does not preclude the possibility for individual irreducible multiplicities $\left\langle \chi^{(n-|\nu|,\nu)}, \beta_S(\Pi_n) \right\rangle_{S_n}$ for fixed S to stabilize sooner than $n \ge 4\max(S)$. □

11 Further Questions and Remarks

11.1 Cohomology of configuration spaces in $\mathbb{R}^d$ need not stabilize fastest

Church's main tool in [3] was the spectral sequence for the inclusion $\mathrm{Conf}(n, X) \hookrightarrow X^n$, converging to $H^*(\mathrm{Conf}(n, X))$, and in particular, Totaro's description [46] of its E_2-page. Totaro noted that $H^*(\mathrm{Conf}(n, \mathbb{R}^d))$ for configurations of points in $\mathbb{R}^d$ is S_n-isomorphic to

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the $p = 0$ column $E_2^{0,*}(\mathrm{Conf}(n,X) \hookrightarrow X^n)$ on the E_2-page, regardless of the choice of X; see [46, Lemma 1].

For this reason, the authors had wondered whether if, after fixing $i \geq 1$, among all connected orientable d-manifolds X with $\dim_{\mathbb{Q}} H^*(X) < \infty$, the cohomology $H^*(\mathrm{Conf}(n,X))$ stabilizes earliest for $X = \mathbb{R}^d$. They thank J. Wiltshire-Gordon for pointing out that this *fails* already when $i = 1$ with $d = 2$ when X is a surface of Genus 1, that is, a two-dimensional torus. Here a direct calculation shows that the two filtration factors $E_\infty^{1,0}$ and $E_\infty^{0,1}$ for $H^1(\mathrm{Conf}(n,X))$ have

- $E_\infty^{0,1} = \ker\left(E_2^{0,1} \xrightarrow{d_2} E_2^{1,2}\right)$ vanishing for $n \geq 2$, and
- $E_\infty^{1,0} = H^1(X^n) = M(\chi^{(1)} \oplus \chi^{(1)})$, stabilizing sharply at $n = 2$.

Thus $H^1(\mathrm{Conf}(n,X))$ stabilizes at $n = 2$, while $H^1(\mathrm{Conf}(n,\mathbb{R}^2))$ stabilizes (sharply) at $n = 4$.

11.2 Tableau model for $\widehat{\mathrm{Lie}}_n^i, \widehat{\mathrm{W}}_n^i$?

Question 11.1. Can one refine the tableau models in Theorem 1.3, for the S_n-irreducible decomposition of $\widehat{\mathrm{Lie}}_n, \widehat{\mathrm{W}}_n$, so as to give a tableau model for each $\widehat{\mathrm{Lie}}_n^i, \widehat{\mathrm{W}}_n^i$ individually? □

In other words, can one model each entry of Tables A1 and A2 via shapes of tableaux, not just the sum across each row? Perhaps the constraints provided by Theorems 1.5 and 1.7 can help in guessing such a model.

Question 11.1 would essentially be answered for both $\widehat{\mathrm{Lie}}_n, \widehat{\mathrm{W}}_n$ if one had a solution to a more basic question that goes back to Thrall [45]; see also [36, Exercise 7.89(i)]:

> What is the explicit Schur function expansion of each $\mathrm{ch}(\mathcal{L}_\lambda)$, that is, the $GL(V)$-irreducible decomposition of each higher Lie representation $\mathcal{L}_\lambda(V)$?

An answer to Question 11.1 would help to address the following question, suggested by computer data. Recall that Theorem 5.1 predicts

$$f_{i,\nu}(n) := \left\langle \chi^{(n-|\nu|,\nu)}, H^{i(d-1)}(\mathrm{Conf}(n,\mathbb{R}^d)) \right\rangle_{S_n}$$

becomes a constant in n for $n \geq |\nu| + i$ (d odd) or $n \geq |\nu| + i + 1$ (d even).

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Question 11.2. For ν with $|\nu| \geq 2$, is there a threshold value $i_0(\nu)$ with the property that for every $i \geq i_0(\nu)$, regarding $f_{i,\nu}(n)$ as a function of n, it stabilizes *sharply* at $n_0 = |\nu| + i$ for d odd, and *sharply* at $n_0 = |\nu| + i + 1$ for d even. □

11.3 Sharp stability for $\beta_S(\Pi_n)$?

Some preliminary analysis of $\beta_S(\Pi_n)$ led us to make the following conjecture.

Conjecture 11.3. Given a subset $S \subset \{1, 2, \ldots, n-2\}$ with $i = \max(S)$, the rank-selected homology S_n-representation $\beta_S(\Pi_n)$ stabilizes sharply at $n = 4i - (|S| - 1)$. □

This would be consistent with the two extreme cases where $|S| = 1$ or $|S| = i$:

- When $S = \{i\}$, Theorem 1.8 showed $\beta_S(\Pi_n)$ stabilizes sharply at $n = 4i$.
- When $S = \{1, 2, \ldots, i\}$, Corollary 5.4 showed $\beta_S(\Pi_n)$ stabilizes sharply at $n = 3i + 1$.

11.4 A precise version of Conjecture 1.5

The Orlik–Solomon algebra $A = \bigoplus_{i=1}^{n-1} A^i$, which was discussed following Proposition 9.1, in conjunction with the S_n-module isomorphism $WH_i(\Pi_n) \cong A^i$, also suggested a sharpening of Wiltshire-Gordon's Conjecture 1.5. There is a well-studied cochain complex structure $(A^\bullet, d)$ on A

$$A^\bullet = (A^0 \stackrel{d}{\to} A^1 \stackrel{d}{\to} \cdots \stackrel{d}{\to} A^{n-2} \stackrel{d}{\to} A^{n-1})$$

whose differential d multiplies by an element $\sum_{1 \leq i < j \leq n} c_{ij} a_{ij}$ in A^1. This complex is exact whenever the coefficients c_{ij} are chosen so that $\sum_{1 \leq i < j \leq n} c_{ij}$ lies in $\mathbb{C}^\times$; see, for example, Dimca and Yuzvinsky [11, Section 5]. Choosing $c_{ij} = 1$ for all i, j makes d into an S_n-equivariant cochain complex. One also obtains an S_n-stable cochain complex structure on $\widehat{\mathrm{W}}_n^\bullet$ as the subcomplex at the bottom of the following decreasing filtration:

$$A^\bullet = \quad F_0(A^\bullet) \supset F_1(A^\bullet) \supset \cdots \supset F_{n-1}(A^\bullet) \supset F_n(A^\bullet) \quad \cong \widehat{\mathrm{W}}_n^\bullet, \tag{55}$$

where $F_p(A^\bullet)$ is the span of the monomials $a_{i_1, j_1} \cdots a_{i_\ell, j_\ell}$ for which $|\{i_1, j_1\} \cup \cdots \{i_\ell, j_\ell\}| \geq p$.

This perspective leads to a natural candidate for a cochain complex of the type suggested by Conjecture 1.5 of Wiltshire-Gordon, namely the one that appears in Theorem 1.6. Indeed it is natural to approach Conjecture 1.5 using the known exactness

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of $(A^\bullet, d)$ together with the spectral sequence associated with the filtration (55). After posting this paper to the arXiv with Theorem 1.6 stated as a conjecture, the authors together with Steven Sam were able to complete a proof of the theorem, contained here in Appendix 1.

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Appendix 1. Proof of Theorem 1.6 (Joint with Steven Sam)

A.1 The Orlik–Solomon algebra

Recall from Section 9 that the Orlik–Solomon algebra of type A_{n-1} is the quotient

$$A(n) := E/I \tag{A.1}$$

of an exterior algebra E on generators $\{a_{ij}\}_{1\leq i<j\leq n}$ (for convenience, we will write $a_{\{i,j\}}$ to mean either a_{ij} or a_{ji} depending on whether $i < j$ or $j < i$), by the ideal I having generators

$$a_{ij}a_{ik} - a_{ij}a_{jk} + a_{ik}a_{jk} = 0 \quad \text{for } 1 \leq i < j < k \leq n.$$

This gives a skew-commutative graded algebra $A(n) = \bigoplus_{i=0}^{n-1} A^i$. One has a finer grading on $A(n)$ indexed by set partitions π in Π_n (see [26, Section 3.1], [11, Section 2.3]) that comes from the direct sum decomposition $E = \bigoplus_{\pi\in\Pi_n} E^\pi$ where E^π is the $\mathbb{C}$-span of all monomials $a_{i_1j_1}\cdots a_{i_\ell j_\ell}$ for which the graph on $\{1, 2, \ldots, n\}$ with edge set $\{\{i_1, j_1\}, \ldots, \{i_\ell, j_\ell\}\}$ has connected components given by the blocks of the set partition π. One checks that the ideal I decomposes as a direct sum $I = \bigoplus_{\pi\in\Pi_n}(I \cap E^\pi)$, and hence $A(n)$ inherits the

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same decomposition

$$A(n) = \bigoplus_{\pi \in \Pi_n} A^\pi \quad \text{where } A^\pi := E^\pi/(I \cap E^\pi),$$
$$WH_i(\Pi_n) \cong A^i = \bigoplus_{\substack{\pi \in \Pi_n \text{ with} \\ n-i \text{ blocks}}} A^\pi.$$

Each of the above direct sum decompositions also is compatible with the S_n-representation defined by $w(a_{ij}) = a_{\{w(i),w(j)\}}$. Multiplication by the S_n-invariant element $\sum_{1\le i<j\le n} a_{ij}$ in A^1 gives a differential d on $A(n) = A^\bullet$ which is S_n-equivariant. As mentioned earlier, this differential on $A^\bullet$ is exact when $n \geq 2$ because its coefficient sum is $\binom{n}{2}$, which is non zero in $\mathbb{C}$; see, for example, Dimca and Yuzvinsky [11, Section 5].

A.2 The filtration and its spectral sequence

Recall also the decreasing filtration (55)

$$A^\bullet = \quad F_0(A^\bullet) \supset F_1(A^\bullet) \supset \cdots \supset F_{n-1}(A^\bullet) \supset F_n(A^\bullet) \quad \cong \widehat{W}^\bullet_n$$

in which $F_p(A^\bullet)$ is the $\mathbb{C}$-span of $\{a_{i_1,j_1}\cdots a_{i_\ell,j_\ell} : |\{i_1,j_1\}\cup\cdots\cup\{i_\ell,j_\ell\}| \geq p\}$, so that

$$F_p(A^\bullet) := \bigoplus_{\substack{\pi \in \Pi_n \text{ with at most} \\ n-p \text{ singletons}}} A^\pi,$$
$$\widehat{W}^\bullet_n = F_n(A^\bullet) \cong \bigoplus_{\substack{\pi \in \Pi_n \text{ with} \\ \text{no singletons}}} A^\pi, \quad \text{where} \quad \widehat{W}^i_n \cong \bigoplus_{\substack{\pi \in \Pi_n \text{ with} \\ n-i \text{ blocks} \\ \text{and no singletons}}} A^\pi.$$

Associated with the decreasing filtration of $A^\bullet$ is a spectral sequence (see, e.g., Spanier [35, Section 9.4, p. 493]) with differentials $E_r^{p,q} \xrightarrow{\delta_r} E_r^{p+r,q-r+1}$, converging to $E_\infty^{p,q} = 0$ since $A^\bullet$ is exact. We next analyze the first two pages E_0, E_1 in this spectral sequence.

A.3 The E_0 and E_1-pages

The spectral sequence starts on its E_0 page with

$$E_0^{p,q} = F_p(A^{p+q})/F_{p+1}(A^{p+q}) \cong \bigoplus_{\substack{\pi \in \Pi_n \text{ with} \\ n-(p+q) \text{ blocks} \\ n-p \text{ singletons}}} A^\pi \cong \bigoplus_{\substack{S \subset \{1,2,\ldots,n\}: \\ |S|=n-p}} \bigoplus_{\substack{\pi \in \Pi_n \text{ with} \\ \text{singleton blocks } S \text{ and} \\ -q \text{ nonsingleton blocks}}} A^\pi, \tag{A.2}$$

and vertical differentials δ_0 induced from d on $A^\bullet$. One can check that the condition in (A.2) that π lies in Π_n with $n-(p+q)$ blocks, and $n-p$ singletons forces $E_0^{p,q} = 0$

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unless $p \geq 0$ and $q \leq 0$ (so the terms lie in the second quadrant), and that (If $p \geq 1$, then there are *some* nonsingleton blocks, namely $-q$ of them, and they partition the p many nonsingleton elements, so one has $1 \leq -q \leq p/2$.) $-1 \geq q \geq -\frac{p}{2}$ for $p \geq 1$.

Example A.1. For $n = 6$, abbreviating $F_p/F_{p+1}(A^\ell) := F_p(A^\ell)/F_{p+1}(A^\ell)$, the E_0-page is

$$\begin{array}{cccccccc}
q\backslash p & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
0 & F_0/F_1(A^0) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & \cdot & \cdot & F_2/F_3(A^1) & F_3/F_4(A^2) & F_4/F_5(A^3) & F_5/F_6(A^4) & F_6(A^5) \\
 & & & & & \uparrow & \uparrow & \uparrow \\
-2 & \cdot & \cdot & \cdot & \cdot & F_4/F_5(A^2) & F_5/F_6(A^3) & F_6(A^4) \\
 & & & & & & & \uparrow \\
-3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & F_6(A^3)
\end{array}$$

□

We next move on to analyze the E_1-page, which has $E_1^{p,q} = H^{p+q}(F_p/F_{p+1})$. In particular, since $F_{n+1} = 0$ this means that the $p = n$ column is $E_1^{n,q} = H^{n+q}(F_n) = H^{n+q}(\hat{W}_n^\bullet)$. There are horizontal differentials $E_1^{p,q} \xrightarrow{\delta_1} E_1^{p+1,q}$.

To understand this further, we analyze each column $E_0^{p,\bullet}$ using (A.2). We consider how the differential d_0 acts on a typical summand A^π on the right in (A.2), where π has S as its set of singleton blocks. Since d_0 is induced from multiplying by $a = \sum_{1\leq i<j\leq n} a_{i,j}$, only terms $a_{i,j}$ with both $i, j \notin S$ are relevant, and the image has the same set S of singleton blocks. This leads to isomorphisms for $p \geq 1$

$$\begin{aligned}
E_0^{p,\bullet} &\cong \bigoplus_{\substack{S\subset\{1,2,\ldots,n\}:\\ |S|=n-p}} \widehat{W}_p^\bullet &&\cong \widehat{W}_p^\bullet \uparrow_p^n \\
E_1^{p,q} &\cong H^{p+q}(\widehat{W}_p^\bullet \uparrow_p^n) &&\cong H^{p+q}(\widehat{W}_p^\bullet) \uparrow_p^n,
\end{aligned}$$

where the first line gives isomorphisms of complexes of $\mathbb{C}$-vector spaces and S_n-representations, respectively, and $C^\bullet \uparrow_p^n$ means $\left(C^\bullet \otimes \mathbf{1}_{S_{n-p}}\right) \uparrow_{S_p\times S_{n-p}}^{S_n}$ for $C^\bullet$ a complex of S_p-representations.

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Example A.2. For $n = 6$ the E_1-page is

$$\begin{array}{cccccccc}
q\backslash p & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
0 & \mathbf{1}_{S_n} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & \cdot & \cdot & H^1(\widehat{W}_2^\bullet)\uparrow_2^6 \to & H^2(\widehat{W}_3^\bullet)\uparrow_3^6 \to & H^3(\hat{W}_4^\bullet)\uparrow_4^6 \to & H^4(\widehat{W}_5^\bullet)\uparrow_5^6 \to & H^5(\widehat{W}_6^\bullet) \\
-2 & \cdot & \cdot & \cdot & \cdot & H^2(\widehat{W}_4^\bullet)\uparrow_4^6 \to & H^3(\widehat{W}_5^\bullet)\uparrow_5^6 \to & H^4(\widehat{W}_6^\bullet) \\
-3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & H^3(\widehat{W}_6^\bullet)
\end{array}$$

□

A.4 The *FI*-structures

Theorem 1.6 is equivalent to these two assertions:

- the only nonvanishing entries on the E_1-page are the upper-left entry $E_1^{0,0}$ and the $q = -1$ row $\{E^{p,-1}\}_{p=2}^n = \{H^{p-1}(\widehat{W}_p)\uparrow_p^n\}_{p=2}^n$, and
- $H^{p-1}(\widehat{W}_p) \cong \chi^{(2,1^{p-2})}$ for $p \geq 1$.

We can glue some of these entries together into a complex, which we will denote $C^\bullet(n)$, using the fact that the E_2 differential $E_2^{p,q} \xrightarrow{\delta_2} E_2^{p+2,q-1}$ sends $E_2^{0,0}$ to $\ker(E_2^{2,-1} \xrightarrow{\delta_1} E_2^{3,-1})$:

$$\begin{array}{ccccccccccccc}
0 \to & E_2^{0,0} & \xrightarrow{\delta_2} & E_2^{2,-1} & \xrightarrow{\delta_1} & E_2^{3,-1} & \xrightarrow{\delta_1} & \cdots & \xrightarrow{\delta_1} & E_2^{n-1,-1} & \xrightarrow{\delta_1} & E_2^{n,-1} & \to 0 \\
& \| & & \| & & \| & & & & \| & & \| & \\
0 \to & \mathbf{1}_{S_n} & \to & H^1(\widehat{W}_2)\uparrow_2^n & \to & H^2(\widehat{W}_3)\uparrow_3^n & \to & \cdots & \to & H^{n-2}(\widehat{W}_{n-1})\uparrow_{n-1}^n & \to & H^{n-1}(\widehat{W}_n) & \to 0 \\
& \| & & \| & & \| & & & & \| & & \| & \\
& C^0(n) & & C^1(n) & & C^2(n) & & \cdots & & C^{n-2}(n) & & C^{n-1}(n) &
\end{array}$$

A crucial step for us will be to eventually show that the complex $C^\bullet(n)$ is exact for $n \geq 2$. The strategy will be to consider *FI-module* structures on the objects involved, along with Schur–Weyl duality, in order to identify $C^\bullet(n)$ with a known exact sequence.

Recall [6, Definition 1.1] that *FI* is a category with objects the finite sets $[n] := \{1, 2, \ldots, n\}$ for $n = 0, 1, 2, \ldots$, and whose morphisms are the injective functions $f : [m] \hookrightarrow [n]$ for $m \leq n$. We denote by FI^{op} its opposite category.

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Returning to the definition of the Orlik–Solomon algebra $A(n) = E/I$ from (A.1), for each injection $f\colon [m] \hookrightarrow [n]$, the map on exterior algebras defined by

$$a_{i,j} \mapsto \begin{cases} a_{\{i',j'\}} & \text{if } \{i,j\} \subset \operatorname{im}(f) \text{ with } f(i') = i,\ f(j') = j, \\ 0 & \text{if either } i \notin \operatorname{im}(f) \text{ or } j \notin \operatorname{im}(f), \end{cases}$$

will send generators of the Orlik–Solomon ideal for $A(n)$ to those for $A(m)$, and will commute with the differentials $d(n)$ on the complexes $A^\bullet(n)$, preserving the filtration pieces $F^p(A(n))$. This gives the following result:

Proposition A.3. $\{A^\bullet(n)\}, \{F^\bullet(A(n))\}, \{E^{\bullet,\bullet}(F(A(n))\}, \{C^\bullet(n)\}$ are functors from FI^{op} into complexes, filtered complexes, spectral sequences, and complexes, respectively. □

Hence, we can speak of the (infinite) complex of FI^{op}-modules

$$C^\bullet = (0 \to C^0 \to C^1 \to C^2 \to \cdots)$$

and consider their $\mathbb{C}$-duals $D_i(n) := \operatorname{Hom}_{\mathbb{C}}(C^i(n), \mathbb{C})$ as a complex of FI-modules

$$D_\bullet = (0 \leftarrow D_0 \leftarrow D_1 \leftarrow D_2 \leftarrow \cdots).$$

To deduce exactness for $C_\bullet$, we will show that $D_\bullet(n)$ is exact for $n \geq 2$. We will compare $D_\bullet$ to the complex of FI-modules $D'_\bullet$ which is *Schur–Weyl dual* in the sense of Sam and Snowden [33, Section 1] to the complex of $GL(\mathbb{C}^\infty)$-modules

$$0 \leftarrow A \leftarrow \mathbb{S}^{(2)} \otimes A \leftarrow \mathbb{S}^{(2,1)} \otimes A \leftarrow \mathbb{S}^{(2,1,1)} \otimes A \leftarrow \cdots \tag{A.3}$$

giving the minimal free resolution (When $A = \mathbb{C}[x_1, \ldots, x_n]$, this resolution (A.3) is a special case of one discussed by Eisenbud *et al.* [12] and also a special case of the Eliahou–Kervaire resolution [13]; see also [25, Section 2.3].) of $\mathfrak{m}^2$ for the irrelevant ideal $\mathfrak{m} = (x_1, x_2, \ldots)$ in the polynomial ring $A := \mathbb{C}[x_1, x_2, \ldots]$; see [33, Example 6.10] with $\alpha = (2)$. Indexing so that D'_i is Schur–Weyl dual to the $GL(\mathbb{C}^\infty)$-module $\mathbb{S}^{(2,1^{i-1})} \otimes A$, then for $n \geq 2$, $D'_\bullet(n)$ is the following exact sequence of $x_1 x_2 \cdots x_n$-weight spaces from (A.3):

$$\begin{array}{ccccccccccccc} 0 \longleftarrow & \mathbf{1}_{S_n} & \longleftarrow & \chi^{(2)}\uparrow_2^n & \leftarrow & \chi^{(2,1)}\uparrow_3^n & \leftarrow \cdots \leftarrow & \chi^{(2,1^{n-3})}\uparrow_{n-1}^n & \leftarrow & \chi^{(2,1^{n-2})} & \longleftarrow & 0 \\ & \| & & \| & & \| & & \| & & \| & & \| \\ & D'_0(n) & & D'_1(n) & & D'_2(n) & \cdots & D'_{n-2}(n) & & D'_{n-1}(n) & & D'_n(n) \end{array} \tag{A.4}$$

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Lemma A.4. One has an isomorphism of the FI-complexes $D_\bullet \cong D'_\bullet$. In particular, for $n \geq 2$,

(a) $H_{n-1}(\widehat{W}^\bullet_n) = \chi^{(2,1^{n-2})}$.

(b) $C^\bullet(n)$ is exact. □

Proof. We use induction on n to prove assertion (a) together with the following assertion equivalent to the FI-complex isomorphism $D_\bullet \cong D'_\bullet$:

(b′) For each $n \geq 1$, one has an isomorphism of the truncated FI-complexes

$$\begin{tikzcd}
0 & D_0 \arrow[l] \arrow[d] & D_1 \arrow[l] \arrow[d] & D_2 \arrow[l] \arrow[d] & \cdots \arrow[l] & D_{n-2} \arrow[l] \arrow[d] & D_{n-1} \arrow[l] \arrow[d] \\
0 & D'_0 \arrow[l] & D'_1 \arrow[l] & D'_2 \arrow[l] & \cdots \arrow[l] & D'_{n-2} \arrow[l] & D'_{n-1} \arrow[l]
\end{tikzcd}.$$

The initial cases $n = 1, 2$ are not hard to check directly.

In the inductive step, assume the assertions (a), (b′) hold for $n-1$, and we will show that they hold for n. We claim that both FI-complexes $D_\bullet, D'_\bullet$ have the property that the truncations to homological degree at most $n-1$ shown in (b′) are completely determined by their values as functors on $[n]$. For $D_\bullet$ this is because $D_i(n)$ is dual to $C^i(n) = H_i(\widehat{W}_{i+1}) \uparrow^n_{i+1}$, and for $D'_\bullet$ this comes from its description in (A.4).

However, we claim that taking the values of the functors on $[n]$ in the diagram (b′) gives the following diagram with both rows exact, and with all solid vertical arrows being isomorphisms:

$$\begin{tikzcd}
0 & \mathbf{1}_{S_n} \arrow[l] \arrow[d] & H^1(\widehat{W}^\bullet_2)\uparrow^n_2 \arrow[l] \arrow[d] & H^2(\widehat{W}^\bullet_3)\uparrow^n_3 \arrow[l] \arrow[d] & \cdots \arrow[l] & H^{n-2}(\widehat{W}^\bullet_{n-1})\uparrow^n_{n-1} \arrow[l] \arrow[d] & H^{n-1}(\widehat{W}^\bullet_n) \arrow[l] \arrow[d, dashed] & 0 \arrow[l] \\
0 & \mathbf{1}_{S_n} \arrow[l] & \chi^{(2)}\uparrow^n_2 \arrow[l] & \chi^{(2,1)}\uparrow^n_3 \arrow[l] & \cdots \arrow[l] & \chi^{(2,1^{n-3})}\uparrow^n_{n-1} \arrow[l] & \chi^{(2,1^{n-2})} \arrow[l] & 0 \arrow[l]
\end{tikzcd} \tag{A.5}$$

In (A.5), the bottom row is the exact sequence (A.4). Its top row is exact in all, except possibly two, entries

$$H^{n-2}(\widehat{W}^\bullet_{n-1})\uparrow^n_{n-1} \quad \text{and} \quad H^{n-1}(\widehat{W}^\bullet_n)$$

using the inductive hypothesis on isomorphism with the bottom row. To finish arguing exactness at these two entries, identify the dual of the top row of (A.5) with $C^\bullet(n)$, and

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also with the $q = -1$ row in the E_1-page of the spectral sequence as in Example A.2, where they appear as the two entries $E_1^{n-1,-1}$ and $E_1^{n,-1}$. Recall that $E_\infty^{\bullet,\bullet} = 0$, and note that the differentials δ_r for $r \geq 2$ that either map into or out of $E_r^{n-1,-1}$ or $E_r^{n,-1}$ are all 0 (as either their source or target is 0), so that $E_r^{n-1,-1} \cong E_{r+1}^{n-1,-1}$ and $E_r^{n,-1} \cong E_{r+1}^{n,-1}$ for all $r \geq 2$. These observations combine to yield the desired result that the differential δ_1 must be exact at both $E_1^{n-1,-1}$ and $E_1^{n,-1}$.

Since both rows in (A.5) are exact, the dotted vertical map is also an isomorphism, and both assertions (a), (b′) for n follow, completing the inductive step. ■

This now lets us easily complete the proof of the theorem.

Proof of Theorem 1.6. We use induction on n, with easy base cases when $n \leq 2$. We already know from Lemma A.4(a) that $H^{n-1}(\widehat{W}_n^\bullet) \cong \chi^{(2,1^{n-2})}$, and so it only remains to show that $H^i(\widehat{W}_n^\bullet)$ vanishes for $i \leq n-2$. In the inductive step, the known vanishing shows that the E_1-page of the spectral sequence looks like this:

$$\begin{array}{cccccccc}
q\backslash p & 0 & 1 & 2 & 3 & \cdots & n-1 & n \\
0 & 1_{S_n} & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
-1 & \cdot & \cdot & \chi^{(2)}\uparrow_2^n \longrightarrow & \chi^{(2,1)}\uparrow_3^n \longrightarrow & \cdots \longrightarrow & \chi^{(2,1^{n-3})}\uparrow_{n-1}^n \longrightarrow & H^{n-1}(\widehat{W}_n^\bullet) \\
-2 & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & H^{n-2}(\widehat{W}_n^\bullet) \\
\vdots & & & & & & & \vdots \\
-\lfloor \frac{n}{2} \rfloor & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & H^{\lceil \frac{n}{2} \rceil}(\widehat{W}_n^\bullet) \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot
\end{array} \tag{A.6}$$

But now we also know from Lemma A.4(b) that $C^\bullet(n)$ is exact, so that the $q = -1$ row of (A.6) is exact after gluing on the $E^{0,0}$ term via δ_2. Thus, most of these entries in the $q = 0$ and $q = -1$ row of E_1 do not survive to the E_2-page; only $E_2^{0,0}$ and $E_2^{2,-1}$ survive, but then die at the E_3-page. Hence in the $p = n$ column, one finds that $E_r^{n,q}$ for $q \leq -2$ have no entries on the pages E_r with $r \geq 1$ that can affect them. Therefore $H^{n+q}(\hat{W}_n^\bullet) = E_1^{n,q} = \cdots = E_\infty^{n,q} = 0$ for each $q \leq -2$, as desired. ■

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Appendix 2. Data on $\widehat{\mathrm{Lie}}_n^i, \widehat{\mathrm{W}}_n^i$

We present some data on the S_n-irreducible decompositions of $\widehat{\mathrm{Lie}}_n^i$ for small n, i to give a sense of the nature of these representations. An irreducible character χ^λ is represented by the Ferrers diagram for λ, so $2(4,2)$ in the table means $+2\chi^{(4,2)}$.

$n\backslash i$	0	1	2	3	4
0	$\varnothing$				
1					
2		$(1,1)$			
3			$(2,1)$		
4			$(2,2)$ $(1,1,1,1)$	$(3,1)$ $(2,1,1)$	
5				$(2,1,1,1)$ $(3,2)$ $(3,1,1)$ $(2,2,1)$	$(4,1)$ $(3,2)$ $(3,1,1)$ $(2,2,1)$ $(2,1,1,1)$
6				$(1,1,1,1,1,1)$ $(2,2,1,1)$ $(3,3)$	$(4,1,1)$ $(2,2,1,1)$ $(2,1,1,1,1)$ $3(3,2,1)$ $3(3,1,1,1)$ $2(2,2,2)$ $2(4,2)$
7					$(4,2,1)$ $(4,3)$ $(2,2,2,1)$ $(3,1,1,1,1)$ $2(3,2,1,1)$ $(2,2,1,1,1)$ $(3,2,2)$ $(3,3,1)$ $(2,1,1,1,1,1)$
8					$(4,4)$ $(3,3,1,1)$ $(2,2,2,2)$ $(2,2,1,1,1,1)$ $(1,1,1,1,1,1,1,1)$

(A1)

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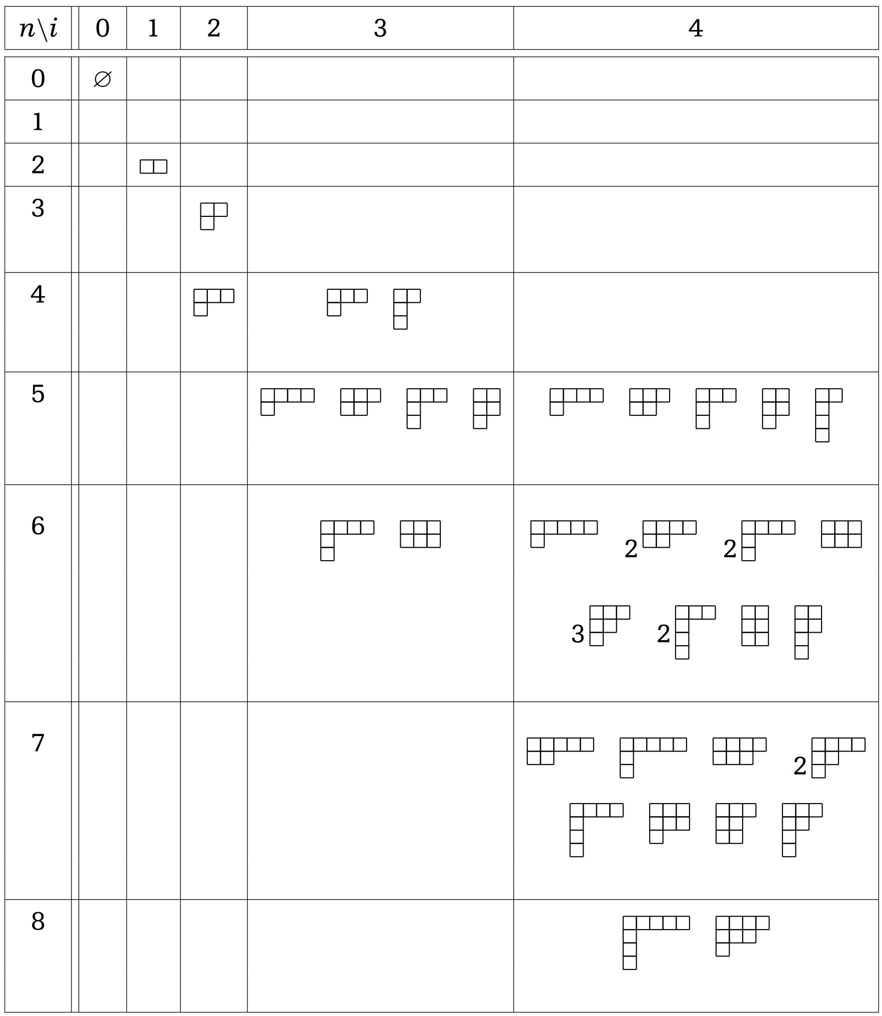
We similarly present some data on the S_n-irreducible decompositions of $\widehat{\mathrm{W}}_n^i$ for small n, i:



(A2)

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