THE COMBINATORICS OF THE BAR RESOLUTION IN GROUP COHOMOLOGY

VICTOR REINER AND PETER WEBB

Abstract. We study a combinatorially-defined double complex structure on the ordered chains of any simplicial complex. Its columns are related to the cell complex K_n whose face poset is isomorphic to the subword ordering on words without repetition from an alphabet of size n. This complex is shellable and as an application we give a representation theoretic interpretation for derangement numbers and a related symmetric function considered by Désarménien and Wachs.

We analyze the two spectral sequences arising from the double complex in the case of the bar resolution for a group. This spectral sequence converges to the cohomology of the group and provides a method for computing group cohomology in terms of the cohomology of subgroups. Its behavior is influenced by the complex of oriented chains of the simplicial complex of finite subsets of the group, and we examine the Ext class of this complex.

1. Introduction

This work is a study of interactions between combinatorics, representation theory and topology. The construction which links everything together is a double complex structure which may be put on the complex of ordered chains of any simplicial complex. We will show that the columns in the double complex turn out to be constructed from shellable CW-complexes which have inherent combinatorial interest: they provide a representation-theoretic interpretation of a symmetric function and other formulas studied by Désarménien and Wachs in the context of derangements. We will also show in the context of the cohomology of groups that the double complex structure gives rise to a spectral sequence which converges to the group cohomology.

1991 *Mathematics Subject Classification.* Primary 57M07; Secondary 05E25, 20J06.

Key words and phrases. bar resolution, group cohomology, derangement, desarrangement, random-to-top shuffle, Tsetlin library.

First author supported by NSF grant DMS–9877047.

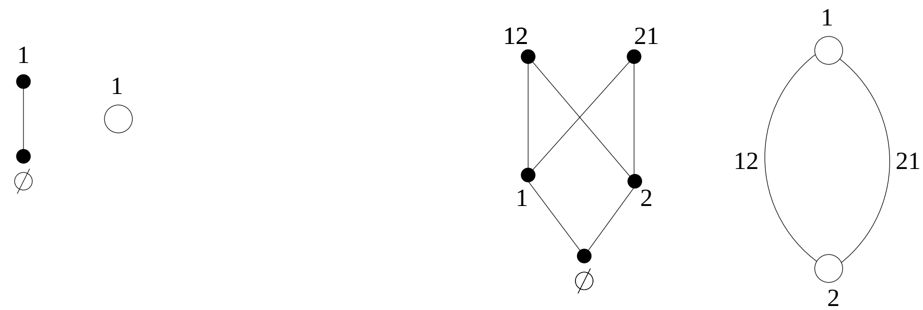

FIGURE 1. The posets P_n and cell complexes K_n for $n = 1, 2$.

Let us first describe the kind of CW-complex out of which we may construct the columns of our double complex. It turns out that these CW-complexes can be described without knowing about the double complex structure to which we will later relate them. Consider the subword ordering P_n on the set of words without repetition from the alphabet $[n] := \{1, 2, \dots, n\}$, defined by

$$a_1 a_2 \cdots a_r \leq b_1 b_2 \cdots b_s$$

if $a_1 a_2 \cdots a_r = b_{i_1} b_{i_2} \cdots b_{i_r}$ for some $1 \leq i_1 < \cdots < i_r \leq s$. Because lower intervals in P_n are isomorphic to Boolean algebras, P_n forms the poset of faces in an $(n-1)$-dimensional regular CW-complex, having all faces isomorphic to simplices (see [4] for more on the combinatorics and topology of such CW-complexes, where they are called *complexes of Boolean type*; in [16] they are called *Boolean complexes*, and in [13, 25] their face posets are called *simplicial posets*). Denote by K_n the regular CW-complex having P_n as its face poset. For small values of n, these CW-complexes are easily constructed by direct experimentation. The posets P_1, P_2 and complexes K_1, K_2 are depicted in Figure 1.

Farmer [15] studied the homotopy type of K_n and showed that it is homotopy equivalent to a wedge of $(n-1)$-spheres, along with a similar property for all of its skeleta. Björner and Wachs [5, Theorem 6.1] reproved this result by showing that the lexicographic order on permutations in the symmetric group $\mathfrak{S}_n$ induces a recursive coatom ordering on P_n and hence a (dual) CL-shelling of K_n. As a consequence, both K_n and the links of any of its faces have homology only in their top dimension, as they are homotopy equivalent to wedges of spheres having this dimension.

The homological properties of these CW-complexes will have implications for our double complex, to be introduced later. For now we point out a combinatorial application, explained in detail in Section 2. Recall that a permutation is a *derangement* if it has no fixed points. It will follow from the shellability of K_n that the rank of its top homology is the number of derangements in $\mathfrak{S}_n$ (see Proposition 2.1). Observe that the permutation action of the symmetric group $\mathfrak{S}_n$ on the alphabet $[n] := \{1, 2, \ldots, n\}$ extends to a cellular action on K_n, and hence to representations of $\mathfrak{S}_n$ on the homology of K_n. We show that this homology representation of $\mathfrak{S}_n$ (taken with $\mathbb{C}$ coefficients) naturally interprets a symmetric function that was introduced by Désarménien and Wachs [11] in their combinatorial analysis of descent sets of derangements: their symmetric function is (up to a twist by the sign character) the image under the Frobenius characteristic map of this homology representation (Theorem 2.4).

Let us now summarize our results as they apply to the cohomology of groups. Given a group G we may consider the simplicial complex Δ whose vertices are the elements of G and in which every finite subset of G is a simplex. We may construct the unnormalized bar resolution for $\mathbb{Z}G$ as $C^{\mathrm{ord}}(\Delta)$, the complex of ordered chains of this simplicial complex, and it may be used to compute the group homology $H_*(G, M)$ and cohomology $H^*(G, M)$ for any module of coefficients M. As part of a more general theory developed in Section 3, we introduce a natural double complex structure on $C^{\mathrm{ord}}(\Delta)$, whose filtrations by rows and by columns give rise to two spectral sequences converging to $H_*(G, M)$, or $H^*(G, M)$ in the cohomology version. One of these spectral sequences has a use in computing group cohomology in a way that we have not seen done before. We remark that the double complex structure of the type we consider has also been used in [28] and [29].

In Section 4 we consider the spectral sequence arising from the filtration by columns. It is here that we make the connection with the combinatorics of the CW-complexes K_n, for the first column on the E^0 page of the spectral sequence is naturally isomorphic to the cellular chain complex for K_n where $n = |G|$, and the remaining columns split into direct summands naturally isomorphic (up to shifts in degree) to cellular chain complexes for links of faces in various K_n. The vanishing of homology below the top dimension for K_n and for links of its faces then gives a simple description for the E^1 page of this spectral sequence, and shows that it stops at the E^2 page (Theorem 4.4).

We investigate the row spectral sequence in Section 5, describing its E^1 page and showing (in the case of finite G) how its terms may be computed using the Möbius function on the lattice of subgroups of G (Proposition 6.1 and sequel). Two appealing features of this spectral sequence in the case of finite G are that it only has a finite number of non-zero rows – $|G|$ of them in fact – and that many of these rows are only non-zero on the left hand edge. The spectral sequence expresses the cohomology of G in terms of the cohomology of the proper subgroups of G, together with the cohomology of G itself, but shifted in degree by $|G| - 1$.

The behavior of this spectral sequence is strongly influenced by the properties of the complex $C^{\mathrm{ori}}(G)$ of oriented chains on G, which when augmented by a map to $\mathbb{Z}$ is an acyclic complex of $\mathbb{Z}G$-modules (see Corollary 7.7). It is a $(|G| - 1)$-fold extension of $\mathbb{Z}$ by a module $\tilde{\mathbb{Z}}$ (namely a copy of $\mathbb{Z}$ on which G acts via a sign representation) which we call the subsets complex of G, and it represents an element $\zeta_G \in \mathrm{Ext}^{|G|-1}_{\mathbb{Z}G}(\mathbb{Z}, \tilde{\mathbb{Z}})$. We study when this element is non-zero and using the row spectral sequence we are able to prove that it is non-zero when G is an elementary abelian p-group, p being any prime (Corollary 7.8). In general ζ_G is an essential element, meaning that it vanishes on restriction to all proper subgroups of G (Theorem 7.5). Building on this, we are able to construct non-zero elements in the cohomology of other groups which are not elementary abelian (Theorem 7.9). Our results actually apply to a more general Ext class ζ_Ω constructed from an arbitrary finite G-set Ω. In the situation when $\Omega = \{1, \ldots, n\}$ is permuted in the natural way by the symmetric group $\mathfrak{S}_n$, we show that the $(n - 1)$-fold extension which we construct with class ζ_Ω is closely related to certain short exact sequences considered by James and Peel [17] (Proposition 7.11). We prove in this situation that ζ_Ω is non-zero precisely when n is a power of a prime (Corollary 7.10). We show also that ζ_Ω can be expressed as a product of terms involving Evens' norm map (Proposition 7.13 and Corollary 7.14).

Several of the sections of this paper can be read more-or-less independently of each other. Our main combinatorial work and its application can be found in Sections 2 and 4, and for Section 4 it is necessary to read Section 3. On the other hand, the reader interested mainly in group cohomology can start at Section 3 and omit much of Section 4.

2. The derangement representation

The goal of this section is to give representation-theoretic intepretations and analogues for combinatorial results on derangements, as mentioned in the introduction. For the reader interested mainly in applications to group cohomology it is possible to skip directly to Section 3. Our notation for representations and symmetric functions will mostly follow [22, 24], to which we refer the reader for basic facts and undefined terms. For topological facts we refer the reader to [20]. We alert the reader to our notation for permutations. Sometimes we write a permutation π of $\{1, \ldots, n\}$ as $\pi = \pi_1 \pi_2 \ldots \pi_n$ where $\pi_i = \pi(i)$ is the image of i under π, and sometimes we write π in cycle notation, depending on which is the more convenient.

We begin by reviewing some results about derangements. Recall from the introduction that a permutation $\pi = \pi_1\pi_2\dots\pi_n$ in $\mathfrak{S}_n$ is a *derangement* if it has no fixed points $\pi_i = i$. Denote by D_n the set of derangements in $\mathfrak{S}_n$, and let $d_n := |D_n|$ (the n^{th} *derangement number*). A simple inclusion-exclusion argument shows that

$$\begin{aligned} d_n &= \sum_{S\subset\{1,2,\dots,n\}} (-1)^{|S|}|\{\pi \in \mathfrak{S}_n : \pi \text{ fixes } S \text{ pointwise}\}| \\ &= \sum_{k=0}^{n}(-1)^k \frac{n!}{k!} \end{aligned} \tag{2.1}$$

from which follows easily the well-known recurrence (see [23, Section 2.2, formula (13)]

$$d_n = nd_{n-1} + (-1)^n. \tag{2.2}$$

Motivated by the desire to explain this recurrence combinatorially Désarménien [10] introduced another subset E_n of the permutations $\mathfrak{S}_n$ (later dubbed *desarrangements* by M. Wachs), defined by the property that the first *ascent* $\pi_i < \pi_{i+1}$ has i even (by convention, we set $\pi_{n+1} = +\infty$ for π in $\mathfrak{S}_n$). He gave a simple bijection showing $|E_n| = |D_n| = d_n$ and a simple combinatorial interpretation of (2.2). Désarménien and Wachs [11] then generalized the equality $|E_n| = |D_n|$ in the following fashion. Given π in $\mathfrak{S}_n$, define its *descent set*

$$\mathrm{Des}(\pi) := \{i : 1 \leq i \leq n-1 : \pi_i > \pi_{i+1}\}.$$

They showed that for any $S \subset [n-1]$,

$$|\{\pi \in D_n : \mathrm{Des}(\pi) = S\}| = |\{\pi \in E_n : \mathrm{Des}(\pi^{-1}) = S\}|. \tag{2.3}$$

The proof in [11] of this fact is a clever use of the theory of symmetric functions: they artifically construct two symmetric functions (see (2.4) below) which encode the numbers on the left- and right-hand sides of (2.3), and then show that these symmetric functions satisfy the same recursion, so that they must coincide. We will denote this symmetric function by $K_n(x)$. Subsequently they provided a purely bijective proof of the same fact in [12]. One of our goals in this section will be to show that the homology of the complex K_n gives a natural representation-theoretic interpretation for $K_n(x)$.

Let χ_n denote the complex character of $\mathfrak{S}_n$ acting on the top reduced homology $\tilde{H}_{n-1}(K_n, \mathbb{C})$, which we proceed to analyze. Recall that the cell complex K_n is defined to be the (unique up to isomorphism) regular CW-complex having face poset P_n. Its $(k-1)$-dimensional cells correspond to words of length k without repetitions from the alphabet $[n]$. The symmetric group $\mathfrak{S}_n$ acts transitively on such words, with the stabilizer of a typical such word isomorphic to the Young subgroup $(\mathfrak{S}_1)^k \times \mathfrak{S}_{n-k}$. Furthermore, when one considers such a word as indexing a basis element in the augmented cellular chain group $C_{k-1}(K_n, \mathbb{C})$, elements in the stabilizer act trivially on this element (that is, they introduce no coefficient). Consequently the character of $\mathfrak{S}_n$ acting on the $C_{k-1}(K_n, \mathbb{C})$ is the induced character $1 \uparrow_{(\mathfrak{S}_1)^k \times \mathfrak{S}_{n-k}}^{\mathfrak{S}_n}$, where 1 denotes the trivial character of the relevant group. One has then the following representation-theoretic analogue of equation (2.1).

Proposition 2.1.

$$\chi_n = \sum_{k=0}^{n} (-1)^{n-k} \; 1 \uparrow_{(\mathfrak{S}_1)^k \times \mathfrak{S}_{n-k}}^{\mathfrak{S}_n}$$

In particular, taking the degree of both sides, we have that

$$\dim_{\mathbb{C}} \tilde{H}_{n-1}(K_n, \mathbb{C}) = \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{(n-k)!} = d_n,$$

the n^{th} derangement number.

Proof. The Hopf trace formula gives the following equality of virtual characters:

$$\sum_{i \geq -1} (-1)^i \chi_{\tilde{H}_i(K_n, \mathbb{C})} = \sum_{i \geq -1} (-1)^i \chi_{C_i(K_n, \mathbb{C})}.$$

Since shellability of K_n implies $\tilde{H}_i(K_n, \mathbb{C}) = 0$ for $i < n$, and the character of $C_i(K_n, \mathbb{C})$ is $1 \uparrow_{(\mathfrak{S}_1)^k \times \mathfrak{S}_{n-k}}^{\mathfrak{S}_n}$, multiplying both sides by $(-1)^{n-1}$ gives the result. □

From this one can show that χ_n satisfies a recurrence generalizing the recurrence (2.2) for d_n:

Proposition 2.2.

$$\chi_n = (1 \otimes \chi_{n-1}) \uparrow_{\mathfrak{S}_1 \times \mathfrak{S}_{n-1}}^{\mathfrak{S}_n} + (-1)^n 1_{\mathfrak{S}_n}.$$

Proof. Use the previous proposition and manipulate as follows:

$$\begin{aligned}
\chi_n &= \sum_{k=0}^{n}(-1)^{n-k}\ 1\uparrow_{(\mathfrak{S}_1)^k\times\mathfrak{S}_{n-k}}^{\mathfrak{S}_n}\\
&= (-1)^n 1_{\mathfrak{S}_n} + \sum_{k=1}^{n}(-1)^{n-k}\ 1\uparrow_{(\mathfrak{S}_1)^k\times\mathfrak{S}_{n-k}}^{\mathfrak{S}_n}\\
&= (-1)^n 1_{\mathfrak{S}_n} + \left(1\otimes\sum_{k=1}^{n}(-1)^{n-k}\ 1\uparrow_{(\mathfrak{S}_1)^{k-1}\times\mathfrak{S}_{n-k}}^{\mathfrak{S}_{n-1}}\right)\uparrow_{\mathfrak{S}_1\times\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}\\
&= (-1)^n 1_{\mathfrak{S}_n} + (1\otimes\chi_{n-1})\uparrow_{\mathfrak{S}_1\times\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}
\end{aligned}$$

□

Our next result gives the decomposition of χ_n into irreducible characters. Recall that the irreducible complex characters χ^λ of $\mathfrak{S}_n$ are indexed by partitions λ of n, and write $\lambda \vdash n$ when λ is a partition of n. Given a *standard Young tableau* (or *SYT*) Q of shape λ, the *descent set* $\mathrm{Des}(Q) \subseteq [n-1]$ is defined to be the set of values i for which $i+1$ appears in a row below the row containing i in Q. If i is not in $\mathrm{Des}(Q)$, say that i is an *ascent* of Q.

Proposition 2.3.

$$\chi_n = \sum_{\lambda\vdash n} |\{\textit{SYT } Q \textit{ of shape } \lambda : Q \textit{ has smallest descent even}\}|\chi^\lambda$$

where, by convention, we say that the smallest descent of Q *is* n *whenever* $\mathrm{Des}(Q) = \emptyset$ *(i.e. when* λ *has only one row).*

Proof. Using the branching rule (or Pieri formula)

$$(1\otimes\chi^\mu)\uparrow_{\mathfrak{S}_1\times\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} = \sum_{\substack{\lambda\supset\mu:\\ |\lambda|=|\mu|+1}} \chi^\lambda,$$

we check that the right-hand side in the proposition satisfies the recurrence for χ_n given in Proposition 2.2. To obtain the set of standard Young tableaux described on the right-hand side for n, starting with those for $n-1$, one must add one new corner cell containing the entry n in all possible ways, and then make two corrections related to the special case of a single row:

- If n is odd, one should not have added a corner cell containing n to the tableau with only row of length $n-1$, as this creates a tableau with one row having first descent n, which is odd.

- If n is even, one should add in the tableaux which has a single row of length n, as this has first descent n, which is even, but cannot be obtained by adding a single cell to a tableaux with first descent even.

Adding the new corner cell in all possible ways corresponds via the branching rule to the first term on the right-hand side in the recurrence of Proposition 2.2, while the two corrections correspond to the $(-1)^n 1_{\mathfrak{S}_n}$ term in this recurrence. $\square$

To define the symmetric function introduced by Désarménien and Wachs [11], we review some of Gessel's theory of (quasi-)symmetric functions (see e.g. [24, §7.19]). For a fixed n, and any subset $S \subset [n-1]$, the *fundamental quasi-symmetric function* L_S is defined by

$$L_S = \sum_{\substack{i_1 \geq i_2 \geq \cdots \geq i_n: \\ i_j > i_{j+1} \ \forall j \in S}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

The two quasi-symmetric functions which Désarménien and Wachs defined can then be written

$$K_n(x) := \sum_{\pi \in D_n} L_{\mathrm{Des}(\pi)} = \sum_{\pi \in E_n} L_{\mathrm{Des}(\pi^{-1})}. \tag{2.4}$$

It is not at all obvious that these are actually *symmetric functions* in the variables $x_1, x_2, \ldots$, nor that they coincide – this is the main point of [11]. It is not hard to see that the $\{L_S\}_{S \subset [n-1]}$ are linearly independent, and therefore the equality of these two symmetric functions gives the equality (2.3) for every $S \subset [n-1]$.

It is well-known [24, Theorem 7.19.7] that the *Schur function* $s_\lambda(x)$ has the following expansion:

$$s_\lambda(x) = \sum_{SYT\ P \text{ of shape } \lambda} L_{\mathrm{Des}(P)}. \tag{2.5}$$

Recall that the Frobenius *characteristic map* ch is a linear isomorphism (even an isometry with respect to appropriate inner products) from the the space of virtual (complex) characters of $\mathfrak{S}_n$ to the space of symmetric functions of degree n, satisfying $\mathrm{ch}(\chi^\lambda) = s_\lambda(x)$. Also recall that the one-dimensional sign character ϵ_n of $\mathfrak{S}_n$ has the property $\epsilon_n \otimes \chi^\lambda = \chi^{\lambda'}$ where λ' is the transpose or conjugate partition to λ.

The next result is the main theorem of this section, interpreting Désarménien and Wachs' symmetric function $K_n(x)$.

Theorem 2.4.

$$K_n(x) = \mathrm{ch}(\epsilon_n \otimes \chi_n)$$

Proof. Starting with the previous proposition, one can manipulate as follows (see justifications below):

$$
\begin{aligned}
&\mathrm{ch}(\epsilon_n \otimes \chi_n)\\
&= \sum_{\lambda \vdash n} |\{SYT\ Q \text{ of shape } \lambda \text{ with smallest descent even}\}| \mathrm{ch}(\epsilon_n \otimes \chi^\lambda)\\
&= \sum_{\lambda \vdash n} |\{SYT\ Q \text{ of shape } \lambda \text{ with smallest descent even}\}| s_{\lambda'}\\
&= \sum_{\lambda \vdash n} |\{SYT\ Q \text{ of shape } \lambda \text{ with smallest ascent even}\}| s_{\lambda}\\
&= \sum_{\substack{\text{pairs of SYT } (P,Q) \text{ of same shape:}\\ Q \text{ has smallest ascent even}}} L_{\mathrm{Des}(P)}\\
&= \sum_{\pi \in E_n} L_{\mathrm{Des}(\pi^{-1})}\\
&= K_n(x)
\end{aligned}
$$

The first equality uses linearity of the map ch. The third equality uses the transpose or conjugation bijection on partitions and tableaux, which swaps ascents and descents in a tableau. The fourth equality uses (2.5). The fifth equality uses the Robinson-Schensted bijection $\pi \mapsto (P, Q)$ between permutations in $\mathfrak{S}_n$ and pairs of standard tableaux of the same shape having n cells, along with one of its important properties [24, Lemma 7.23.1]:

$$
\begin{aligned}
\mathrm{Des}(P) &= \mathrm{Des}(\pi^{-1})\\
\mathrm{Des}(Q) &= \mathrm{Des}(\pi).
\end{aligned}
$$

□

We conclude this section by briefly drawing attention to some connections with related areas in combinatorics and probability. We deliberately omit definitions of the terms we use, which may be found in [9, 21, 27].

The top boundary operator of the complex of injective words K_n, upon tensoring with the sign representation, may be identified with the random-to-top shuffling operator on the group algebra of the symmetric group. In this context, Proposition 2.1 was obtained independently by Correll [9, Theorem 3.0.7]. Phatarfod [21] showed that this operator has eigenvalues $0, 1, \ldots, n$, and that the k-eigenspace has dimension equal to the number of permutations in $\mathfrak{S}_n$ with k fixed points. As a particular case, when $k = 0$ we may deduce that the top homology of K_n has dimension equal to the number of derangements.

Uyemura-Reyes [27] refined Phatarfod's result by describing the $\mathfrak{S}_n$-module structure of the eigenspaces of more general shuffling operators in terms of induced representations.

3. Oriented and ordered chains, and a double complex

The goal of this section is to introduce a double complex structure on the bar resolution used in computing the homology or cohomology of a finite group, and describe the two spectral sequences which arise from filtering it by rows and by columns. We will eventually show that the columns in this double complex are very closely related to the topology of the complex K_n.

We begin however in a slightly more general setting, by placing a bigrading on the ordered simplicial chains used to compute the (ordinary, non-reduced) homology of any simplicial complex [20, Chapter 1 §13]. A bigrading of this type appears also in [28] and [29].

Let Δ be an abstract simplicial complex on vertex set V, so that we may construct its *oriented chain complex* $C^{\text{ori}}(\Delta)$ [20, Chapter 1 §5] and *ordered chain complex* [20, Chapter 1 §13] $C^{\text{ord}}(\Delta)$. The oriented chain complex may be described by placing the vertices V in order $v_1 < \cdots < v_n$ and taking $C^{\text{ori}}_r(\Delta)$ to have $\mathbb{Z}$-basis the symbols $[v_{i_0}, \dots, v_{i_r}]$ where $\{v_{i_0}, \dots, v_{i_r}\}$ is a simplex in Δ and where we require $v_{i_0} < \cdots < v_{i_r}$. Should we happen to see the elements $v_{i_0}, \dots, v_{i_r}$ in the wrong order we identify $[v_{i_0}, \dots, v_{i_r}]$ with $\text{sign}(\sigma)[v_{\sigma(i_0)}, \dots, v_{\sigma(i_r)}]$ for every permutation σ, and take this to be zero if terms are repeated. The ordered chain complex $C^{\text{ord}}(\Delta)$ has $\mathbb{Z}$-basis given by the symbols $(v_{i_0}, \dots, v_{i_r})$ where $\{v_{i_0}, \dots, v_{i_r}\}$ are the vertices of a simplex of Δ, but we have the possibility that there may be repeats among the symbols. We will refer to $(v_{i_0}, \dots, v_{i_r})$ as an *ordered chain*.

There is a chain map $u : C^{\text{ord}}(\Delta) \to C^{\text{ori}}(\Delta)$ specified $\mathbb{Z}$-linearly by $(v_{i_0}, \dots, v_{i_r}) \mapsto [v_{i_0}, \dots, v_{i_r}]$, and we recall the following result from simplicial homology.

Theorem 3.1. [20, Chapter 1, Theorem 13.6], [18, Theorem 4.3.9] *This map is a chain homotopy equivalence.* □

We now describe a double complex structure which may be put upon the ordered chain complex. Let $C^{\text{ord}}_{r,s} = C_{r,s}$ be the span of those $(v_{i_0}, \dots, v_{i_{r+s}})$ in $C^{\text{ord}}(\Delta)$ such that $|\{v_{i_0}, \dots, v_{i_{r+s}}\}| = s + 1$. Then $C_n = \bigoplus_{r+s=n} C_{r,s}$ and if δ denotes the dimension of Δ, we may position the summands on a grid which has $\delta + 1$ rows:

$$\begin{array}{cccc} C_{0,\delta} & C_{1,\delta} & C_{2,\delta} & \cdots \\ \vdots & & \vdots & \\ C_{0,1} & C_{1,1} & C_{2,1} & \cdots \\ C_{0,0} & C_{1,0} & C_{2,0} & \cdots \end{array}$$

On the bottom row, $C_{n,0} = \mathbb{Z}V$ having as $\mathbb{Z}$-basis the elements which are $(n+1)$-tuples $(v, v, v, \ldots, v)$ for $v \in V$. On the left edge, $C_{0,n}$ has $\mathbb{Z}$-basis the $(n+1)$-tuples of distinct elements of V which are the vertices of a simplex. In the boundary map

$$d(v_{i_0}, \ldots, v_{i_{r+s}}) = \sum_{j=0}^{r+s} (-1)^j (v_{i_0}, \ldots, \hat{v}_{i_j}, \ldots, v_{i_{r+s}})$$

the support of each term in the sum is either $\{v_{i_0}, \ldots, v_{i_{r+s}}\}$ or a set of size one smaller, and so

$$d(C_{r,s}) \subseteq C_{r-1,s} + C_{r,s-1}.$$

It follows that $C_{r,s}$ is a double complex. The rows and columns are all chain complexes, and the squares (anti-)commute.

We first observe that the double complex diagram for C^{ord} may be extended to give a commutative diagram in which the left hand edge is C^{ori}, as follows:

$$\begin{array}{ccccccccc} C^{\text{ori}}_\delta & \longleftarrow & C_{0,\delta} & \longleftarrow & C_{1,\delta} & \longleftarrow & C_{2,\delta} & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \vdots & & \vdots & & \vdots & & \vdots & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ C^{\text{ori}}_1 & \longleftarrow & C_{0,1} & \longleftarrow & C_{1,1} & \longleftarrow & C_{2,1} & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ C^{\text{ori}}_0 & \longleftarrow & C_{0,0} & \longleftarrow & C_{1,0} & \longleftarrow & C_{2,0} & \cdots \end{array} \tag{3.1}$$

This works simply because the chain map $u : C^{\text{ord}} \to C^{\text{ori}}$ considered previously sends $C_{r,s}$ to zero if $r > 0$.

We now begin to examine the properties of this double complex in generality. Our immediate goal is Theorem 3.4 which states that diagram 3.1 is a proper resolution in the sense of Mac Lane [19]. The proof depends on a refined analysis of the usual proof of Theorem 3.1. We proceed by means of two intermediate lemmas.

It is standard (see [20, Chapter 1, Theorem 13.6], [18, Theorem 4.3.9]) to prove that the map $u : C^{\mathrm{ord}}(\Delta) \to C^{\mathrm{ori}}(\Delta)$ is a homotopy equivalence by defining a homotopy inverse v such that $uv = 1$, and constructing a chain homotopy

$$T : C^{\mathrm{ord}}_t(\Delta) \to C^{\mathrm{ord}}_{t+1}(\Delta)$$

satisfying

$$dT + Td = 1 - vu. \tag{3.2}$$

We observe here that it is possible to choose the homotopy T so as to preserve the above double complex structure.

Lemma 3.2. *There exists a homotopy inverse v to u and a chain-homotopy T as above, satisfying $uv = 1$ and* (3.2)*, along with the extra property that*

$$T : C^{\mathrm{ord}}_{r,s}(\Delta) \to C^{\mathrm{ord}}_{r+1,s}(\Delta). \tag{3.3}$$

Proof. The usual homotopy inverse v is defined linearly by

$$v([v_{i_0}, \dots, v_{i_t}]) = (v_{i_0}, \dots, v_{i_t})$$

when $i_0 < \cdots < i_t$, and clearly satisfies $uv = 1$.

The usual proof of the existence of T is an acyclic carrier argument which proceeds by induction on t. We will also proceed in this way, using a stronger induction hypothesis than usual. For each ordered chain $\sigma = (v_{i_0}, \dots, v_{i_t})$ let us define the *support* of σ to be $\mathrm{supp}(\sigma) = \{v_{i_0}, \dots, v_{i_t}\}$ and denote by $2^{\mathrm{supp}(\sigma)}$ the simplex whose faces are the subsets of $\mathrm{supp}(\sigma)$. We will prove by induction that for each $t \geq 0$ there is a map $T : C^{\mathrm{ord}}_t(\Delta) \to C^{\mathrm{ord}}_{t+1}(\Delta)$ satisfying 3.2 and 3.3 together with the further property:

$$\text{for each ordered chain } \sigma, \quad T(\sigma) \subseteq C^{\mathrm{ord}}(2^{\mathrm{supp}(\sigma)}). \tag{3.4}$$

We readily see that conditions (3.3) and (3.4) taken together may be straightforwardly expressed by saying that $T(\sigma)$ is always a linear combination of ordered chains, all of whose supports exactly equal $\mathrm{supp}(\sigma)$.

We check in the case $t = 0$ that we may choose $T = 0$ to satisfy these conditions. Supposing now that $t > 0$ and we have defined T on $C^{\mathrm{ord}}_{t'}(\Delta)$ when $t' < t$, we need to define $T(v_{i_0}, \dots, v_{i_t})$ in such a way that

$$dT(v_{i_0}, \dots, v_{i_t}) = (v_{i_0}, \dots, v_{i_t}) - uv(v_{i_0}, \dots, v_{i_t}) - Td(v_{i_0}, \dots, v_{i_t}).$$

As in the usual proof, a straightforward calculation using the inductive hypothesis of (3.2) shows that the right-hand side is a cycle, which we denote z. It is clear from the induction hypothesis that every ordered chain occurring in z has support in $2^{\mathrm{supp}(\sigma)}$, where $\sigma = (v_{i_0}, \dots, v_{i_t})$, but closer inspection shows that, in fact, the support sets of these ordered chains are all either $\mathrm{supp}(\sigma)$ or of co-cardinality 1 in $\mathrm{supp}(\sigma)$.

To finish the usual proof, one notes that the simplex $2^{\mathrm{supp}(\sigma)}$ is acyclic, so that one can find a chain $c \in C^{ord}_{t+1}(2^{\mathrm{supp}(\sigma)})$ having $dc = z$. For example, one can explicitly construct such a c as follows. Pick a fixed vertex w in $\mathrm{supp}(\sigma)$ and linearly define a map $C^{ord}_t(2^{\mathrm{supp}(\sigma)}) \to C^{ord}_{t+1}(2^{\mathrm{supp}(\sigma)})$ via $w * (v_{i_0}, \ldots, v_{i_t}) = (w, v_{i_0}, \ldots, v_{i_t})$. It is straightforward to check that

$$d(w * a) = a - w * da \tag{3.5}$$

for all a in $C^{ord}(2^V)$, and hence since z is a cycle, taking $T(v_{i_0}, \ldots, v_{i_t}) = w * z$ does the job.

For our purposes, we want to choose $T(\sigma)$ with the extra property that every one of its terms has support $\mathrm{supp}(\sigma)$, and unfortunately $w * z$ may not satisfy this extra condition. The proof is completed by the following lemma.

Lemma 3.3. *Let V be a finite set and let z be a cycle in $C^{ord}_t(2^V)$ such that every ordered chain in z either has support V or has support of co-cardinality 1 in V. Then it is possible to choose c in $C^{ord}_{t+1}(2^V)$ with $dc = z$ such that every ordered chain in c has support V.*

We remark that one cannot further relax the condition on the co-cardinality of the support in the preceding proposition: if one takes $V = \{v_1, v_2, v_3\}$, then $z = (v_1, v_1)$ is a cycle whose unique ordered chain has support of co-cardinality 2, and $z \neq dc$ for any c whose ordered chains all have support V.

Proof. To prove the lemma, order the elements of V as $v_1, \ldots, v_s$, and let q be the largest index such that every ordered chain in z has support containing $\{v_1, \ldots, v_q\}$ (so $q = 0$ unless every ordered chain in z has v_1 in its support). One proves the proposition by induction on $|V| - q$. When this quantity is 0, every ordered chain in z has V as its support, and then taking $c = w * z$ (for *any* choice of w in V) will work.

In the inductive step, assume that every ordered chain in z contains $\{v_1, \ldots, v_q\}$. Write $z = a + b$ where a is the sum of all terms in z whose support does *not* contain v_{q+1}, and b consists of the remaining terms (those whose support contains therefore $\{v_1, \ldots, v_q, v_{q+1}\}$). We wish to replace z by a new cycle z' defined as follows:

$$\begin{aligned} z' &:= z - d(v_{q+1} * a) \\ &= a + b - (a - v_{q+1} * da) \quad \text{using (3.5)} \\ &= b - v_{q+1} * da \end{aligned}$$

Note that indeed z' is again a cycle. Note also that z' differs from z by the boundary of $v_{q+1} * a$, which is a chain having all ordered chains with support V. (Since a only contains ordered chains which avoid v_{q+1} in their support and have co-cardinality at most 1, they must all have support exactly $V - \{v_{q+1}\}$). Therefore we can safely replace z by z'.

However, we claim that induction applies to z', i.e. that all its ordered chains have support containing $\{v_1, \ldots, v_q, v_{q+1}\}$. To see this, recall that $z' = b - v_{q+1} * da$, and note that it is true by definition for the ordered chains contained in b. For the ordered chains in $v_{q+1} * da$, first observe that $da = -db$ since $z = a+b$ was a cycle. Note that every ordered chain in db must have support containing at least one subset of $\{v_1, \ldots, v_q, v_{q+1}\}$ of cardinality q, but since these ordered chains also occur in $da(= -db)$, they cannot contain v_{q+1}. Therefore every ordered chain in da or db has support containing $\{v_1, \ldots, v_q\}$, and the desired assertion for ordered chains in $v_{q+1} * da$ follows. This completes the proof of the proposition, and the lemma. □

Theorem 3.4. *Diagram 3.1 is a proper resolution of C^{ori} by complexes in the sense of Mac Lane* [19, Chapter XII §11]. □

The definition of a *proper resolution* of complexes is that it is an exact sequence of complexes such that in each degree the complexes of boundaries and of cycles are also exact.

Proof. Let us write the horizontal differentials in (3.1) as d^{hor} and the vertical differentials as d^{vert}, so that the total differential is $d = d^{\mathrm{hor}} + d^{\mathrm{vert}}$. We show that there is a mapping τ of degree $+1$ defined on each row of (3.1) so that $d^{\mathrm{hor}}\tau + \tau d^{\mathrm{hor}} = 1$ and so that τ commutes with the vertical differentials: $d^{\mathrm{vert}}\tau + \tau d^{\mathrm{vert}} = 0$. In fact this mapping τ is simply the restriction of T to each row (which makes sense by condition (3.3)) except on the left hand edge of (3.1) where we define τ to be v. Then the claimed equations are another way of writing the equation $dT + Td = 1 - vu$. From this we deduce that τ restricts to a mapping on the complexes of cycles with respect to d^{vert} and also on the complexes of boundaries with respect to d^{vert}, and on each of these complexes $d^{\mathrm{hor}}\tau + \tau d^{\mathrm{hor}} = 1$ still holds. Thus all of these complexes are chain homotopy equivalent to the zero complex and so are acyclic. □

Theorem 3.4 has consequences for the homology groups of the vertical complexes. Let $H^{\mathrm{vert}}_{r,s}(\Delta)$ denote the homology in the r^{th} column in 3.1 with respect to its vertical differential at the (r, s) position.

Corollary 3.5. *The homology groups $H^{\text{vert}}_{*,*}(\Delta)$ resolve (horizontally) the simplicial homology groups $H_*(\Delta)$. That is, for each s, the sequence*

$$0 \leftarrow H_s(\Delta) \leftarrow H^{\text{vert}}_{0,s}(\Delta) \leftarrow H^{\text{vert}}_{1,s}(\Delta) \leftarrow \dots \tag{3.6}$$

is exact.

Proof. See [19, Chapter XII Proposition 11.2]. □

We now consider the extent to which our double complex construction is natural, and show that it gives a double complex structure on the bar resolution for a group. Recall that if Δ, Δ' are simplicial complexes on vertex sets V, V', respectively, a set map $f : V \to V'$ is called *simplicial* if it takes faces of Δ to faces of Δ'. Simplicial maps induce homomorphisms of the chain complexes $C^{\text{ori}}, C^{\text{ord}}$ and homology groups via

$$f(v_{i_0}, \dots, v_{i_r}) = (f(v_{i_0}), \dots, f(v_{i_r}))$$
$$f[v_{i_0}, \dots, v_{i_r}] = [f(v_{i_0}), \dots, f(v_{i_r})].$$

One sees that f will respect the double complex structure $C^{\text{ord}}_{r,s}$ if and only if f is *dimension-preserving*, that is, for every face σ in Δ, the face $f(\sigma)$ in Δ' has the same dimension (cardinality). One can check that the homology groups $H^{vert}_{*,*}(\Delta)$ and exact sequences (3.6) are functorial with respect to dimension-preserving simplicial maps.

An important special case of dimension-preserving simplicial maps occurs when G is a group of simplicial automorphisms of Δ. In this situation G acts on the chain groups in both C^{ori} and C^{ord} by the formulas

$$g(v_{i_0}, \dots, v_{i_r}) = (gv_{i_0}, \dots, gv_{i_r})$$
$$g[v_{i_0}, \dots, v_{i_r}] = [gv_{i_0}, \dots, gv_{i_r}]$$

so that these are chain complexes of $\mathbb{Z}G$-modules. We point this out because we do not assume the condition that the stabilizer of each simplex must necessarily stabilize its vertices, which often is assumed when equivariant questions are considered. We warn the reader that in our situation the equivariant type of the complexes C^{ori} and C^{ord} may be changed if we replace Δ by a G-complex equivariantly homotopy equivalent to Δ (for example, the barycentric subdivision). Furthermore, whereas the map $C^{\text{ord}} \to C^{\text{ori}}$ is a map of chain complexes of $\mathbb{Z}G$-modules which is an ordinary chain homotopy equivalence, it need not be an equivariant homotopy equivalence. An example illustrating distinct equivariant types may be found by considering the action of a cyclic group of order 2 on a single edge, interchanging the two vertices.

If G acts simplicially on Δ then the stabilizer of a basis element $(v_{i_0}, \dots, v_{i_r})$ of C^{ord} is the intersection of the stabilizers of the v_{i_j}, and if any of these stabilizers happens to be the identity then this basis element lies in a free G-orbit. We will be particularly interested in the case where G permutes a set Ω and Δ is the simplicial complex whose simplices are the finite subsets of Ω (so that when Ω is finite, Δ is a single simplex). In this case, since Δ is contractible both of the simplicial chain complexes C^{ori} and C^{ord} are acyclic, except in dimension 0 where the homology is the trivial module $\mathbb{Z}$. As a consequence of these observations we have the following well-known result.

Proposition 3.6. *Suppose that Δ is a simplicial complex whose vertices are permuted freely by G and such that every finite subset of the vertices is a simplex. Then C^{ord} is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. When the set of vertices is G with the regular G-action, C^{ord} is a copy of the unnormalized bar resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

Proof. For a description of the unnormalized bar resolution see [19, Chapter 4 §5]. We observe simply that when Δ has G as its vertices, the basis elements of C^{ord} exactly correspond to the basis elements of the unnormalized bar resolution. □

4. The column spectral sequence in the (co)homology of groups

In this section we assume that Δ is the simplicial complex which has as its vertices the elements of a group G, and in which every finite subset is a simplex, so that Δ is contractible. We have seen that in this situation C^{ord} is a resolution of $\mathbb{Z}$ by free $\mathbb{Z}G$-modules and that it may be written as a double complex. When M is a $\mathbb{Z}G$-module, applying the functor $\_\_ \otimes_{\mathbb{Z}G} M$ to the double complex for C^{ord} gives a double complex whose total complex is $C^{\mathrm{ord}} \otimes_{\mathbb{Z}G} M$, with terms $C_{r,s} \otimes_{\mathbb{Z}G} M$. The homology of the total complex is $H_*(G, M)$. Filtering the double complex by columns, or by rows, we obtain two spectral sequences, each converging to $H_*(G, M)$. In a similar way, applying the functor $\mathrm{Hom}_{\mathbb{Z}G}(\quad, M)$ we obtain two spectral sequences, each converging to $H^*(G, M)$.

We begin by analyzing the spectral sequence obtained from filtering by columns. We will see that it stops at the E^2 page. The spectral sequence obtained from filtering by rows is the subject of the next section, and the reader interested mainly in this second spectral sequence (which has an application to computing group cohomology) can skip directly to Section 5 at this point.

The first thing we need to do is to determine what the columns of the double complex for the bar resolution $C^{\mathrm{ord}}(G)$ look like in order to identify the E^0 page of the spectral sequence. It is here that the cell complex K_n of words without repetitions and its shellability play a crucial role. We will assume in the next result that G is finite, and immediately afterwards deduce what happens when G is infinite.

Proposition 4.1. *Let Δ be the simplicial complex in which the vertices are a finite set Ω and in which every subset is a simplex. Then the vertical complexes $C^{\text{vert}}_{r,*}$ obtained from the double complex for C^{ord} all have zero homology, except in the top degree $|\Omega|-1$, and except for the left vertical edge $C^{\text{vert}}_{0,*}$ which additionally has homology $\mathbb{Z}$ in dimension 0.*

Proof. We wish to identify these vertical complexes with the cellular homology complexes of links of faces in K_n for various values of n. Given a basis element $w = (v_{i_0}, \dots, v_{i_{r+s}})$ in $C_{r,s}$, let $\text{Repeats}(w) = u_1 \cdots u_t$ denote the *repeats subword* of w, consisting of those letters which appear at least twice in w. It is easy to check that the vertical boundary map sends w to a combination of words w' which satisfy $\text{Repeats}(w') = \text{Repeats}(w)$, and consequently the vertical complex $C^{\text{vert}}_{r,*}$ is a direct sum of complexes $C^{\text{vert,u}}_{*}$ having as $\mathbb{Z}$-basis the elements w with fixed repeats subword u.

We claim that as a complex $C^{\text{vert,u}}_{*}$ is isomorphic (up to a shift in degree) to the cellular chain complex for the link of the face corresponding to the word $12\cdots t$ in K_n where $n = |\Omega| + r$ and t is the length of u. Recall [13, §3.3] that this link is defined to be the unique (up to isomorphism) regular CW-complex having face poset given by the words without repetitions from the alphabet $[n]$ which contain the word $12\cdots t$ as a subword. To make the identification, let Ω' be the set of $|\Omega| - t + r$ letters in Ω which do not occur in u, and choose an arbitrary bijection $\phi : \Omega' \to \{t+1, t+2, \dots, |\Omega| + r\}$. Then given a word w having $\text{Repeats}(w) = u$, replace the entries in w from u by the letters $1, 2, \dots, t$ from left-to-right, and replace each letter v of w from Ω' by $\phi(v)$. By abuse of notation, call this new word $\phi(w)$. For example, let

$$\begin{aligned}
\Omega &= \{a, b, c, d, e, f\} \\
u &= bebbe \\
t &= 5 \\
r &= 3 \\
\Omega' &= \{a, c, d, f\}
\end{aligned}$$

If we choose the bijection $\phi : \Omega' \to \{6, 7, 8, 9\}$ to be

$$\begin{aligned} \phi : a &\mapsto 6 \\ c &\mapsto 7 \\ d &\mapsto 8 \\ f &\mapsto 9, \end{aligned}$$

then on some typical words w with $\mathrm{Repeats}(w) = u = bebbe$ the map ϕ looks like

$$\begin{aligned} \phi(\mathbf{bebbe}) &= \mathbf{12345} \\ \phi(\mathbf{b}\mathrm{cf}\mathbf{eb}\mathrm{a}\mathbf{be}) &= \mathbf{1}79\mathbf{23}6\mathbf{45} \\ \phi(\mathbf{be}\mathrm{d}\mathbf{bb}\mathrm{a}\mathbf{e}\mathrm{fc}) &= \mathbf{12}8\mathbf{34}6\mathbf{5}97 \end{aligned}$$

Note that the repeats word u indexes a basis element in $C^{\mathrm{vert,u}}_{t-r-1}$, while its image $\phi(u) = 12 \cdots t$ indexes a basis element corresponding to the empty face (which has dimension -1) in $\mathrm{link}_{K_n} 12 \cdots t$. Thus we see that the summand $C^{\mathrm{vert,u}}_{*}$ in the vertical complex $C^{\mathrm{vert}}_{r,*}$ is identified with the augmented cellular chain complex $\tilde{C}_*(\mathrm{link}_{K_n} 12 \cdots t)$ up to a shift in degree by $t - r$, that is $C^{\mathrm{vert,u}}_{i} \cong \tilde{C}_{i-(t-r)}(\mathrm{link}_{K_n} 12 \cdots t)$. We conclude that

$$H^{\mathrm{vert,u}}_{i} \cong \tilde{H}_{i-(t-r)}(\mathrm{link}_{K_n} 12 \cdots t).$$

A special case occurs when $r = 0$ (i.e. u is empty, so $t = 0$), as there is no group $C^{\mathrm{vert},\emptyset}_{-1}$ occurring in the leftmost column. Here one obtains an identification with the ordinary (unaugmented) cellular chain complex $\tilde{C}_*(K_n)$ (with no shift in degree) and hence

$$H^{\mathrm{vert}}_{0,i} = H^{\mathrm{vert},\emptyset}_{i} \cong H_i(K_n).$$

As mentioned earlier, since K_n is shellable, it and each of its links have only reduced homology in the top dimension. As $\mathrm{link}_{K_n} 12 \cdots t$ has dimension

$$(n - 1) - t = (|\Omega| + r - 1) - t = (|\Omega| - 1) - (t - r),$$

this implies $\tilde{H}_{i-(t-r)}(\mathrm{link}_{K_n} 12 \cdots t)$ vanishes for $i < |\Omega| - 1$, and consequently each vertical complex has no homology below dimension $|\Omega| - 1$ (with the exception of $H_0 = \mathbb{Z}$ occurring in the 0^{th} column, due to the fact that K_n is connected). □

Corollary 4.2. *Let Δ be an infinite simplicial complex in which every finite subset of the set of vertices is a simplex. Then the vertical complexes $C^{\mathrm{vert}}_{r,*}$ obtained from the double complex for C^{ord} are all acyclic, except for the left vertical edge $C^{\mathrm{vert}}_{0,*}$ which has homology $\mathbb{Z}$ in dimension 0.*

Proof. Assuming that $(r,s) \neq (0,0)$, if $z \in C^{\mathrm{vert}}_{r,s}$ is a cycle then it is a linear combination of simplices which lie in some finite subsimplex of Δ, and so provided the number of vertices of this subsimplex is at least $s+2$ it is in the image of the vertical boundary map by Proposition 4.1. Since Δ is contractible, and the vertical complexes form the E^1 page of a spectral sequence which converges to $H_*(\Delta)$, the left vertical edge $C^{\mathrm{vert}}_{0,*}$ has homology $\mathbb{Z}$ in dimension 0. □

We are now able to specify the form of the E^1 page of the spectral sequence arising from the filtration of the double complex by columns. We assume that G permutes Ω freely and that Δ is the simplicial complex with vertices Ω and in which every finite subset of Ω is a simplex. Let us first deal with the case when G is infinite.

Theorem 4.3. *Let G be an infinite group which freely permutes a set Ω, and consider the columns of the double complex of ordered chains of Δ as above. The left-hand vertical column $C^{\mathrm{vert}}_{0,*}$, whose modules have as bases the ordered chains without repeats, is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. The spectral sequence obtained by filtering the double complex $C^{\mathrm{ord}} \otimes_{\mathbb{Z}G} M$ by columns has E^1 page which is non-zero only on the left edge:*

$$E^1_{r,s} = \begin{cases} H_s(G,M) & \text{if } r=0, \\ 0 & \text{if } r \geq 1 \end{cases}$$

Proof. We have seen that the left edge is an acyclic complex except in degree 0 where its homology is $\mathbb{Z}$. Since G permutes Ω freely, the modules which appear are free. This establishes the first statement and also the form of $E^1_{r,s}$ when $r=0$. When $r>0$ each vertical complex $C^{\mathrm{vert}}_{r,*}$ is an acyclic complex of free $\mathbb{Z}G$-modules which is bounded below, so it splits completely and after applying $\_\_ \otimes_{\mathbb{Z}G} M$ has zero homology. □

We now deal with the case of a finite group.

Theorem 4.4. *Let G be a finite group which freely permutes a finite set Ω. The spectral sequence obtained by filtering the double complex $C^{\mathrm{ord}} \otimes_{\mathbb{Z}G} M$ by columns has E^1-page*

$$\begin{array}{ccccccc}
K & \longleftarrow & P_{|\Omega|} \otimes M & \longleftarrow & P_{|\Omega|+1} \otimes M & \longleftarrow & \cdots \\
H_{|\Omega|-2}(G,M) & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots \\
\vdots & & \vdots & & \vdots & & \\
H_0(G,M) & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots
\end{array}$$

where

$$0 \leftarrow \mathbb{Z} \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$$

is a projective resolution of $\mathbb{Z}$ *and* $K = \mathrm{Ker}(P_{|\Omega|-1} \otimes M \to P_{|\Omega|-2} \otimes M)$. *It stops at the* E^2*-page.*

Proof. As in the proof of Theorem 4.3, when $r \geq 1$ the vertical complexes $C^{\mathrm{ord}}_{r,*}$ split completely as complexes of $\mathbb{Z}G$-modules, since they are acyclic except in their top dimension, and they are complexes of free $\mathbb{Z}G$-modules. It follows from this that the top homologies are projective $\mathbb{Z}G$-modules when $r \geq 1$. By Corollary 3.5 these top homologies form a projective resolution over $\mathbb{Z}G$ of the top homology of the left vertical edge.

On the other hand, from Proposition 4.1 the left vertical edge complex is the start of a resolution of $\mathbb{Z}$ by free $\mathbb{Z}G$-modules, and so its top homology is the syzygy $X = \mathrm{Ker}(C^{\mathrm{vert}}_{0,|\Omega|-1} \to C^{\mathrm{vert}}_{0,|\Omega|-2})$ of the trivial module. We write $P_s = C^{\mathrm{vert}}_{0,s}$ when $0 \leq s \leq |\Omega| - 1$. On applying $\_\_ \otimes_{\mathbb{Z}G} M$ to the double complex and taking homology with respect to the vertical differentials we obtain the group homology $H_n(G, M)$ on the left hand edge except in the top dimension $|\Omega| - 1$ where the homology is K. In the other vertical columns we obtain $P_{(|\Omega|-1)+r} \otimes_{\mathbb{Z}G} M$ where

$$0 \leftarrow X \leftarrow P_{|\Omega|} \leftarrow P_{|\Omega|+1} \leftarrow \cdots$$

is the projective resolution of X given by the top homologies of the columns. □

5. The row spectral sequence in the (co)homology of groups

We now turn to the other spectral sequence indicated at the start of Section 4, obtained by filtering by rows. We again specialize to the case where Δ is the simplicial complex whose vertices are a set Ω permuted regularly by G and in which every finite subset is a simplex, and we let C^{ord} and C^{ori} be the ordered and oriented chain complexes of Δ. From the discussion at the start of Section 4 we have two spectral sequences, each of which converges to the cohomology of G.

In the rest of this section we describe the 0 and 1 pages of the spectral sequence obtained by filtering by rows, showing how it may be computed and indicating its application to group cohomology. To start with we need to describe the rows of the double complex C^{ord} to obtain the E^0-page of this spectral sequence. We see from Theorem 3.4 that each row $C^{\mathrm{ord}}_{*,n}$ is in fact an acyclic complex except at the end, where the homology is C^{ori}_n. Since the modules in C^{ord} are free $\mathbb{Z}G$-modules, each row is thus a projective resolution of the corresponding module in C^{ori}. We wish to describe the modules C_n^{ori}.

Lemma 5.1. *Let Δ be the simplicial complex of finite subsets of G. We may write*

$$C_n^{\text{ori}} = \bigoplus_j \tilde{\mathbb{Z}} \uparrow_{H_{n,j}}^G$$

as $\mathbb{Z}G$-modules, where j ranges over the orbits of the action of G on subsets of G of size $n+1$, $H_{n,j}$ is the stabilizer of a subset in such an orbit, and $\tilde{\mathbb{Z}} = \mathbb{Z}\cdot[v_{i_0},\dots,v_{i_n}]$ is the $\mathbb{Z}H_{n,j}$-module of $\mathbb{Z}$-rank 1 where the action of $H_{n,j}$ is given by the sign representation of $H_{n,j}$ acting on the subset $\{v_{i_0},\dots,v_{i_n}\}$.

Proof. From its definition C_n^{ori} is the direct sum of the abelian groups $\mathbb{Z}\cdot[v_{i_0},\dots,v_{i_n}]$ which are permuted by G, and the stabilizer of such a direct summand in this action is precisely the set-wise stabilizer of $\{v_{i_0},\dots,v_{i_n}\}$. The action of the stabilizer $H_{n,j}$ on $\mathbb{Z}\cdot[v_{i_0},\dots,v_{i_n}]$ is via the sign of the permutation representation of $H_{n,j}$ on $\{v_{i_0},\dots,v_{i_n}\}$. Specifying this information is exactly the same as giving a decomposition as a direct sum of induced modules as claimed. □

As a consequence, if we filter the double complex $C^{\text{ord}} \otimes_{\mathbb{Z}G} M$ by rows we get a spectral sequence whose E^0-page consists of projective resolutions of $\bigoplus \tilde{\mathbb{Z}} \uparrow_{H_{n,j}}^G$ tensored with M. We have an analogous description of $\operatorname{Hom}_{\mathbb{Z}G}(C^{\text{ord}}, M)$. Accordingly we have:

Proposition 5.2. *Let M be a $\mathbb{Z}G$-module. There are 1st quadrant spectral sequences with first pages*

$$E^1_{r,s} = \bigoplus_j \operatorname{Tor}_r^{\mathbb{Z}H_{s,j}}(\tilde{\mathbb{Z}}, M) \Rightarrow H_{r+s}(G,M)$$

and

$$E_1^{r,s} = \bigoplus_j \operatorname{Ext}^r_{\mathbb{Z}H_{s,j}}(\tilde{\mathbb{Z}}, M) \Rightarrow H^{r+s}(G,M)$$

where for each n, $C_n^{\text{ori}} = \bigoplus \tilde{\mathbb{Z}} \uparrow_{H_{n,j}}^G$ is a decomposition of the module of ordered subsets of G of size $n+1$.

Proof. By the previous discussion the rows of the E^1-page of the homology spectral sequence have terms $\operatorname{Tor}_r^{\mathbb{Z}G}(C_s^{\text{ori}}, M)$. Using the Eckmann-Schapiro lemma [8, Proposition X.7.3] we obtain the stated result. □

We point out that these spectral sequences are isomorphic from the first page onwards to the hyperhomology and hypercohomology spectral sequences associated to the complex $C^{\text{ori}} \otimes_{\mathbb{Z}} M$, as described in [7, Section VII.5]. However, we do not need to know this for the purposes of the present exposition.

We now describe how the terms in these spectral sequences may be computed, illustrating the calculation with an example. The first step is to find the orbits of the action of G on the subsets of G of size $n+1$ for each n. We start with two very straightforward observations. The second is often set as an elementary exercise in group theory.

Lemma 5.3. *A subgroup H of G stabilizes a subset $\{v_{i_0}, \dots, v_{i_n}\}$ if and only if the subset is a union of cosets of H:*

$$\{v_{i_0}, \dots, v_{i_n}\} = Hx_1 \cup \cdots \cup Hx_t$$

for some $x_1, \dots, x_t \in G$. Consequently, the stabilizer of such a subset is the largest subgroup H for which the subset is a union of cosets of H. □

Lemma 5.4. *Let the finite group H act freely on a finite set W. The sign representation of H in this action is non-trivial if and only if H has a non-identity cyclic Sylow 2-subgroup and the number of orbits of H on W is odd.*

Proof. Any element $h \in H$ acts on W as the product of a number of cycles of length $|\langle h \rangle|$. The action has sign -1 if and only if the number of cycles is odd and $|\langle h \rangle|$ is even. This forces $\langle h \rangle$ to be a Sylow 2-subgroup of H, and the number of H-orbits on W to be odd; and conversely in this situation h acts with sign -1. □

At this point we present as an example the structure of the spectral sequence when $G = D_8$ is the dihedral group of order 8 and $M = \mathbb{Z}$ is the trivial module. This is a small example which we use to illustrate the principle behind our constructions, the calculations being more transparent than in a larger example. Towards the end of this section we indicate in more abstract terms the properties which have been exemplified.

Let us give names to the subgroups of G as follows: there are three subgroups A, B, C of order 4, one of which (C, say) is cyclic. Let D and E be representatives of the two non-central conjugacy classes of subgroups of order 2, and let F denote the center of G. We claim that as a complex of $\mathbb{Z}G$-modules C^{ori} is

(5.1)
$$
\begin{array}{ccc}
C_7^{\mathrm{ori}} & \cong & \mathbb{Z} \\
\downarrow & & \downarrow \\
C_6^{\mathrm{ori}} & \cong & \mathbb{Z}G \\
\downarrow & & \downarrow \\
C_5^{\mathrm{ori}} & \cong & (\tilde{\mathbb{Z}}\uparrow_D^G)^2 \oplus (\tilde{\mathbb{Z}}\uparrow_E^G)^2 \oplus \tilde{\mathbb{Z}}\uparrow_F^G \oplus \mathbb{Z}G \\
\downarrow & & \downarrow \\
C_4^{\mathrm{ori}} & \cong & \mathbb{Z}G^7 \\
\downarrow & & \downarrow \\
C_3^{\mathrm{ori}} & \cong & \mathbb{Z}\uparrow_A^G \oplus \mathbb{Z}\uparrow_B^G \oplus \tilde{\mathbb{Z}}\uparrow_C^G \oplus (\mathbb{Z}\uparrow_D^G)^2 \oplus (\mathbb{Z}\uparrow_E^G)^2 \oplus \mathbb{Z}G^6 \\
\downarrow & & \downarrow \\
C_2^{\mathrm{ori}} & \cong & \mathbb{Z}G^7 \\
\downarrow & & \downarrow \\
C_1^{\mathrm{ori}} & \cong & (\tilde{\mathbb{Z}}\uparrow_D^G)^2 \oplus (\tilde{\mathbb{Z}}\uparrow_E^G)^2 \oplus \tilde{\mathbb{Z}}\uparrow_F^G \oplus \mathbb{Z}G \\
\downarrow & & \downarrow \\
C_0^{\mathrm{ori}} & \cong & \mathbb{Z}G
\end{array}
$$

We have omitted a $\sim$ over a $\mathbb{Z}$ here when the action of the stabilizer subgroup via the sign representation is trivial, the occurrence of a non-trivial action being predicted by Lemma 5.4. The decomposition of the modules in C^{ori} may be computed by *ad hoc* means using Lemma 5.3. However, we will indicate in Section 6 how this calculation may be done in an efficient fashion.

We now present the E_1-page of the cohomology spectral sequence which converges to $H^*(D_8, \mathbb{Z})$. The cohomology of the cyclic groups which appear here is computed by elementary methods as in [8] and for the groups A and B we may use the Künneth formula.

$$
\begin{array}{ccccccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & H^0 & H^1 & H^2 & H^3 & H^4 & H^5 & H^6 & H^7 & \cdots \\
 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & \mathbb{Z} & C_2^5 & 0 & C_2^5 & 0 & C_2^5 & 0 & C_2^5 & \cdots \\
 & \mathbb{Z}^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & \mathbb{Z}^{12} & C_2 & C_2^8 & C_2^3 & C_2^{10} & C_2^5 & C_2^{12} & C_2^7 & \cdots \\
 & \mathbb{Z}^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & \mathbb{Z} & C_2^5 & 0 & C_2^5 & 0 & C_2^5 & 0 & C_2^5 & \cdots \\
s\Big\uparrow & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & \xrightarrow{r} & & & & & & & &
\end{array}
$$

The E^1 page of the spectral sequence for $H^*(D_8, \mathbb{Z})$.

To simplify the notation here we have written C_2 to denote a cyclic group of order 2, and H^i to denote the cohomology group $H^i(D_8, \mathbb{Z})$. Note that

$$
\begin{aligned}
H^0(G, \mathbb{Z}) &= \mathbb{Z} \\
H^1(G, \mathbb{Z}) &= 0 \\
H^2(G, \mathbb{Z}) &\cong G/G'
\end{aligned}
$$

and $H^3(G, \mathbb{Z})$ is isomorphic to the Schur multiplier of G, for any finite group G (use the integral duality theorem [8, XII.6.6] together with the usual results for homology).

This example illustrates a number of properties of this spectral sequence, which we now describe.

Proposition 5.5. *For every finite group G, if* g.c.d.$(s+1, |G|) = 1$ *then in row s of the cohomology spectral sequence of Proposition 5.2 we have*

$$
E_1^{r,s} = \begin{cases} M^t & \text{if } r = 0 \\ 0 & \text{if } r \geq 1 \end{cases}
$$

where $t = \binom{|G|}{s+1}/|G|$.

Proof. This follows from Lemma 5.3, because in this situation every subset of size $s+1$ must have trivial stabilizer. Since C^{ori}_{s+1} is a free module of $\mathbb{Z}$-rank $\binom{|G|}{s+1}$, it is a free $\mathbb{Z}G$-module of rank t. The terms in row s are thus $\text{Ext}^r_{\mathbb{Z}G}(\mathbb{Z}G^t, M)$ as in Proposition 5.2. When $r \geq 1$ these groups vanish, and $\text{Ext}^0_{\mathbb{Z}G}(\mathbb{Z}G^t, M) \cong \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}G^t, M) \cong M^t$. □

We see from this that the spectral sequence has many rows which are zero, except at the left edge.

Proposition 5.6. *The top non-zero row of the spectral sequence occurs when* $s = |G|-1$*, and on this top row we have* $E_1^{r,|G|-1} = \text{Ext}^r_{\mathbb{Z}G}(\tilde{\mathbb{Z}}, M)$.

Proof. The fact that the spectral sequence is only non-zero in rows up to row $|G|-1$ is immediate from the fact that C^{ori} is zero in degree $|G|$ and above. The identification of the top row arises because $C^{\text{ori}}_{|G|} = \tilde{\mathbb{Z}}$. □

We note that unless G has a non-identity cyclic Sylow 2-subgroup then $\tilde{\mathbb{Z}} = \mathbb{Z}$ by Lemma 5.4. Provided G does not have such a subgroup we thus have $E_1^{r,|G|-1} = H^r(G, M)$ on the top row.

Proposition 5.7. *In case* $M = \mathbb{Z}$ *and* G *does not have a non-identity cyclic Sylow 2-subgroup, the differential* $d : E_1^{0,|G|-2} \to E_1^{0,|G|-1}$ *is multiplication by* $|G| : \mathbb{Z} \to \mathbb{Z}$.

Proof. Here the differential is the functor $\text{Hom}_{\mathbb{Z}G}(\quad, M)$ applied to the top boundary map $d_{|G|-1} : C^{\text{ori}}_{|G|-1} \to C^{\text{ori}}_{|G|-2}$ where $C^{\text{ori}}_{|G|-1} \cong \tilde{\mathbb{Z}} \cong \mathbb{Z}$ and $C^{\text{ori}}_{|G|-2} \cong \mathbb{Z}G$. Under these isomorphisms, (as one may check in a technically elementary fashion) the boundary map embeds $\mathbb{Z}$ as multiples of the element $\sum_{g \in G} g$. Since $\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{Z})$ is generated by the augmentation map, which sends $\sum_{g \in G} g$ to $|G|$, the map $\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{Z}) \to \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$ is multiplication by $|G|$. □

To illustrate these properties we show how to use the spectral sequence to deduce lower bounds on the sizes of certain groups $H^n(D_8, \mathbb{Z})$. In each diagonal strip specified by $r + s = n$ when n is odd, there are only two non-zero terms when $n \geq 5$ (and only one when $n = 1$ or 3). The position of these terms is such that elements in the middle term $E_1^{r,3}$ when $r \geq 3$ is odd can only be killed in later pages of the spectral sequence on the E_4-page by a differential mapping them to something non-zero in the top row. We conclude that when $t \geq 5$ the term $E_t^{r,3}$ is constant, and for $r \geq 5$ and odd, it has size at least as big as $|E_1^{r,3}|/|E_1^{r-3,7}|$. For example, using the fact that $H^2(D_8, \mathbb{Z}) \cong D_8/D_8' \cong C_2 \times C_2$, we deduce

$$|H^8(D_8, \mathbb{Z})| \geq 2^3.$$

We may modify this argument when $r = 3$, using Proposition 5.7 to deduce that on the E_2 page $E_2^{0,7}$ is cyclic of order 8, and hence the term $E_1^{3,3}$ can only be reduced on later pages by a map to this cyclic group on the E_4 page. We deduce that

$$|H^6(D_8, \mathbb{Z})| \geq 2^2.$$

According to [14] the correct values are

$$|H^6(D_8, \mathbb{Z})| = 2^4$$
$$|H^8(D_8, \mathbb{Z})| = 2^6$$

so our estimates are not particularly sharp, but at least they have been obtained without very much difficulty.

We will show in Section 7 how this spectral sequence may be used in a more theoretical way to determine when the Ext class determined by the complex C^{ori} of subsets of G is non-zero.

6. Computing the decomposition of the subsets complex

In this section we will assume that G is finite. We will describe how the orbit decomposition of G acting on subsets of a G-set may be done in an efficient manner, and hence how the $\mathbb{Z}G$-module decomposition of the oriented chain complex C^{ori} considered in the last section may be obtained. We follow the approach of [30] and summarize the notation used there. The actions on the subsets of various sizes provide an example of what in [30] we call a *graded G-set*, namely a set partitioned as

$$\Psi = \Psi(0) \cup \Psi(1) \cup \Psi(2) \cup \cdots$$

in which each $\Psi(i)$ is a G-set. We let $B(G)$ denote the *Burnside ring* of G with $\mathbb{Q}$ coefficients (see [6]) and write $\Psi(i)$ also for the element of $B(G)$ which this G-set represents. In general, we define a power series

$$P_\Psi(t) = \sum_{i=0}^{\infty} \Psi(i) t^i$$

in the variable t with coefficients in $B(G)$ which we call the *Poincaré series* of Ψ. This power series is simply a useful tool for manipulating the information present in Ψ. In the case of the subsets of a G-set Ω we may let $\Psi(i)$ be the G-set consisting of subsets of size i, and when Ω is finite P_Ψ is a polynomial. In order to obtain the orbit decomposition of each $\Psi(i)$, our aim is to express P_Ψ in terms of polynomials in the transitive G-sets G/H which form a basis of $B(G)$.

We may describe the graded G-set of subsets of Ω as follows. Let $\Xi = \{0, 1\}$ regarded as a graded set with $\deg(0) = 0$ and $\deg(1) = 1$

and Poincaré series $f(t) = 1 + t$. We write $\mathrm{Map}(\Omega, \Xi)$ for the set of all functions $\phi : \Omega \to \Xi$ and define the degree of ϕ to be $\deg(\phi) = \sum_{g \in G} \deg(\phi(g))$. Then $\mathrm{Map}(\Omega, \Xi)$ becomes a left G-set by means of ${}^g\phi(x) = \phi(g^{-1}x)$ and in our particular situation it is isomorphic to the graded G-set of subsets of Ω.

We now quote Proposition 1.4 of [30].

Proposition 6.1. *Let Ξ be a graded set with Poincaré series f and let Ω be a finite G-set. Then*

$$P_{\mathrm{Map}(\Omega,\Xi)} = \sum_{K \leq J \leq G} \frac{G/K \cdot \mu(K, J) f_J}{|G : K|}$$

where for each subgroup J of G if $\Omega = \Omega_1 \cup \cdots \cup \Omega_n$ is the decomposition of Ω into J-orbits then $f_J(t) = f(t^{|\Omega_1|}) \cdots f(t^{|\Omega_n|})$, and μ is the Möbius function on the poset of subgroups of G.

Specializing to our setting where $\Omega = G$ and $\Xi = \{0, 1\}$, if we fix a subgroup K in this sum we obtain a polynomial whose coefficients are the multiplicities of the G-set G/K in the action on subsets of G of various sizes. For example, with $G = D_8$ a non-central subgroup D of order 2 of G has two subgroups properly containing it, namely the one previously called A and G itself, so that $\mu(D, D) = 1$, $\mu(D, A) = -1$ and $\mu(D, G) = 0$. Thus the coefficient of G/D in P_Ψ is

$$\frac{1}{4}(1 \cdot (1 + t^2)^4 - 1 \cdot (1 + t^4)^2 + 0 \cdot (1 + t^8)^1) = t^2 + t^4 + t^6.$$

Each subgroup of G conjugate to D gives a similar contribution, and since D has one other conjugate in G, there are 2 orbits isomorphic to G/D in the action of G on each of subsets of size 2, 4 and 6, and none in the other cases. Furthermore, D acts with sign $+1$ on subsets of size 4, and non-trivial sign on subsets of size 2 and 6 and so we have accounted for the terms $(\tilde{\mathbb{Z}} \uparrow_D^G)^2$ and $(\mathbb{Z} \uparrow_D^G)^2$ which appear in our description of the chain complex C^{ori} for D_8.

We see from this that an important ingredient in describing the $\mathbb{Z}G$-module structure of C^{ori} is a knowledge of the Möbius function for the poset of subgroups of G.

7. The Ext class of the subsets complex

We again assume in this section that G is finite, and let C^{ord} and C^{ori} be the ordered and oriented chain complexes of the simplicial complex whose vertices are a finite set Ω permuted by G, and in which every subset is a simplex. We have already seen that the structure of C^{ori} plays a crucial role in determining the form of the spectral sequence coming from the row filtration considered in Section 5. We now analyze this complex further. We will work in the generality that Ω is any finite G-set, and we consider the simplicial complex which is a single simplex whose vertices are the elements of Ω. The oriented chain complex of this simplicial complex augmented by a copy of the trivial module is an exact sequence of $\mathbb{Z}G$-modules of the form

$$\mathcal{E}^{\Omega} = \quad 0 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z}\Omega \leftarrow \cdots \leftarrow \mathbb{Z}\Omega \leftarrow \tilde{\mathbb{Z}} \leftarrow 0 \tag{7.1}$$

which represents a class $\zeta_\Omega \in \operatorname{Ext}^{|\Omega|-1}_{\mathbb{Z}G}(\mathbb{Z}, \tilde{\mathbb{Z}})$ and which we call the *subsets complex* of Ω. In particular for each subgroup H of G, on taking $\Omega = G/H$ we obtain a canonically defined element $\zeta_{G/H}$, and when $H = 1$ and G does not have a non-identity Sylow 2-subgroup (see Lemma 5.4) the canonical class ζ_G lies in $H^{|G|-1}(G, \mathbb{Z})$. We will now examine these cohomology classes, and investigate their significance for the spectral sequence.

We first analyze the form of $\mathcal{E}^{\Omega}$ when Ω is not a transitive G-set.

Lemma 7.1. *Let Ω and Ψ be G-sets. Then $\mathcal{E}^{\Omega \cup \Psi} \cong (\mathcal{E}^{\Omega} \otimes \mathcal{E}^{\Psi})[1]$, where on the right we mean the total complex of the tensor product, shifted by one degree.*

Proof. We may obtain basis elements of the chain groups in $\mathcal{E}^{\Omega\cup\Psi}$ by ordering the elements of $\Omega \cup \Psi$. Order them so that the elements of Ψ come after the elements of Ω. Now the basis elements of $\mathcal{E}^{\Omega\cup\Psi}_{n+1}$ are lists $[\omega_0, \cdots, \omega_r, \psi_0, \cdots, \psi_s]$ where the $\omega_i \in \Omega$ and $\psi_j \in \Psi$ are taken in order and $r+s = n$. These biject with the basis elements $[\omega_0, \cdots, \omega_r] \otimes [\psi_0, \cdots, \psi_s]$ of $\bigoplus_{r+s=n} \mathcal{E}^{\Omega}_r \otimes \mathcal{E}^{\Psi}_s$, and this bijection commutes with the boundary maps. □

Before the next result we briefly review how an exact sequence gives rise to an Ext class. Given R-modules M, N recall that $\operatorname{Ext}^*_R(M, N)$ can be computed from any projective resolution of M by applying the functor $\operatorname{Hom}_R(\quad, N)$ and taking homology of the resulting cochain complex. Given an exact sequence of R-modules

$$0 \leftarrow M \leftarrow M_0 \leftarrow \cdots \leftarrow M_{r-1} \leftarrow N \leftarrow 0 \tag{7.2}$$

one associates a class in $\operatorname{Ext}^r_R(M, N)$ as follows. Lift the identity map $M \to M$ along a projective resolution as shown

$$\begin{array}{ccccccccccccc}
0 & \longleftarrow & M & \longleftarrow & P_0 & \longleftarrow & \cdots & \longleftarrow & P_r & \longleftarrow & P_{r+1} & \longleftarrow & \cdots \\
 & & \| & & \downarrow & & & & {\scriptstyle f}\downarrow & & \downarrow & & \\
0 & \longleftarrow & M & \longleftarrow & M_0 & \longleftarrow & \cdots & \longleftarrow & N & \longleftarrow & 0 & &
\end{array}$$

The map $f \in \operatorname{Hom}_R(P_r, N)$ turns out to give rise to a well-defined class $[f]$ in $\operatorname{Ext}^r_R(M, N)$ which represents the sequence (7.2).

Proposition 7.2. *Let R be a commutative ring and suppose that*

$$\mathcal{A} = 0 \leftarrow A_{-1} \leftarrow A_0 \leftarrow \cdots \leftarrow A_t \leftarrow 0$$
$$\mathcal{B} = 0 \leftarrow B_{-1} \leftarrow B_0 \leftarrow \cdots \leftarrow B_u \leftarrow 0$$

are two exact sequences of RG-modules. Suppose that all the modules B_j are projective as R-modules. Then the total complex of $\mathcal{A} \otimes_R \mathcal{B}$ is acyclic, and the class in $\operatorname{Ext}^{t+u+1}_{RG}(A_{-1} \otimes B_{-1}, A_t \otimes B_u)$ of the total complex of $\mathcal{A} \otimes_R \mathcal{B}$ is zero.

Proof. Since the modules B_j are all projective, as a complex of R-modules the complex $\mathcal{B}$ is a direct sum of complexes of the form

$$\cdots \leftarrow 0 \leftarrow X \overset{\cong}{\leftarrow} X \leftarrow 0 \leftarrow \cdots$$

and so $\mathcal{A} \otimes_R \mathcal{B}$ is acyclic.

To show that its Ext class is zero, take a projective resolution

$$0 \leftarrow A_{-1} \otimes_R B_{-1} \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$$

and consider the diagram

$$\begin{array}{ccccccccc}
A_{-1} \otimes B_{-1} & \leftarrow & P_0 & \leftarrow & \cdots & \leftarrow & P_t & \leftarrow & P_{t+1} \\
\| & & \downarrow & & & & \downarrow & & \downarrow \\
A_{-1} \otimes B_{-1} & \leftarrow & A_0 \otimes B_{-1} & \leftarrow & \cdots & \leftarrow & A_t \otimes B_{-1} & \leftarrow & 0 \\
\uparrow & & \uparrow & & & & \uparrow & & \\
A_{-1} \otimes B_0 & \leftarrow & A_0 \otimes B_0 & \leftarrow & \cdots & \leftarrow & A_t \otimes B_0 & & \\
\uparrow & & \uparrow & & & & \uparrow & & \\
\vdots & & \vdots & & & & \vdots & & \\
\uparrow & & \uparrow & & & & \uparrow & & \\
A_{-1} \otimes B_u & \leftarrow & A_0 \otimes B_u & \leftarrow & \cdots & \leftarrow & A_t \otimes B_u & &
\end{array}$$

In this diagram the top row is the projective resolution of $A_{-1} \otimes_R B_{-1}$, and underneath it is $\mathcal{A} \otimes_R \mathcal{B}$. The top row of $\mathcal{A} \otimes_R \mathcal{B}$ is $\mathcal{A} \otimes_R B_{-1}$ and this is acyclic because B_{-1} is projective as an R-module. We may lift maps from the resolution of $A_{-1} \otimes_R B_{-1}$ along this row as shown. This is also a lifting from the resolution to $\mathcal{A} \otimes_R \mathcal{B}$. We reach a stage where the map from P_{t+1} is zero and so we can continue the lifting with zero maps. The final lift to $A_t \otimes B_u$ is zero and so the Ext class is zero. □

We immediately obtain the following corollary.

Corollary 7.3. *If Ω is not a transitive G-set then $\zeta_\Omega = 0$*

Proof. We simply put together the last two results. □

Building on this we obtain the following consequence:

Corollary 7.4. *Suppose that Ω is a G-set for which $|\Omega|$ is not a power of some prime. Then $\zeta_\Omega = 0$.*

Proof. Since $\bigoplus_P \operatorname{Res}^G_P : H^*(G, \tilde{\mathbb{Z}}) \to \bigoplus_P H^*(P, \tilde{\mathbb{Z}})$ is injective, where the sum is taken over Sylow p-subgroups P of G for different primes p (see [8, XII.10.1]) in order to check that $\zeta_\Omega \neq 0$ it suffices to check it on restriction to each Sylow p-subgroup P of G. But $\operatorname{Res}^G_P(\zeta_\Omega) = \zeta_{\Omega\downarrow^G_P}$ and if $|\Omega|$ is not a prime power then $\Omega \downarrow^G_P$ is never a transitive P-set. Thus by Corollary 7.3 $\operatorname{Res}^G_P(\zeta_\Omega)$ is always zero. □

We now go further and examine the structure of ζ_Ω when Ω is a transitive G-set. Any such Ω is isomorphic to a coset space G/K for some subgroup K of G. For the next theorem, recall that an element $\zeta \in \operatorname{Ext}^n_{\mathbb{Z}G}(A, B)$ is said to be *essential* if $\operatorname{Res}^G_H(\zeta) = 0$ for every proper subgroup H of G. For a discussion of the role of essential elements in cohomology, see for example [1].

Theorem 7.5. *Let H and K be subgroups of G. Then*

$$\operatorname{Res}^G_H \zeta_{G/K} = \begin{cases} \zeta_{H/H\cap K} & \text{if } HK = G, \\ 0 & \text{otherwise.} \end{cases}$$

Hence if $K \subseteq \Phi(G)$ then $\zeta_{G/K}$ is essential, where $\Phi(G)$ denotes the Frattini subgroup of G.

Proof. We examine the orbits of H on G/K. Evidently there is just one orbit if and only if $HK = G$, so that if $HK \neq G$ then $\operatorname{Res}^G_H \zeta_{G/K} = 0$ by Corollary 7.3. On the other hand if $HK = G$ then $(G/K)\downarrow^G_H$ is a transitive H-set with stabilizer $\operatorname{Stab}_H(K) = H \cap K$, so is isomorphic to $H/H \cap K$. This shows that $\operatorname{Res}^G_H \zeta_{G/K} = \zeta_{H/H\cap K}$ in this case. The Frattini subgroup has the property that for no proper subgroup H of G is $H\Phi(G) = G$, and the final statement follows from this. □

The above theorem provides a way to construct essential elements in cohomology, and we obtain, for instance, that ζ_G is always essential. The trouble is that these elements may be zero, and indeed they often are. We will see in Corollary 7.8 that when G is an elementary abelian p-group for some prime p then $\zeta_G \neq 0$. Also when $G = C_4$ is cyclic of order 4 we may show by direct calculation that $\zeta_G \neq 0$. At the time of writing, these are the only groups for which we know that $\zeta_G \neq 0$, and we have been led to wonder whether there are indeed any others.

We will show now that it is also of interest from a different point of view if ζ_G does happen to be zero, since this condition implies a simplification of the row spectral sequence we considered in Section 5. The effect of the simplification is that when $\zeta_G = 0$ then no part of the top row of the spectral sequence survives to infinity (and the converse is also true), as will be expressed in Corollary 7.7. In any case, when $\zeta_G \neq 0$ we have information about the spectral sequence given in the next lemma which is useful for computation.

Lemma 7.6. *Let M be a $\mathbb{Z}G$-module and let $n \geq |G|-1$. The following subgroups of $H^n(G, M)$ are equal:*
(1) The cohomology classes represented by cocycles in $\mathrm{Hom}_{\mathbb{Z}G}(C_n^{\mathrm{ord}}, M)$ which vanish on $C_{n,0}^{\mathrm{ord}} \oplus \cdots \oplus C_{n-|G|+2,|G|-2}^{\mathrm{ord}}$ (i.e. which have support only on the top row of the double complex for C^{ord}),
(2) $E_\infty^{n-|G|+1,|G|-1}$, regarded as a subgroup of $H^n(G, M)$, and
(3) $\zeta_G \cdot \mathrm{Ext}_{\mathbb{Z}G}^{n-|G|+1}(\tilde{\mathbb{Z}}, M)$ where $\zeta_G \in \mathrm{Ext}_{\mathbb{Z}G}^{|G|-1}(\mathbb{Z}, \tilde{\mathbb{Z}})$ is the class of the subsets complex.

Proof. Equality of (1) and (2): the filtration on the cochain complex $\mathrm{Hom}_{\mathbb{Z}G}(C^{\mathrm{ord}}, M)$ given by its rows is finite, and the bottom term in the filtration consists of the top row of the double complex. Cocycles defined on the top row of C^{ord} remain cocycles in $\mathrm{Hom}_{\mathbb{Z}G}(C^{\mathrm{ord}}, M)$, and it comes from the construction of the spectral sequence that they represent the elements in the top row of every page of the spectral sequence from the first on. This establishes the equality.

Equality of (1) and (3): suppose that $\eta \in \mathrm{Ext}_{\mathbb{Z}G}^{n-|G|+1}(\tilde{\mathbb{Z}}, M)$. We show first that every cohomology element of the form $\zeta_G \cup \eta$ is represented by a cocycle in the top row of $\mathrm{Hom}_{\mathbb{Z}G}(C^{\mathrm{ord}}, M)$. For, suppose η is represented by an extension

$$\eta : \qquad 0 \leftarrow \tilde{\mathbb{Z}} \leftarrow A_0 \leftarrow A_1 \leftarrow \cdots \leftarrow A_t \leftarrow M \leftarrow 0$$

where $t = n - |G|$. Then $\zeta_G \cup \eta$ is represented by the Yoneda splice of $\mathcal{E}^G$ and the above extension and it is represented by a cocycle which may be obtained by lifting maps from the projective resolution C^{ord} as in the following diagram:

$$
\begin{array}{ccccccccccc}
 & & A_0 & \leftarrow & A_1 & \leftarrow & \cdots & \leftarrow & M & \leftarrow & 0 \\
 & \swarrow & \uparrow & & \uparrow & & & & \uparrow & & \\
\tilde{\mathbb{Z}} & \leftarrow & C^{\text{ord}}_{0,|G|-1} & \leftarrow & C^{\text{ord}}_{1,|G|-1} & \leftarrow & \cdots & \leftarrow & C^{\text{ord}}_{t+1,|G|-1} & \leftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
\mathbb{Z}G & \leftarrow & C^{\text{ord}}_{0,0} & \leftarrow & C^{\text{ord}}_{1,0} & \leftarrow & \cdots & \leftarrow & C^{\text{ord}}_{t+1,0} & \leftarrow & \cdots \\
\downarrow & & & & & & & & & & \\
\mathbb{Z} & & & & & & & & & &
\end{array}
$$

The bottom right portion of this diagram is the double complex for C^{ord}. At the left edge we have $C^{\text{ori}} = \mathcal{E}^G$ and along the top the sequence which represents η. The fact that we may lift along this sequence using only maps with support on the top row of C^{ord} may be seen on considering that the top row is a projective resolution of $\tilde{\mathbb{Z}}$, and it is a quotient complex of C^{ord}, so we may lift first along the quotient complex, and then compose with the quotient homomorphism from C^{ord}.

Conversely we see from the same picture that every cocycle with support in the top row represents a cohomology class of the form $\zeta_G \cup \eta$; for we may take η to be represented by the extension obtained from the top row of C^{ord} by a pushout construction using the cocycle. □

Corollary 7.7. *The following are equivalent:*
(1) For some r, $E_\infty^{r,|G|-1} \neq 0$,
(2) $E_\infty^{0,|G|-1} \neq 0$, and
(3) $\zeta_G \neq 0$ in $\operatorname{Ext}^{|G|-1}_{\mathbb{Z}G}(\mathbb{Z}, \tilde{\mathbb{Z}})$.

Proof. Clearly (2) implies (1). (3) implies (2) because $\zeta_G \in E_\infty^{0,|G|-1}$ by Lemma 7.6. Finally (1) implies (3) because any element of $E_\infty^{r,|G|-1}$ can be written $\zeta_G \cup \eta$ by Lemma 7.6 and if $\zeta_G = 0$ then $E_\infty^{r,|G|-1} = 0$ for every r. □

In what follows we will refer to the notion of the complexity at a prime p of a group G. This is the Krull dimension of the even degree part of the cohomology ring of G with $\mathbb{Z}/p\mathbb{Z}$ coefficients. The dimensions of the $\mathbb{Z}/p\mathbb{Z}$-vector spaces $H^n(G, \mathbb{Z}/p\mathbb{Z})$ have a polynomial rate of growth which has degree 1 less than the complexity. See [2] for an overview of this theory.

Corollary 7.8. *If $\zeta_G = 0$ then the complexity of G at each prime p is at most the maximum complexity of the proper subgroups of G. Consequently, if G is an elementary abelian p-subgroup for some prime p, then $\zeta_G \neq 0$.*

Proof. Taking $M = \mathbb{Z}/p\mathbb{Z}$ in the spectral sequence, if $\zeta_G = 0$ then by Lemma 7.6, the growth of $H^n(G, \mathbb{Z}/p\mathbb{Z})$ is bounded by the growth of the terms in the spectral sequence which lie below the top row, and these terms are direct sums of Ext groups taken over proper subgroups of G. This shows that the complexity of G at the prime p is at most the maximum complexity of a proper subgroup of G. However, the complexity of an elementary abelian group of rank d is equal to d, and this is larger than the complexity of any of its proper subgroups. Thus when G is an elementary abelian p-group, $\zeta_G \neq 0$. □

We may combine this corollary with Theorem 7.5 to produce the following result.

Theorem 7.9. *Let G be a group with subgroups H and E such that*
(1) E is an elementary abelian p-group for some prime p,
(2) $H \cap E = 1$, and
(3) $|G| = |H||E|$.
Then the class $\zeta_{G/H} \in \mathrm{Ext}_{\mathbb{Z}G}^{|G:H|-1}(\mathbb{Z}, \tilde{\mathbb{Z}})$ is non-zero.

Proof. From Theorem 7.5 and Corollary 7.8 it is immediate that

$$\mathrm{Res}_E^G\, \zeta_{G/H} = \zeta_E \neq 0$$

which implies the result. □

In the statement of Theorem 7.5, in the presence of conditions (1) and (2) it is equivalent to replace (3) by the condition $G = HE$. These conditions are very often satisfied. Any subgroup of the symmetric group S_{p^n} which contains an elementary abelian subgroup acting regularly satisfies these conditions, taking H to be the stabilizer of a point. For example, S_{p^n} itself satisfies these conditions. For another example consider the extraspecial p-group of exponent p and order p^3 when p is an odd prime (i.e. the Sylow p-subgroup of $GL(3,p)$). Taking H to be any non-central subgroup of order p gives a non-zero class $\zeta_{G/H} \in H^{p^2-1}(G, \mathbb{Z})$, and taking H to be a subgroup of order p^2 also gives a non-zero class $\zeta_{G/H} \in H^{p-1}(G, \mathbb{Z})$.

We conclude this section by examining the subsets complex in an important special case, namely the situation in which Ω is permuted by the full symmetric group on Ω. We illustrate the techniques we have developed by determining precisely when $\zeta_\Omega \neq 0$.

Corollary 7.10. *Let $\Omega = \{1, \dots, n\}$ be (faithfully) permuted by $\mathfrak{S}_n$. Then $\zeta_\Omega \neq 0$ if and only if n is a prime power.*

Proof. If n is a prime power then $\zeta_\Omega \neq 0$ by Theorem 7.9 and the comments which follow it; and if n is not a prime power then $\zeta_\Omega = 0$ by Corollary 7.4. □

It turns out that as a complex of $\mathbb{Z}\mathfrak{S}_n$-modules, the subsets complex $\mathcal{E}^\Omega$ where $\Omega = \{1, \dots, n\}$ is the Yoneda splice of some well-known short exact sequences of Specht modules considered by James and Peel [17], as we next explain. To any set D consisting of n points in the integer plane $\mathbb{N} \times \mathbb{N}$ one can associate a $\mathbb{Z}\mathfrak{S}_n$-module called the *Specht module* $\mathcal{S}^D$. In the case where D is a Ferrers diagram for some partition of n, these are the usual Specht modules used in constructing the indecomposable representations of $\mathfrak{S}_n$. One can also check that when D^i is the union of i points in a column with another $n-i$ points in a single row, sharing no other rows or columns, $\mathcal{S}^{D^i}$ is equivalent to the chain group C^{ori}_{i-1} in $\mathcal{E}^\Omega$. Furthermore, a very easy special case of [17, Theorem 2.4] gives a short sequence of $\mathbb{Z}\mathfrak{S}_n$-modules

$$0 \to \mathcal{S}^{(n-i,1^i)} \to \mathcal{S}^{D^i} \to \mathcal{S}^{(n-i+1,1^{i-1})} \to 0. \tag{7.3}$$

The following proposition is then a routine check of the definitions:

Proposition 7.11. *Under the identification $C^{\text{ori}}_{i-1} \cong \mathcal{S}^{D^i}$, the short exact sequence*

$$0 \to \ker(d_{i+1}) \to C^{\text{ori}}_i \to \operatorname{im}(d_i) \to 0$$

coincides with the sequence (7.3). *Hence the subsets complex $\mathcal{E}^\Omega$ is the Yoneda splice of these short exact sequences of Specht modules.*

Corollary 7.12. *When n is a power of a prime p, none of the sequences 7.3 is split as a sequence of $\mathbb{Z}\mathfrak{S}_n$-modules.*

Proof. If any of the sequences 7.3 were split then the Ext class of C^{ori} would be zero, by Proposition 7.11. By Corollary 7.10 this is not the case. □

In fact we may strengthen the statement of Corollary 7.12 to say that when n is a power of p the reduction mod p of none of the sequences 7.3 is split. This is because Corollary 7.8, on which our proof that $\zeta_\Omega \neq 0$ depends, comes from a property of the cohomology $H^*(G, \mathbb{Z}/p\mathbb{Z})$ when G is an elementary abelian p-group. We may compute this cohomology by working throughout with $\mathbb{Z}/p\mathbb{Z}$ as the ground ring, and hence deduce that the Ext class of $\mathcal{E}^\Omega \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is non-zero. We now complete the argument as above.

We conclude this section of results on the subsets complex by showing that it can be expressed as a product in a certain way. More specifically, when K and H are subgroups of G with $K \leq H$ then, to within signs, the Ext class $\zeta_{G/K}$ is the cup product of $\zeta_{G/H}$ and $\mathrm{Norm}_H^G(\zeta_{H/K})$ where Norm_H^G is Evens' norm map. We start our account of this by recalling the construction of the norm, and we will follow the description given in [2].

Supose that H is a subgroup of G. In its regular left action on itself, G preserves the partition $G = x_1H \cup \cdots \cup x_nH$ given by the left cosets of H (we will take $x_1 = 1$). The full group of permutations of G which preserve this partition is a wreath product $\mathfrak{S}_n \wr \mathfrak{S}_{|H|}$, where $n = |G : H|$, and so G is isomorphic to a subgroup of this group. In fact this subgroup is contained in $\mathfrak{S}_n \wr H$ where H is identified as a subgroup of $\mathfrak{S}_{|H|}$ via its left regular representation. We thus have an embedding $i : G \hookrightarrow \mathfrak{S}_n \wr H$.

Given a chain complex C on which H acts, we may define an action of $\mathfrak{S}_n \wr H$ on $C^{\otimes n}$ by allowing each factor H of the base group H^n to act on the corresponding factor C in the tensor product in the specified manner, and on the remaining factors in the tensor product as the identity. We let a permutation $\pi \in \mathfrak{S}_n$ act on $C^{\otimes n}$ as follows:

$$\pi(c_1 \otimes \cdots \otimes c_n) = (-1)^\nu c_{\pi^{-1}(1)} \otimes \cdots \otimes c_{\pi^{-1}(n)}$$

where

$$\nu = \sum_{\substack{j<k \\ \pi(j)>\pi(k)}} \deg(c_j)\deg(c_k).$$

This action of the wreath product restricts to give an action of G on $C^{\otimes n}$. We will be interested in the situation where C only has non-zero homology in degree zero, so appears as the truncation of a multiple extension

$$\mathcal{C} = \quad 0 \leftarrow A \leftarrow C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_s \leftarrow 0$$

for some $\mathbb{Z}H$-module A. We will furthermore suppose that every module is a free abelian group, so that the sequence is totally split as abelian groups. In this case (according to the description in [2]) $C^{\otimes n}$ has the tensor-induced module $A\updownarrow_\otimes^G$ as its degree zero homology and is otherwise acyclic. The top degree term in $C^{\otimes n}$ is the module $C_s\updownarrow_\otimes^G \otimes \mathbb{Z}^{(s)}$ where $\mathbb{Z}^{(s)}$ is the restriction to G of the $\mathbb{Z}[\mathfrak{S}_n \wr H]$-module $\mathbb{Z}$ on which $H \times \cdots \times H$ acts trivially and $\mathfrak{S}_n$ acts trivially if s is even, and via the sign representation of s is odd. Extending the usual notation, let us write $\mathrm{Norm}_H^G(\mathcal{C})$ for the extension of $\mathbb{Z}G$-modules

$$0 \leftarrow A{\uparrow}_{\otimes}^{G} \leftarrow \cdots \leftarrow C_s{\uparrow}_{\otimes}^{G} \otimes \mathbb{Z}^{(s)} \leftarrow 0$$

just described. Then the Ext class of $\mathrm{Norm}_H^G(\mathcal{C})$ is the norm of the Ext class of $\mathcal{C}$.

Proposition 7.13. *Suppose that K and H are subgroups of G with $K \leq H$. Then the subset complex $\mathcal{E}^{G/K}$ is equivalent to the Yoneda splice of $\mathcal{E}^{G/H}$ and $\mathrm{Norm}_H^G(\mathcal{E}^{H/K}) \otimes \tilde{\mathbb{Z}}_{G/H}$. Here equivalence means that the two extensions represent the same Ext class, and $\tilde{\mathbb{Z}}_{G/H}$ denotes the $\mathbb{Z}G$-module $\mathbb{Z}$ on which G acts via the sign of the permutation action on G/H.*

Proof. We start by putting a double complex structure on $\mathcal{E}^{G/K}$ in a similar way to what we have done before with ordered chain complexes. The degree t term C_t of $\mathcal{E}^{G/K}$ is spanned by symbols $[g_0K, \ldots, g_tK]$ and we will write $C_{r,s}$ for the span of those $[g_0K, \ldots, g_{r+s}K]$ for which $|\{g_0H, \ldots, g_{r+s}H\}| = s + 1$. Then evidently

$$C_t = \bigoplus_{r+s=t} C_{r,s}$$

and

$$d(C_{r,s}) \subseteq C_{r-1,s} \oplus C_{r,s-1}$$

since when we omit a term g_iK from $[g_0K, \ldots, g_tK]$, the number of H-cosets represented by the remaining K-cosets either stays the same or goes down by 1. We have thus expressed $\mathcal{E}^{G/K}$ as a double complex.

There is a morphism of G-sets $\phi : G/K \to G/H$ specified by $gK \mapsto gH$ which gives rise to a map of the G-simplicial complexes of subsets of these G-sets, and hence to a morphism of the augmented oriented chain complexes $\phi : \mathcal{E}^{G/K} \to \mathcal{E}^{G/H}$. Now $\phi([g_0K, \ldots, g_tK]) = [g_0H, \ldots, g_tH]$ is zero unless these $t + 1$ cosets of H are distinct, and so $\phi(C_{r,s}) = 0$ unless $r = 0$. We thus have a map from the left edge of the double complex to $\mathcal{E}^{G/H}$, which we may picture as follows:

$$
\begin{array}{ccccccc}
\tilde{\mathbb{Z}}_{G/H} & \leftarrow & C_{0,n-1} & \leftarrow & C_{1,n-1} & \leftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{Z}[G/H] & \leftarrow & C_{0,n-2} & \leftarrow & C_{1,n-2} & \leftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{Z}[G/H] & \leftarrow & C_{0,0} & \leftarrow & C_{1,0} & \leftarrow & \cdots \\
\downarrow & & \downarrow & & & & \\
\mathbb{Z} & = & \mathbb{Z} & & & &
\end{array}
\tag{7.4}
$$

The left hand column in this diagram is $\mathcal{E}^{G/H}$ and the terms to the right of this are the double complex for $\mathcal{E}^{G/K}$. Since the construction we have just performed depends only on the partition of G/K given by the left cosets of H, and this is preserved by the wreath product $\mathfrak{S}_n \wr H$, we have in fact a diagram of $\mathbb{Z}[\mathfrak{S}_n \wr H]$-modules.

Let us now examine the top row $C_{*,n-1} \to \tilde{\mathbb{Z}}_{G/H}$ in diagram 7.4. We claim that it is isomorphic to $\mathrm{Norm}_H^G(\mathcal{E}^{H/K}) \otimes \tilde{\mathbb{Z}}_{G/H}$ and demonstrate this by producing an isomorphism

$$\theta : \mathrm{Norm}_H^G(\mathcal{E}^{H/K}) \to (C_{*,n-1} \to \tilde{\mathbb{Z}}_{G/H}) \otimes \tilde{\mathbb{Z}}_{G/H}$$

defined on the basis elements of $\mathrm{Norm}_H^G(\mathcal{E}^{H/K})$ in degree ≥ 0 by

$$\theta([a_{1,0}K, \ldots, a_{1,d_1}K] \otimes \cdots \otimes [a_{n,0}K, \ldots, a_{n,d_n}K]) =$$
$$(-1)^{d_2+d_4+d_6+\cdots}[x_1a_{1,0}K, \ldots, x_1a_{1,d_1}K, \ldots, x_na_{n,d_n}K] \otimes 1.$$

Here the $a_{i,j}$ belong to H and $1 = x_1, x_2, \ldots, x_n$ are representatives of the cosets $G = x_1H \cup \cdots \cup x_nH$. The tensor product element to which θ is applied in the above definition is a basis element of $\mathrm{Norm}_H^G(\mathcal{E}^{H/K})$ in degree $d_1 + \cdots + d_n$. We observe that the image of θ does indeed lie in the top row of the double complex, since the terms in the top row are the span of symbols $[g_0K, \ldots, g_tK]$ in which every coset of H is represented, and this is guaranteed by the construction.

It is evident that in each degree θ determines an isomorphism of abelian groups, since basis elements are mapped bijectively to basis elements or their negatives. What remains is to check that θ commutes with the boundary maps of these complexes, and that it commutes with the action of G. In verifying the latter it is convenient to verify that θ

commutes with the action of the larger group $\mathfrak{S}_n \wr H$ since this allows us to break up the calculation into separate checks for the base group H^n and for $\mathfrak{S}_n$, and the latter may be done by considering the effect of the Coxeter generators $(i, i+1)$ for the symmetric group. These checks are complicated but routine, and we leave the details to the reader.

At this stage we conclude that the top row $C_{*,n-1}$ of diagram 7.4 is acyclic except for its zero homology $\tilde{\mathbb{Z}}_{G/H}$. The Yoneda splice of $\mathcal{E}^{G/H}$ with $\mathrm{Norm}_H^G(\mathcal{E}^{H/K}) \otimes \tilde{\mathbb{Z}}_{G/H}$ is now obtained by splicing the left vertical column of diagram 7.4 with the top row of that diagram. We see that there is a morphism of chain complexes from $\mathcal{E}^{G/K}$ to this splice given by ϕ on the left edge (except for the top term), the identity on the top row, and zero on the remaining terms of the double complex. This morphism is the identity on the terms at each end, and so we have an equivalence of complexes, as claimed. □

By iterating the product decomposition of 7.13 we immediately obtain the following result.

Corollary 7.14. *Let $K = K_0 \leq K_1 \leq K_2 \leq \cdots \leq K_t = G$ be a chain of subgroups of G. Then $\mathcal{E}^{G/K}$ is equivalent to the Yoneda splice of the complexes $\mathrm{Norm}_{K_i}^G(\mathcal{E}^{K_i/K_{i-1}}) \otimes \tilde{\mathbb{Z}}_{G/K_i}$ where $1 \leq i \leq t$.*

8. Remarks and questions

8.1. **The flag f-vector, h-vector of K_n.** Note that every maximal face in the complex K_n has the same vertex set $[n]$, and therefore K_n is an example of a *balanced complex*, so that one can speak of its *flag (or fine) f-vector and h-vector* (see [26, Chapter III §4]). These quantities are easy to compute, either directly, or via the shelling of K_n, and the comparison of answers has a curious enumerative consequence.

For any $S \subset [n]$, the flag f-vector entry $\alpha_{K_n}(S)$ is defined to be the number of faces of K_n having vertex set S, while the flag h-vector entry $\beta_{K_n}(S)$ is defined by the inclusion-exclusion-like formula

$$\beta_{K_n}(S) = \sum_{T \subset S} (-1)^{|S-T|} \alpha_{K_n}(T).$$

In the case of K_n it is easy to see that

$$\begin{aligned} \alpha_{K_n}(S) &= |S|! = |\{\pi \in \mathfrak{S}_n : [n] - \mathrm{Fix}(\pi) \subset S\}| \\ \beta_{K_n}(S) &= |\{\pi \in \mathfrak{S}_n : [n] - \mathrm{Fix}(\pi) = S\}|. \end{aligned} \tag{8.1}$$

where $\mathrm{Fix}(\pi)$ denotes the fixed point set of π.

On the other hand, from the fact that the lexicographic ordering on permutations in $\mathfrak{S}_n$ induces a shelling order on the facets of K_n, one can immediately deduce (cf. [26, Proposition III.2.3])

$$\beta_{K_n}(S) = |\{\pi \in \mathfrak{S}_n : [n] - \mathrm{Opt}(\pi) = S\}| \tag{8.2}$$

where we define the *optimal set* $\mathrm{Opt}(\pi)$ to be those values i in $[n]$ for which π is the lexicographically smallest among all permutations having $[n]-\{i\}$ in the same relative order as π. For example, $\mathrm{Opt}(37246851) = \{3, 4, 6\}$.

Comparing (8.1) and (8.2) shows that for all $S \subset [n]$,

$$|\{\pi \in \mathfrak{S}_n : \mathrm{Opt}(\pi) = S\}| = |\{\pi \in \mathfrak{S}_n : \mathrm{Fix}(\pi) = S\}|.$$

This begs for a bijection $\phi : \mathfrak{S}_n \to \mathfrak{S}_n$ having $\mathrm{Opt}(\phi(\pi)) = \mathrm{Fix}(\pi)$. The authors thank M. Wachs for pointing out that such a bijection is given by a version of Foata and Schützenberger's "transformation fondamentale" (see e.g. [23, §1.3]). Given π, write it in cycle notation, with the largest element of each cycle coming first in the cycle, and with the cycles listed in increasing order of their largest elements. Then erase the parentheses to obtain $\phi(\pi)$. For example, if we take

$$\pi = \begin{pmatrix} 123456789 \\ 792486513 \end{pmatrix} = (4)(6)(8175)(932)$$

then $\phi(\pi) = 468175932$. To see why this works, note that $\mathrm{Opt}(\pi)$ is exactly the set of values in π which are left-to-right maxima having another left-to-right maximum immediately to their right. Under the map ϕ, left-to-right maxima in $\phi(\pi)$ correspond to the left parentheses in cycle notation for π, and hence two adjacent left-to-right maxima correspond to singleton cycles (fixed points) of π.

8.2. **More on group cohomology.** We make a few more comments about the Ext classes ζ_G, only sketching the arguments. First we point out that ζ_G is acted upon by the automorphism group $\mathrm{Aut}(G)$ (in its functorial action on Ext groups) as multiplication by the sign of the permution action of $\mathrm{Aut}(G)$ on G. We may explain this as follows. Given any $\mathbb{Z}G$-module M and $\alpha \in \mathrm{Aut}\, G$ we may produce a new module ${}^{\alpha}M$ which has the same set as M and a new G-action specified by $g \cdot m = ({}^{\alpha^{-1}}g)m$. Then for every $\alpha \in \mathrm{Aut}\, G$ there is an isomorphism of $\mathbb{Z}G$-modules $\mathbb{Z}G \cong {}^{\alpha}\mathbb{Z}G$ specified by $x \mapsto {}^{\alpha^{-1}}x$, and this isomorphism extends to an isomorphism of chain complexes $\mathcal{E}^G \cong {}^{\alpha}\mathcal{E}^G$ given on oriented chains by $[g_1, \dots, g_t] \mapsto [{}^{\alpha^{-1}}g_1, \dots, {}^{\alpha^{-1}}g_t]$. Now ${}^{\alpha}\tilde{\mathbb{Z}} = \tilde{\mathbb{Z}}$, and the Ext class of ${}^{\alpha}\mathcal{E}^G$ is determined from that of $\mathcal{E}^G$ by applying the morphism $\tilde{\mathbb{Z}} \to {}^{\alpha}\tilde{\mathbb{Z}}$ which is the top morphism in the isomorphism $\mathcal{E}^G \to {}^{\alpha}\mathcal{E}^G$. Since $\tilde{\mathbb{Z}}$ is spanned by an element $[g_1, \dots, g_{|G|}]$ in which every element of G appears just once, this morphism is multiplication by the sign of the permutation α.

Suppose now that $G = C_2^d$ is an elementary abelian 2-group and we reduce $\mathcal{E}^G$ modulo 2, giving an Ext class which we again denote $\zeta_G \in H^{|G|-1}(G, \mathbb{F}_2)$. Since $-1 = +1$ in $\mathbb{F}_2$ we deduce that ζ_G is fixed by $\mathrm{Aut}(G) = GL(d, 2)$, and so lies in the ring of Dickson invariants of the polynomial algebra $H^*(G, \mathbb{F}_2)$ (see [3] for background on the Dickson invariants). Recall also from Corollary 7.8, whose conclusion is also valid on reduction mod p, that $\zeta_G \neq 0$. Since the degrees of the generators of the Dickson invariants are 2^d-1, 2^d-2, 2^d-4, ... and all of these are even except the highest one, it follows that $\mathcal{E}^G$ represents the top degree generator of the Dickson invariants.

When we try the same thing with $G = C_p^d$ where p is an odd prime we run into the difficulty that ζ_G is not fixed by $\mathrm{Aut}(G)$, even regarding it as an element of $H^*(G, \mathbb{F}_p)$. However, we may dualize the complex $\mathcal{E}^G \otimes_{\mathbb{Z}} \mathbb{F}_p$ to give a complex $(\mathcal{E}^G \otimes_{\mathbb{Z}} \mathbb{F}_p)^* = \mathrm{Hom}_{\mathbb{F}_p}(\mathcal{E}^G \otimes_{\mathbb{Z}} \mathbb{F}_p, \mathbb{F}_p)$ which represents an element $\zeta_G^* \in \mathrm{Ext}_{\mathbb{F}_pG}^{|G|-1}(\tilde{\mathbb{F}}_p, \mathbb{F}_p)$. Now the cup product $\zeta_G^*\zeta_G \in \mathrm{Ext}_{\mathbb{F}_pG}^{2|G|-2}(\mathbb{F}_p, \mathbb{F}_p)$ is fixed by $\mathrm{Aut}(G)$ (since $\mathrm{Aut}(G)$ acts on it by the square of the sign representation) and represents a top degree generator of the Dickson invariants by similar arguments to the previous ones.

It has been pointed out to us (and we are particularly grateful to Michael Mandell for providing a proof) that the Ext classes ζ_Ω are in fact Euler classes of representations. Specifically, if we decompose the real permutation module on Ω as $\mathbb{R}\Omega = \mathbb{R} \oplus A$ (thus defining the representation A), then ζ_Ω is the Euler class of A. We have not used this approach in our treatment here, but if we accept this identification then a number of results follow from general properties of Euler classes. For instance, the Euler class of a direct sum of representations is the product of the Euler classes, and the Euler class of the trivial representation is zero, so that if Ω is not a transitive G-set then $\zeta_\Omega = 0$ because in this situation the representation A has the trivial representation as a summand. This was our Corollary 7.3. We may also obtain a part of our Proposition 7.13 in this way. If K is a subgroup of H then the representation A for the G-set G/H is a summand of the corresponding representation for G/K, and so we may deduce that $\zeta_{G/H}$ is a factor of $\zeta_{G/K}$. Identifying the remaining factor in terms of Evens' norm is perhaps more difficult.

We conclude by stating again a question raised just after Theorem 7.5: what are the finite groups G for which $\zeta_G \neq 0$? The only G for which we know this to be true are the elementary abelian p-groups and the cyclic group of order 4. On the other hand we know in a number of other cases that ζ_G is in fact zero. For example if G is a quaternion

2-group then $H^{|G|-1}(G, \mathbb{Z}) = 0$ (see [8]) which forces $\zeta_G = 0$. Furthermore, any group with no non-zero essential elements in cohomology must necessarily have $\zeta_G = 0$ by Theorem 7.5 (see [1] for a discussion of such groups). A resolution of this problem would have consequences regardless of what the answer turns out to be. If $\zeta_G = 0$ we obtain spectral sequence information using Corollary 7.7, and if $\zeta_G \neq 0$ then we have non-zero essential elements in cohomology, which is again of interest.

Acknowledgments

The first author thanks Michelle Wachs for helpful conversations. The second author is grateful to the Mathematics Department of the University of Virginia for its hospitality during the time much of this research was undertaken. He thanks Jean Lannes and Larry Smith for their very helpful comments, and Michael Mandell for supplying a proof that the cohomology classes ζ_Ω are indeed Euler classes.

References

[1] A. Adem, D. Karagueuzian, Essential cohomology of finite groups. *Comment. Math. Helv.* **72** (1997), 101–109.

[2] D.J. Benson, Representations and cohomology II: cohomology of groups and modules, *Cambridge Studies in Advanced Mathematics* **31**, Cambridge University Press, Cambridge, 1991.

[3] D.J. Benson, Polynomial invariants of finite groups, *London Math. Soc. Lecture Notes* **190**, Cambridge University Press, Cambridge, 1993.

[4] A. Björner, Posets, regular CW complexes and Bruhat order, *European J. Combin.* **5** (1984), 7–16.

[5] A. Björner and M. Wachs, On lexicographically shellable posets, *Trans. Amer. Math. Soc.* **277** (1983), 323–341.

[6] S. Bouc, Burnside rings. Handbook of algebra, Vol. 2, 739–804, North-Holland, Amsterdam, 2000.

[7] K.S. Brown, Cohomology of groups, Springer-Verlag New York 1982.

[8] H. Cartan and S. Eilenberg, Homological algebra, Princeton University Press, 1956.

[9] W.L. Correll, Jr., On the invariant factors and module structure of the kernel of the Varchenko matrix, Ph.D. dissertation, Univ. of Michigan, Ann Arbor, 2002.

[10] J. Désarménien, Une autre interprétation du nombre de dérangements, *Sem. Lotharing. Combin.* **8** (1984), 11–16.

[11] J. Désarménien and M. Wachs, Descentes des dérangements et mot circulaires, *Sem. Lotharing. Combin.* **19** (1988), 13–21.

[12] J. Désarménien and M. Wachs, Descent classes of permutations with a given number of fixed points, *J. Combin. Theory Ser. A* **64** (1993), 311–328.

[13] A.M. Duval, Free resolutions of simplicial posets, *J. Algebra* **188** (1997), 363–399.

[14] L. Evens, On the Chern classes of representations of finite groups, *Trans. Amer. Math. Soc.* **115** (1965), 180–193.

[15] F.D. Farmer, Cellular homology for posets, *Math. Japon.* **23** (1978/79), 607–613.

[16] A.M. Garsia and D. Stanton, Group actions of Stanley-Reisner rings and invariants of permutation groups, *Adv. in Math.* **51** (1984), 107–201.

[17] G.D. James, M.H. Peel, Specht series for skew representations of symmetric groups. *J. Algebra* **56** (1979), 343–364.

[18] C.R.F. Maunder, Algebraic topology. Dover, Mineola, NY, 1996.

[19] S. Mac Lane, Homology, Die Grundlehren der mathematischen Wissenschaften, Bd. 114, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[20] J.R. Munkres, Elements of algebraic topology, Addison-Wesley, Menlo Park, CA, 1984.

[21] R.M. Phatarfod, On the matrix occurring in a linear search problem, *J. Appl. Probab.* **28** (1991), 336–346.

[22] B.E. Sagan, The symmetric group. Representations, combinatorial algorithms, and symmetric functions, Wadsworth & Brooks/Cole, Pacific Grove, CA, 1991.

[23] R.P. Stanley, Enumerative combinatorics. Vol. 1. *Cambridge Studies in Advanced Mathematics* **49**. Cambridge University Press, Cambridge,1997.

[24] R.P. Stanley, Enumerative combinatorics. Vol. 2. *Cambridge Studies in Advanced Mathematics* **62**, Cambridge University Press, Cambridge, 1999.

[25] R.P. Stanley, f-vectors and h-vectors of simplicial posets, *J. Pure Appl. Algebra* 71 (1991),319–331.

[26] R.P. Stanley, Combinatorics and commutative algebra, 2nd edition, *Progress in Mathematics* **41**. Birkhäuser, Boston, MA, 1996.

[27] J.-C. Uyemura-Reyes, Random walk, semi-direct products, and card shuffling, Ph.D. dissertation, Stanford Univ., 2002.

[28] R. Villarroel-Flores, Equivariant homotopy type of categories and preordered sets, Ph.D. thesis, University of Minnesota 1999.

[29] R. Villarroel-Flores and P.J. Webb, Some split exact sequences in the cohomology of groups, *Topology* to appear.

[30] P.J. Webb, Graded G-sets, symmetric powers of permutation modules, and the cohomology of wreath products, *Contemporary Mathematics* **146** (1993), 441–452.

E-mail address: `reiner@math.umn.edu`

E-mail address: `webb@math.umn.edu`

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA