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Weyl Group q-Kreweras Numbers and Cyclic Sieving

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Abstract. Catalan numbers are known to count noncrossing set partitions, while Narayana and Kreweras numbers refine this count according to the number of blocks in the set partition, and by its collection of block sizes. Motivated by reflection group generalizations of Catalan numbers and their q-analogues, this paper concerns a definition of q-Kreweras numbers for finite Weyl groups W, refining the q-Catalan numbers for W, and arising from work of the second author. We give explicit formulas in all types for the q-Kreweras numbers. In the classical types A, B, C, we also record formulas for the q-Narayana numbers and in the process show that the formulas depend only on the Weyl group (that is, they coincide in types B and C). In addition, we verify that in the classical types A, B, C, D the q-Kreweras numbers obey the expected cyclic sieving phenomena when evaluated at appropriate roots of unity.

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1. Introduction

This paper examines polynomials in q, generalizing what are sometimes called the *Kreweras numbers*, as refinements of the *Catalan numbers*. The q-Kreweras numbers arose as by-products of work of the second author [43] on nilpotent orbits in a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ under the action of the associated connected algebraic group G. Their definition, using the Lusztig-Shoji algorithm in Springer theory, is reviewed in Sect. 2 below. One of our main motivations was that these q-Kreweras numbers turn out to answer a question of the first author and Bessis from [8, Sect. 6]; see Sect. 1.2 and Theorem 1.7 below.

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More specifically, one has polynomials in q defined for certain positive integral parameters m (see below) and for each nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ and each *local system* on $\mathcal{O}$ that arises in the Springer correspondence. Let Φ be the root system of $\mathfrak{g}$ relative to a Cartan subalgebra $\mathfrak{h}$. What we call here the *q-Kreweras numbers* $\mathrm{Krew}(\Phi, \mathcal{O}, m; q)$ correspond to the *trivial* local system on $\mathcal{O}$. In this paper, we show three new results about the polynomials $\mathrm{Krew}(\Phi, \mathcal{O}, m; q)$, namely Theorems 1.5, 1.6, 1.7 below, which will be proven in Sects. 3, 4, and 6, respectively. In types A, B, C, we also discuss a definition of the q-Narayana numbers $\mathrm{Nar}(\Phi, m, k; q)$, which are sums of the q-Kreweras numbers depending on a statistic on $\mathcal{O}$, and establish Theorem 1.10 in Sect. 5. We will take up the q-Narayana numbers for other types in a sequel paper.

The further parameter m in the definitions of $\mathrm{Krew}(\Phi, \mathcal{O}, m; q)$ and $\mathrm{Nar}(\Phi, m, k; q)$ is a positive integer that is *very good for* Φ in the terminology of [43, Sect. 5]; this amounts to m being relatively prime to the *Coxeter number* h in types A, E, F, G, and the (weaker) condition of m being *odd* in the classical types B, C, D. See Ito and Okada [23] for further characterizations of this condition.

Let W be the Weyl group of Φ. Since $\mathfrak{g}$ is simple, W acts irreducibly on $\mathfrak{h}$, which is called the reflection representation of W and denoted by V. Let $r = \dim V$, the rank of $\mathfrak{g}$. Let $d_1 \leq \cdots \leq d_r$ be the degrees of any set of fundamental invariants for the action of W on the polynomials $S := \mathrm{Sym}(V^*)$ on V. The Coxeter number h of Φ is equal to d_r. Define

$$\mathrm{Cat}(W, m; q) := \prod_{i=1}^{r} \frac{[m-1+d_i]_q}{[d_i]_q}, \tag{1.1}$$

where $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$. This is known to be a polynomial in $\mathbb{N}[q]$ for all very good m. Results from [43, Sect. 5.3] imply a summation formula

$$\mathrm{Cat}(W, m; q) = \sum_{\mathcal{O}} \mathrm{Krew}(\Phi, \mathcal{O}, m; q) \tag{1.2}$$

as $\mathcal{O}$ runs through the nilpotent orbits in $\mathfrak{g}$, which is a generalization of known results for the specialization at $q = 1$, as we now recall.

In type A_{n-1}, so that m is very good if $\gcd(m, n) = 1$, this $\mathrm{Cat}(W, m; q)$ is the *rational q-Catalan number* considered, for example, by Armstrong, Rhoades and Williams [2]. At the specializations $m = h + 1 = n + 1$ and $q = 1$, these become the *Catalan numbers*

$$C_n := \frac{1}{n+1}\binom{2n}{n} \tag{1.3}$$

which have a plethora of combinatorial interpretations (see Stanley [46] and [47, Exer. 6.19]), some restricting to interpretations of the successive refinements by the *Narayana numbers* $N(n, k)$ and the *Kreweras numbers* $\mathrm{Krew}(\lambda)$:

$$C_n = \sum_{k=1}^{n} N(n,k) \qquad \text{where} \quad N(n,k) := \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1},$$

$$N(n,k) = \sum_{\substack{\lambda \in \mathcal{P}(n):\\ \ell(\lambda)=k}} \mathrm{Krew}(\lambda) \quad \text{where} \quad \mathrm{Krew}(\lambda) := \frac{1}{n+1}\binom{n+1}{n-k, \mu_1(\lambda), \mu_2(\lambda), \ldots, \mu_n(\lambda)}.$$

Here, $\mathcal{P}(n)$ is the set of all partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ of the number n and $\mu_j(\lambda)$ denotes the multiplicity of the number j among the parts $\{\lambda_i\}$. The number of parts $\ell =: \ell(\lambda)$ of λ is called the *length* of λ.

Kreweras [28] originally interpreted $C_n, N(n,k), \mathrm{Krew}(\lambda)$ in terms of the set $NC(n)$ of *noncrossing set partitions* of $\{1,2,\ldots,n\}$ arranged circularly in the plane; that is, partitions for which the convex hull of the blocks are pairwise disjoint. The Catalan number C_n counts the whole set $NC(n)$, while the Narayana number $N(n,k)$ counts those noncrossing set partitions with exactly k blocks, and the Kreweras number $\mathrm{Krew}(\lambda)$ counts those for which λ lists their block sizes. The following table illustrates these interpretations for $n = 4$.

Noncrossing partitions	λ	$\mathrm{Krew}(\lambda)$	k	$N(n,k)$
1234	(4)	$\frac{1}{5}\binom{5}{4,1} = 1$	1	$\frac{1}{1}\binom{3}{0}\binom{4}{0} = 1$
123−4 124−3 134−2 1−234	$(3,1)$	$\frac{1}{5}\binom{5}{3,1,1} = 4$	2	$\frac{1}{2}\binom{3}{1}\binom{4}{1} = 6$
12−34 14−23	$(2,2)$	$\frac{1}{5}\binom{5}{3,2} = 2$		
12−3−4 13−2−4 14−2−3 1−23−4 1−24−3 1−2−34	$(2,1,1)$	$\frac{1}{5}\binom{5}{2,2,1} = 6$	3	$\frac{1}{3}\binom{3}{2}\binom{4}{2} = 6$
1−2−3−4	$(1,1,1,1)$	$\frac{1}{5}\binom{5}{1,4} = 1$	4	$\frac{1}{4}\binom{3}{3}\binom{4}{3} = 1$

1.1. Generalizations to Other W and Other m

1.1.1. Generalizing Noncrossing Partitions to Other W and $m = sh + 1$. The set of noncrossing set partitions $NC(n)$ has a generalization to all Weyl groups and for any parameter m of the form $sh + 1$ where $s \in \mathbb{N}$. The case of $m = h + 1$ was introduced by Bessis [6].[1]

Definition 1.1. Consider the Cayley graph for W with respect to the generating set of *all* reflections in W. Fix a *Coxeter element* c in W. Then $NC(W)$ is defined to be the set of w in W that lie along a shortest path between the identity element and c in this Cayley graph. We regard $NC(W)$ as a partially ordered set in which $x \leq y$ if there is a shortest path from the identity to c in this Cayley graph that passes first through x and then through y.

Bessis [6] showed that the cardinality of $NC(W)$ equals $\mathrm{Cat}(W, h+1, q = 1)$, generalizing Kreweras's original interpretation of the Catalan numbers C_n counting the number of noncrossing set partitions $NC(n)$ in type A_{n-1}.

Armstrong [1] defined a generalization of $NC(W)$ for each positive integer s, inspired by Edelman's s-*divisible noncrossing partitions* [18, Sect. 4].

[1] In fact, Bessis's work in [6] deals not just with Weyl groups, but all finite real reflection groups, and his later work in [7] deals more generally with the class of *well-generated complex reflection groups*. See work of Gordon and Griffeth [21] for definitions of the Catalan and q-Catalan numbers that apply to *all* complex reflection groups.

Definition 1.2. Let $NC^{(s)}(W)$ denote the set of all s-element multichains $w_1 \leq \cdots \leq w_s$ in $NC(W)$.

Armstrong showed that cardinality of $NC^{(s)}(W)$ equals $\mathrm{Cat}(W, sh+1, q=1)$. Note that when $s=1$ one recovers the set $NC(W) = NC^{(1)}(W)$.

1.1.2. Generalizing the Kreweras Numbers to Other W and Very Good m. The Kreweras numbers $\mathrm{Krew}(\lambda)$ in type A_{n-1} have a generalization to any W and any very good m.

Let $X \subset V$ be the common fixed points of a set of reflections in W. Then the pointwise-stabilizer subgroup W_X of X in W is a parabolic subgroup of W. The normalizer $N(W_X)$ of W_X within W is then the setwise stabilizer of X within W. We are interested in X and W_X up to W-conjugacy, so we set $[X] := W \cdot X$ for the W-orbit of X.

Associated to the subspace X is a hyperplane arrangement, obtained by considering the hyperplanes in X of the form $V^w \cap X$ where w is a reflection in W and V^w denotes the pointwise-stabilizer of w on V. The characteristic polynomial of a hyperplane arrangement [34] is an important invariant, denoted by $p_X(t)$ for the hyperplane arrangement we are considering in X.

Using work of Orlik–Solomon [33], we know from [42] that when m is very good for W,

$$\sum_{[X]} \tfrac{p_X(m)}{[N(W_X):W_X]} \cdot 1_{W_X}^W \tag{1.4}$$

is a representation of W whose character takes the value $m^{\dim V^w}$ at $w \in W$. Moreover, by Shepard–Todd [39], proved uniformly by Solomon [41], we know that the multiplicity of the trivial representation in (1.4) is $\mathrm{Cat}(W, m, q=1)$, which implies that

$$\mathrm{Cat}(W, m, q=1) = \sum_{[X]} \tfrac{p_X(m)}{[N(W_X):W_X]}. \tag{1.5}$$

In fact, $p_X(t)$ takes a simple form. It is a monic polynomial with positive roots

$$m_1(X), m_2(X), \ldots, m_{\dim(X)}(X), \tag{1.6}$$

which were first calculated in [33]. The fact that $p_X(t)$ has this form is a consequence of this hyperplane arrangement being *free* (see [48]), proved by Orlik-Terao [34] and then uniformly by Broer [12] and Douglass [17].

In type A_{n-1}, W_X is, up to conjugacy, just a Young subgroup of S_n of the form $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k}$, which allows us to associate the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n to W_X. Then $p_X(t) = (t-1)(t-2)\cdots(t-k+1)$ and $\frac{p_X(n+1)}{[N(W_X):W_X]}$ coincides with $\mathrm{Krew}(\lambda)$. Based on this fact and (1.5), Athanasiadis and Reiner [4] considered

$$\mathrm{Krew}(W, [X], m) := \tfrac{p_X(m)}{[N(W_X):W_X]} = \tfrac{1}{[N(W_X):W_X]} \prod_{i=1}^{\dim(X)} (m - m_i(X))$$

as the Kreweras numbers for arbitrary W and very good m. They then showed that $\mathrm{Krew}(W,[X],h+1)$ equals the cardinality of the set of $w \in NC(W)$ with $V^w \in [X]$ [4, Theorem 6.3]. In classical (A,B,C,D) and dihedral (I) types, work of Rhoades [37] showed more generally that $\mathrm{Krew}(W,[X],sh+1)$ counts the elements $w_1 \leq \cdots \leq w_s$ in $NC^{(s)}(W)$ with V^{w_1} in $[X]$; this remains open in the exceptional types E, F, H.

1.2. Cyclic Sieving

Armstrong defined a natural action of the cyclic group $\mathbb{Z}/sh\mathbb{Z}$ on $NC^{(s)}(W)$. In [8] it was shown that this gives an instance of the *cyclic sieving phenomenon* introduced in [36] on $NC^{(s)}(W)$. To state it, for a positive integer d, let $\omega_d := e^{\frac{2\pi i}{d}}$, a primitive d^{th} root-of-unity.

Theorem 1.3 ([8]). *For $m = sh+1$, one has that $\mathrm{Cat}(W,m;q=\omega_d)$ counts those*

$$w_1 \leq \cdots \leq w_s \in NC^{(s)}(W)$$

that are fixed under the action of an element of order d in the $\mathbb{Z}/sh\mathbb{Z}$-action.

In [8, Sect. 6], it was asked how to produce the q-Kreweras numbers, $\mathrm{Krew}(\Phi,X,m;q)$, polynomials that would evaluate to the Kreweras numbers $\mathrm{Krew}(W,[X],m)$ at $q=1$, but more generally would have the following property: $\mathrm{Krew}(\Phi,X,m;q=\omega_d)$ counts the elements $w_1 \leq \cdots \leq w_s \in NC^{(s)}(W)$ with $V^{w_1} \in [X]$ which are additionally fixed under the action of an element of order d in the $\mathbb{Z}/sh\mathbb{Z}$-action. Such a result would generalize Theorem 1.3.

1.3. The q-Kreweras Numbers

In work of the second author [43], a polynomial in the variable q, denoted $f_{e,\phi}(m;q)$, is introduced for a nilpotent element $e \in \mathfrak{g}$, an irreducible representation ϕ of the component group of e arising in the Springer correspondence, and a very good m (see Sect. 2). The definition involves a graded version of the representation in (1.4) and only depends on the nilpotent orbit $\mathcal{O}$ containing e.

Given X as before, the centralizer in $\mathfrak{g}$ of $X \subset \mathfrak{h}$ is a Levi subalgebra, denoted $\mathfrak{l}_X$, which contains $\mathfrak{h}$ and whose Weyl group identifies with W_X. We write $\mathcal{O}_X$ for the unique nilpotent orbit which contains elements that are principal nilpotent in $\mathfrak{l}_X$. The definition of $\mathcal{O}_X$ only depends on $[X]$. We say that $\mathcal{O}_X$, and each of the elements it contains, is principal-in-a-Levi.

Now when $e \in \mathcal{O}_X$, then $f_{e,\phi}(m;1) = \mathrm{Krew}(W,[X],m)$. Setting ϕ to be trivial, we also have

$$\mathrm{Cat}(W,m;q) = \sum f_{e,1}(m;q),$$

where the sum is over a set of representatives e from each nilpotent orbit. These two results led us to the following definition of the q-Kreweras numbers for each nilpotent orbit $\mathcal{O}$

$$\mathrm{Krew}(\Phi,\mathcal{O},m;q) := f_{e,1}(m;q) \text{ where } e \in \mathcal{O}$$

and to conjecture

Conjecture 1.4. *For $m = sh+1$, and for each W-orbit $[X]$,* $\mathrm{Krew}(\Phi, \mathcal{O}_X, m; q = \omega_d)$ *counts those*

$$w_1 \leq \cdots \leq w_s \in NC^{(s)}(W)$$

which are fixed under the action of an element of order d in the $\mathbb{Z}/sh\mathbb{Z}$-action and have $V^{w_1} \in [X]$.

1.4. Results

1.4.1. Formulas for the *q*-Kreweras Numbers. The formulas for $f_{e,\phi}$ for general ϕ in the classical groups are given in Propositions 3.5 and 3.7. The formulas for $f_{e,\phi}$ in the exceptional groups are tabulated in Sect. 3.6. Theorem 1.5 below summarizes the formulas for the q-Kreweras numbers (that is, ϕ is trivial) in the classical types.

Recall that the nilpotent orbits in the classical Lie algebras can be parametrized by the number partitions λ obeying certain restrictions by considering the defining representation of $\mathfrak{g}$ and taking the Jordan form for an element in the orbit. For such a partition λ, we write $\mathcal{O}_\lambda$ for the corresponding orbit in $\mathfrak{g}$.

- In type A_{n-1}, that is, $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, nilpotent orbits are parametrized by all partitions λ of n; as before, denote this set of partitions by $\mathcal{P}(n)$.
- In type B_n, that is, $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$, nilpotent orbits are parametrized by partitions λ of $2n+1$ having $\mu_j(\lambda)$ even for j even; denote this set of partitions by $\mathcal{P}_B(2n+1)$.
- In type C_n, that is, $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$, nilpotent orbits are parametrized by all partitions λ of $2n$ having $\mu_j(\lambda)$ even for j odd; denote this set of partitions by $\mathcal{P}_C(2n)$.
- In type D_n, that is, $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C})$, nilpotent orbits under the orthogonal group $\mathrm{O}_{2n}(\mathbb{C})$ are parametrized by partitions λ of $2n$ having $\mu_j(\lambda)$ even for j even; denote this set of partitions by $\mathcal{P}_D(2n)$. We denote by $\mathcal{O}_\lambda$ the orbit under $\mathrm{O}_{2n}(\mathbb{C})$. Then $\mathcal{O}_\lambda$ is a single $\mathrm{SO}_{2n}(\mathbb{C})$-orbit unless λ has only even parts, in which case $\mathcal{O}_\lambda$ splits into two orbits under $\mathrm{SO}_{2n}(\mathbb{C})$. Both of these orbits have the same q-Kreweras numbers, given by 1/2 times the formula shown in Theorem 1.5 (Type D_n).

For the classical groups of types A, B, C, D the polynomial $\mathrm{Krew}(\Phi, \mathcal{O}, m; q)$ is expressed using *q-multinomials* which are defined as follows: for $\nu = (\nu_1, \ldots, \nu_t) \in \mathbb{N}^t$ with $|\nu| := \sum_i \nu_i \leq n$, let

$$\begin{bmatrix} n \\ \nu \end{bmatrix}_q := \begin{bmatrix} n \\ \nu, n-|\nu| \end{bmatrix}_q := \frac{[n]!_q}{[\nu_1]!_q \cdots [\nu_t]!_q [n-|\nu|]!_q},$$

where $[n]!_q := [n]_q [n-1]_q \cdots [1]_q$, and define the left side to be zero whenever $|\nu| > n$.

Letting λ' denote the *conjugate* or *transpose* partition of λ, define

$$c(\lambda) := \sum_j \lambda'_j \lambda'_{j+1}.$$

Theorem 1.5. (Type A_{n-1}) *For* $\lambda \in \mathcal{P}(n)$ *and for* $\gcd(m,n)=1$*, one has*

$$\mathrm{Krew}(A_{n-1},\mathcal{O}_\lambda,m;q) = q^{m(n-\ell(\lambda))-c(\lambda)} \frac{1}{[m]_q} \begin{bmatrix} m \\ \mu(\lambda) \end{bmatrix}_q .$$

In types B, C, D, the formulas are similar, replacing various parameters by roughly half their values. Introduce the notation $\hat{N} := \lfloor N/2 \rfloor$ for $N \in \mathbb{N}$, and for $\nu = (\nu_1, \nu_2, \ldots)$, set $\hat{\nu} := (\hat{\nu}_1, \hat{\nu}_2, \ldots)$. Using "$a \equiv b$" to abbreviate "$a = b \bmod 2$", define the following quantities

$$L(\lambda) := \#\{j \in \mathbb{N} \colon \mu_j(\lambda) \text{ odd}\},$$

$$\psi(n,m,\lambda) := m(n-\hat{\ell}(\lambda)) - \frac{c(\lambda)}{2} - \frac{L(\lambda)}{4},$$

$$\tau_\epsilon(\lambda) := \frac{1}{2} \sum_{\substack{j \equiv \epsilon \\ \mu_j \equiv 0}} \mu_j(\lambda), \text{ where } \epsilon := \begin{cases} 0 & \text{in type } C, \\ 1 & \text{in types } B \text{ and } D. \end{cases}$$

Theorem 1.5. (Type B_n) *For* $\lambda \in \mathcal{P}_B(2n+1)$ *and for* m *odd, one has*

$$\mathrm{Krew}(B_n,\mathcal{O}_\lambda,m;q) = q^{\psi(n,m,\lambda)+\tau_1(\lambda)+\frac{1}{4}} \prod_{i=1}^{\hat{L}(\lambda)} (q^{m-2i+1}-1) \begin{bmatrix} \hat{m}-\hat{L}(\lambda) \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2} .$$

For the type C formula, additionally define

$$\delta(\lambda) := \begin{cases} \frac{1}{4} - \frac{\ell(\lambda)}{2} & \text{for } \ell(\lambda) \text{ odd}, \\ 0 & \text{for } \ell(\lambda) \text{ even}. \end{cases}$$

Theorem 1.5. (Type C_n) *For* $\lambda \in \mathcal{P}_C(2n)$ *and for* m *odd, one has*

$$\mathrm{Krew}(C_n,\mathcal{O}_\lambda,m;q) = q^{\psi(n,m,\lambda)+\tau_0(\lambda)+\delta(\lambda)} \prod_{i=1}^{\hat{L}(\lambda)} (q^{m-2i+1}-1) \begin{bmatrix} \hat{m}-\hat{L}(\lambda) \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2} .$$

In type D_n, the multiplicity $\mu_1(\lambda)$ of the part 1 in λ plays a special role, and we also define

$$\mu_{\geq 2}(\lambda) := (\mu_2(\lambda), \mu_3(\lambda), \ldots).$$

Theorem 1.5. (Type D_n) *For* $\lambda \in \mathcal{P}_D(2n)$ *and* m *odd,* $\mathrm{Krew}(D_n,\mathcal{O}_\lambda,m;q)$ *is* $q^{\psi(n,m,\lambda)+\tau_1(\lambda)}$ *times:*

$$\begin{cases} q^{m-\hat{\ell}(\lambda)+1} \displaystyle\prod_{i=1}^{\hat{L}(\lambda)-1} (q^{m-2i+1}-1) \cdot \begin{bmatrix} \hat{m}+1-\hat{L}(\lambda) \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2} & \text{if } \mu_1(\lambda) \text{ is odd}, \\ q^{\hat{\ell}(\lambda)-\mu_1(\lambda)} \displaystyle\prod_{i=1}^{\hat{L}(\lambda)} (q^{m-2i+1}-1) \cdot \begin{bmatrix} \hat{m}-\hat{L}(\lambda) \\ \hat{\mu}_{\geq 2}(\lambda) \end{bmatrix}_{q^2} \begin{bmatrix} \hat{m}+1-\hat{L}(\lambda)-|\hat{\mu}_{\geq 2}(\lambda)| \\ \hat{\mu}_1(\lambda) \end{bmatrix}_{q^2} & \\ & \text{if } \mu_1(\lambda) \text{ even and } \hat{L}(\lambda) \geq 1, \\ q^{\hat{\ell}(\lambda)-\tau_1(\lambda)} \begin{bmatrix} \hat{m} \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2} + q^{\hat{\ell}(\lambda)-\mu_1(\lambda)} \begin{bmatrix} \hat{m} \\ \hat{\mu}_{\geq 2}(\lambda) \end{bmatrix}_{q^2} \begin{bmatrix} \hat{m}+1-|\hat{\mu}_{\geq 2}(\lambda)| \\ \hat{\mu}_1(\lambda) \end{bmatrix}_{q^2} & \\ & \text{if } \hat{L}(\lambda) = 0. \end{cases}$$

1.4.2. Divisibility and Positivity Properties of the q-Kreweras Numbers. Using our explicit formulas for the $f_{e,\phi}$, we gather some of their properties in the following Theorem 1.6. Its statement will be slightly less precise for a fairly short list of ill-behaved nilpotent orbits occurring inside the exceptional types F_4, E_6, E_7, and E_8, given here by their Bala–Carter notation [14]:

$$\begin{aligned} &F_4(a_3),\ E_6(a_3),\ E_6(a_3)+A_1,\ E_7(a_5),\ E_7(a_3),\ E_8(a_7),\ E_8(a_6),\\ &\quad E_8(b_5),\ E_8(a_4),\ E_8(a_3). \end{aligned} \tag{1.7}$$

Let $R = \operatorname{rank}(Z_G(e))$. Recall that e is principal-in-a-Levi if $e \in \mathcal{O}_X$ for some X. We will denote by $H^*(\mathcal{B}_e)$ the cohomology of the Springer fiber for $e \in \mathcal{O}$, regarded as a W-representation, which will play a central role in the definition of the q-Kreweras numbers in Sect. 2.

Theorem 1.6. *Let e be a nilpotent element* **not** *among the ill-behaved orbits from* (1.7), *and assume that $f_{e,\phi}$ is not identically zero. Then there exists $L, c \in \mathbb{N}$, independent of ϕ, such that*

$$f_{e,\phi}(m;q) = \prod_{j=1}^{L} (q^{m+1-2j} - 1) \cdot q^{cm} \cdot g_\phi(m;q),$$

where $g_\phi(m;q)$ is the sum of at most two products of the form $q^{-z} \prod_{i=1}^{R} \frac{[m-a_i]_q}{[b_i]_q}$ for some $a_i, b_i, z \in \mathbb{N}$. Moreover, we have the following properties

(i) *For each very good m, the polynomial $q^{cm} \cdot g_\phi(m;q)$ lies in $\mathbb{N}[q]$.*
(ii) *The rank r of $\mathfrak{g}$ equals $L + c + R$.*
(iii) *The multiplicity of V in the W-representation $H^*(\mathcal{B}_e)$ is $r - c$.*
(iv) *If e is principal-in-a-Levi, then $L = 0$. In particular, $f_{e,\phi}(m;q) \in \mathbb{N}[q]$ for each very good m.*
(v) *If e is not principal-in-a-Levi, then $L \geq 1$. In the exceptional types it always happens that $L = 1$.*

Even when e is one of the ill-behaved orbits from (1.7), *if one further specializes to the case $\phi = 1$, then the polynomial $f_{e,1}(m;q)$ is always nonzero, and still has properties* (i), (ii), (iv), (v) *listed above.*[2]

It follows that the q-Kreweras numbers $f_{e,1}(m;q)$ are never identically zero, that they have nonnegative coefficients as polynomials in q if e is principal-in-a-Levi, and are divisible by $q^{m-1} - 1$ otherwise. These facts are used in establishing the cyclic sieving property. The proof of Theorem 1.6 is given in Sect. 4.

1.4.3. Cyclic Sieving. We also show that the cyclic sieving property holds in the classical types in Sect. 6.

Theorem 1.7. *Conjecture 1.4 holds in all of the classical types A, B, C, D.*

[2] Examination of the tables in Sect. 3.6 shows that for e in an ill-behaved nilpotent orbit, for certain $\phi \neq 1$ one still has a factorization of $f_{e,\phi}(m;q)$ as in the theorem, but with $-g_\phi(m;q)$ in $\mathbb{N}[q]$. Also, for such e, property (iii) fails even if $\phi = 1$. Instead, the value $r - c$ is the multiplicity of V in the $A(e)$-invariants in $H^*(\mathcal{B}_e)$.

1.4.4. q-Narayana Numbers. We are able to establish a q-version of the Narayana numbers in types A, B, C. We take up the question of the q-Narayana numbers for other types in a sequel paper. From Theorem 1.6 we have that the lowest degree q-monomial in $\mathrm{Krew}(\Phi, \mathcal{O}, m; q)$ equals q^{cm-z} for some $c, z \in \mathbb{N}$.

Definition 1.8. Define $d(\mathcal{O}) = r - c$.

As long as e does not belong to one of the orbits in (1.7), then by Theorem 1.6 (iii), $d(\mathcal{O})$ equals the multiplicity of V in $H^*(\mathcal{B}_e)$. In particular, this holds whenever $\mathfrak{g}$ is of classical type or e is principal-in-a-Levi. On the other hand, when e belongs to one of the orbits in (1.7), then $d(\mathcal{O})$ equals the multiplicity of V in $A(e)$-invariants of $H^*(\mathcal{B}_e)$. Using the parameter $d(\mathcal{O})$, we obtain a good definition for a q-version of the Narayana numbers in types A, B, C.

Definition 1.9. (*Types* A, B, C) The *q-Narayana number* for $k = 0, 1, 2, \ldots, r$ and very good m is given by

$$\mathrm{Nar}(\Phi, m, k; q) := \sum_{\substack{\text{nilpotent orbits } \mathcal{O}: \\ d(\mathcal{O})=k}} \mathrm{Krew}(\Phi, \mathcal{O}, m; q). \tag{1.8}$$

Theorem 1.10. *The q-Narayana polynomials have the following formulas in types A, B, C:*

- *For type A_{n-1}, when* $\gcd(m, n) = 1$ *one has for $0 \le k \le n-1$ that*
$$\mathrm{Nar}(A_{n-1}, m, k; q) = q^{(n-1-k)(m-1-k)} \frac{1}{[k+1]_q} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} m-1 \\ k \end{bmatrix}_q.$$
- *For either type B_n or type C_n, when m is odd one has for $0 \le k \le n$ that*
$$\mathrm{Nar}(B_n, m, k; q) = \mathrm{Nar}(C_n, m, k; q) = (q^2)^{(n-k)(\hat{m}-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \begin{bmatrix} \hat{m} \\ k \end{bmatrix}_{q^2}.$$

When $m = h + 1 = n + 1$ in type A_{n-1}, the polynomial $\mathrm{Nar}(A_{n-1}, h + 1, k; q)$ equals a q-Narayana number considered by Furlinger and Hofbauer [19] and Bränden [11].

Theorem 1.10 shows that, in the instance where two root systems $\Phi = B_n, C_n$ are associated with the same Weyl group W, it turns out that $\mathrm{Nar}(\Phi, m, k; q)$ depends only on W, and not on Φ, even though the polynomials $\mathrm{Krew}(\Phi, \mathcal{O}, m; q)$ for the two root systems are not the same (they are not even indexed by the same set). It is also interesting to note that, although the polynomials $\mathrm{Krew}(\Phi, \mathcal{O}, m; q)$ can have negative integral coefficients in types B and C, the formulas above for $\mathrm{Nar}(\Phi, m, k; q)$ exhibit them as polynomials in q with *nonnegative* coefficients, that is, lying in $\mathbb{N}[q]$.

Theorem 1.10 is proved in Sect. 5.

2. Reviewing $f_{e,\phi}$ and the q-Kreweras Numbers $f_{e,1}$

In this section, we detail some results from [43]. Recall that $S = \mathrm{Sym}(V^*)$ is the graded ring of polynomials on the reflection representation V of W and

$r = \dim V$. When m is very good for Φ, it is known [5,20] that S contains a *homogeneous system of parameters* $\theta^{(m)} = (\theta_1^{(m)}, \ldots, \theta_r^{(m)})$, all of degree m, whose span is W-stable and carries a representation isomorphic to V. This implies by [41] that the W-invariant subspace of the finite-dimensional quotient ring $S/(\theta^{(m)})$ is a graded vector space whose Hilbert series is $\mathrm{Cat}(W, m; q)$ and in particular shows that $\mathrm{Cat}(W, m; q)$ is polynomial. A main result of [43] is that $S/(\theta^{(m)})$, as a graded W-module, is an integral combination of certain finite-dimensional graded W-representations (and their graded shifts), related to the Green functions that arise in the representation theory of Chevalley groups.

To be more precise, let G be the connected simple algebraic group of adjoint type over an algebraically closed field $\mathbf{k}$ of good characteristic p attached to the root system Φ. Let $\mathfrak{g}$ be its Lie algebra. For a nilpotent element $e \in \mathfrak{g}$, let $\mathcal{B}_e$ be the variety of Borel subalgebras containing e, known as a Springer fiber. Let $Z_G(e)$ be the centralizer of e in G and let $A(e) := Z_G(e)/Z_G^\circ(e)$ be the component group of e. Then the l-adic cohomology $H^*(\mathcal{B}_e)$ carries a representation of $W \times A(e)$ [22,31], originally defined by Springer [45]. Denote the irreducible $\overline{\mathbb{Q}}_l$-representations of $A(e)$ by $\hat{A}(e)$. For $\phi \in \hat{A}(e)$ define the finite-dimensional, graded representations $Q_{e,\phi}$ so that

$$H^*(\mathcal{B}_e) = \sum_{\phi \in \hat{A}(e)} Q_{e,\phi} \otimes \phi,$$

as graded representations of $W \times A(e)$. The cohomology of $\mathcal{B}_e$ vanishes in odd degrees and we grade $Q_{e,\phi}$ by putting q in cohomological degree two. Then (as Frobenius series)

$$S/(\theta^{(m)}) = \sum_{e,\phi} f_{e,\phi}(m; q) Q_{e,\phi} \tag{2.1}$$

where $f_{e,\phi}(m; q) \in \mathbb{Z}[q]$ and the sum is over a set of representatives e of the nilpotent orbits in $\mathfrak{g}$ and those $\phi \in \hat{A}(e)$ that appear in the Springer correspondence.

There is an explicit formula [43, Eq. 18] for $f_{e,\phi}(m; q)$ that involves (1) the cardinality of the rational nilpotent orbits in $\mathfrak{g}$ for a finite subfield $\mathbb{F}_q$ of $\mathbf{k}$; and (2) the Frobenius series of the reflection representation V in $Q_{e,\phi}$.

Define $\{(m_1, \pi_1), (m_2, \pi_2), \ldots, (m_\kappa, \pi_\kappa)\}$, where $m_j \in \mathbb{Z}_{\geq 0}$ and $\pi_j \in \hat{A}(e)$, by

$$\sum_{j \geq 0} q^j \langle H^{2j}(\mathcal{B}_e), V \rangle_W = q^{m_1}\pi_1 + q^{m_2}\pi_2 + \cdots + q^{m_\kappa}\pi_\kappa, \tag{2.2}$$

where the pairing is the usual inner product for W and the result is viewed as a Frobenius series for $A(e)$. It turns out that at most one of the π_j's is non-trivial and we set π_κ to be this non-trivial representation of $A(e)$ when it occurs. When e belongs to $\mathcal{O}_X$, it is known [29,44] that the m_i's coincide with the $m_i(X)$ from 1.6.

Let $G(\mathbb{F}_q) \subset G$ be the $\mathbb{F}_q$-points of G with respect to a split Frobenius endomorphism F. Let c denote a conjugacy class in $A(e)$ and e_c a representative from the rational $G(\mathbb{F}_q)$-orbit in $\mathfrak{g}$ over $\mathbb{F}_q$ corresponding to the pair (e, c). Set $d_\kappa = \dim(\pi_\kappa)$ and $M = m_1 + \cdots + m_{\kappa-1} + d_\kappa \cdot m_\kappa$. Then

$$f_{e,\phi}(m;q) = q^{m(r-\kappa-d_\kappa+1)+M} \prod_{j=1}^{\kappa-1} (q^{m-m_j} - 1) \cdot \left(\sum_{i=0}^{d_\kappa} (-1)^{d_\kappa - i} q^{i(m-m_\kappa)} \sum_c \frac{\wedge^{d_\kappa - i} \pi_\kappa(c) \phi(c)}{|Z_{G^F}(e_c)|} \right). \tag{2.3}$$

When π_κ is trivial, the above expression simplifies to

$$f_{e,\phi}(m;q) = q^{m(r-\kappa)+\sum m_j} \prod_{j=1}^{\kappa} \left(q^{m-m_j} - 1\right) \left(\sum_c \frac{\phi(c)}{|Z_{G^F}(e_c)|} \right). \tag{2.4}$$

In the present paper, we are primarily concerned with the Frobenius series of $S/(\theta^{(m)})$ at the trivial representation of W. Since the trivial representation of W only occurs in $Q_{e,\phi}$ for the trivial local system ϕ and then only once in degree zero [10], we obtain the identity

$$\mathrm{Cat}(W, m; q) = \sum_e f_{e,1}(m;q). \tag{2.5}$$

By [43, Theorem 15]

$$f_{e,\phi}(m; q = 1) = \mathrm{Krew}(W, m, [X]) \tag{2.6}$$

when e is conjugate to a principal nilpotent element in $\mathfrak{l}_X$. When e is not of that form, on the other hand, $f_{e,\phi}(m; q = 1) = 0$. In light of these results, it is reasonable to think of $f_{e,1}(m;q)$ as q-Kreweras numbers, where there is one such polynomial for each nilpotent orbit in $\mathfrak{g}$.

Definition 2.1. The q-Kreweras numbers for Φ are defined to be

$$\mathrm{Krew}(\Phi, \mathcal{O}, m; q) = f_{e,1}(m;q)$$

for $e \in \mathcal{O}$.

We will write down the formulas for $f_{e,\phi}(m;q)$, and hence $\mathrm{Krew}(\Phi, \mathcal{O}, m; q)$ in Sect. 3.1 for type A_{n-1}, Sect. 3.5 for the other classical types, and Sect. 3.6 for the exceptional types.

3. Computing $f_{e,\phi}$ and the Proof of Theorem 1.5

We use the following notation in this section. For a partition λ, let $\mu_j := \mu_j(\lambda)$, which is the number of parts of λ of size j. For a nilpotent element $e \in \mathfrak{g}$, let $d = \dim Z_G(e)$ and d^u denote the dimension of a maximal unipotent subgroup of $Z_G(e)$.

3.1. Type A

For simplicity we will work with $G = GL_n(\bar{\mathbb{F}}_q)$ and adjust our results for the case where G is adjoint. Recall from the introduction that the nilpotent G-orbits in $\mathfrak{g}$ are parametrized by the set $\mathcal{P}(n)$ of partitions of n, with $\lambda \in \mathcal{P}(n)$ corresponding to the sizes of the Jordan blocks of any element in the nilpotent orbit $\mathcal{O}_\lambda$ indexed by λ. Let the Frobenius map F consist of raising each matrix element to the q-th power, giving the standard split structure on G, so that $G^F = GL_n(\mathbb{F}_q)$ and $\mathfrak{g}^F = \mathfrak{gl}_n(\mathbb{F}_q)$. Then it is known, say using the rational canonical form, that nilpotent G^F-orbits on $\mathfrak{g}^F$ are indexed by $\mathcal{P}(n)$; in other words, the rational points of $\mathcal{O}_\lambda$ remain a single orbit under G^F.

We wish to compute

$$\mathrm{Krew}(A_{n-1}, \mathcal{O}_\lambda, m; q) := f_{e,1}(m; q)$$

where $e := e_\lambda \in \mathcal{O}_\lambda$ is a rational element. Since all $A(e)$ are trivial in $GL_n(\bar{\mathbb{F}}_q)$, the computation of $f_{e,1}$ in (2.4) reduces to calculating $m_1, \ldots, m_\kappa$ and the value of $|Z_{G^F}(e)|$. First, according to [29] we have

$$\kappa = \ell(\lambda) - 1 \quad \text{and} \quad m_j = j. \tag{3.1}$$

Thus (2.4), with $r = n - 1$, becomes

$$f_{e,1} = q^{m(n-\ell(\lambda))+\binom{\ell(\lambda)}{2}} \cdot \frac{q-1}{|Z_{G^F}(e)|} \prod_{j=1}^{\ell(\lambda)-1} \left(q^{m-j} - 1\right), \tag{3.2}$$

where the extra factor of $q - 1$ accounts for the center of $GL_n(\mathbb{F}_q)$ since the formulas in (2.4) are presented relative to an adjoint group.

Next, we need to compute $|Z_{G^F}(e)|$. As a variety $Z_G(e)$ is isomorphic to the product of an affine space (its unipotent radical) and a maximal reductive part of the centralizer of $Z_G(e)$. The reductive part is isomorphic to

$$Z_{\mathrm{red}} := \prod_j GL_{\mu_j}(\bar{\mathbb{F}}_q).$$

Up to isomorphism, the affine space and each factor in the reductive part carry the standard action of F, so $Z_{G^F}(e)$ is isomorphic over $\mathbb{F}_q$ to

$$\mathbb{F}_q^{d_1^u} \times \prod_j GL_{\mu_j}(\mathbb{F}_q).$$

Since

$$|GL_r(\mathbb{F}_q)| = q^{\binom{r}{2}}(q-1)^r [r]!_q,$$

it follows that

$$|Z_{G^F}(e)| = q^{d^u}(q-1)^{\ell(\lambda)} \prod_j [\mu_j]!_q. \tag{3.3}$$

The Borel subgroup of Z_{red} has dimension $\sum \binom{\mu_j+1}{2}$, so $d^u = d - \sum \binom{\mu_j+1}{2}$. Now $d = \sum (\lambda_i')^2$ (see [14]) and so a calculation gives

$$d^u = \sum (\lambda_i')^2 - \sum \binom{\mu_j+1}{2} = \binom{\ell(\lambda)}{2} + \mathrm{c}(\lambda) \tag{3.4}$$

where

$$\mathrm{c}(\lambda) := \sum_{j \geq 1} \lambda'_j \lambda'_{j+1}.$$

Plugging (3.3) and (3.4) into (3.2) yields

$$\begin{aligned} f_{e,1} &= q^{m(n-\ell(\lambda))-\mathrm{c}(\lambda)} \frac{1}{\prod_j [\mu_j]!_q} \cdot \frac{[m-1]!_q}{[m-\ell(\lambda)]!_q} \\ &= q^{m(n-\ell(\lambda))-c(\lambda)} \frac{1}{[m]_q} \begin{bmatrix} m \\ \mu(\lambda) \end{bmatrix}_q, \end{aligned}$$

as asserted in Theorem 1.5(Type A).

Remark 3.1. The formula for $|Z_{G^F}(e)|$ could also be obtained by looking up the value $|\mathcal{O}_e^F| = \frac{|G^F|}{|Z_{G^F}(e)|}$ in, for example, [16].

3.2. Preparation for Types B, C, D

From (2.3), in order to compute $f_{e,\phi}$, we need to evaluate the sum

$$\sum_c \frac{\phi(c)}{|Z_{G^F}(e_c)|}$$

as c runs over representatives of the conjugacy classes in $A(e)$. The component group $A(e)$ is elementary abelian, hence we will have occasion to use the following two lemmas.

First, we introduce some notation for working with elementary abelian groups and their characters. Let $v, w \in (\mathbb{F}_2)^s$. Write $v = (v_i), w = (w_i)$ relative to the standard basis and denote the usual dot product by $\langle v, w\rangle := \sum_{i=1}^s v_i w_i$. For each $w \in \mathbb{F}_2^s$, define the character $\phi_w \in \widehat{\mathbb{F}_2^s}$ by

$$\phi_w(v) = (-1)^{\langle v,w\rangle} \in \mathbb{Q}.$$

Every character of $\mathbb{F}_2^s$ is of the form ϕ_w for a unique w.

Let $x_1, x_2, \ldots, x_s$ be a set of s variables. For $a \in \mathbb{F}_2$ and a variable y, we evaluate y^a to 1 or y according to whether $a = 0$ or $a = 1$, respectively, in $\mathbb{F}_2$. Then the next two lemmas have similar proofs; we omit the proof of the first, since it is very similar to the proof of the second, but simpler.

Lemma 3.2. *Let ϕ be a character of $\mathbb{F}_2^s$. Let t be a nonnegative integer with $0 \leq t \leq s$. Then*

$$\sum_{v \in \mathbb{F}_2^s} \phi(v) \prod_{i=1}^t (x_i + (-1)^{v_i}) = \begin{cases} 2^s \prod_{i=1}^t x_i^{w_i+1} & \text{if } w_j = 0 \text{ for all } j > t, \\ 0 & \text{otherwise,} \end{cases}$$

where w is the unique element of $\mathbb{F}_2^s$ for which $\phi = \phi_w$.

We remark that this identity also holds in the degenerate case where $s = 0$.

Let K denote the subgroup of $\mathbb{F}_2^s$ consisting of those $v \in \mathbb{F}_2^s$ with $\sum v_i = 0$. Any character $\phi \in \widehat{K}$ is now equal to the restriction of ϕ_w for two values of $w \in \mathbb{F}_2^s$, call them w', w'', where $w' + w'' = (1, 1, \ldots, 1)$.

Lemma 3.3. *Let ϕ be a character of K and $s > 0$. Let t be a nonnegative integer with $0 \le t \le s$. Then*

$$\sum_{v\in K} \phi_w(v) \prod_{i=1}^{t} (x_i + (-1)^{v_i})$$
$$= \begin{cases} 2^{s-1}\left(\prod_{i=1}^{s} x_i^{w_i} + \prod_{i=1}^{s} x_i^{w_i+1}\right) & \text{if } t = s, \\ 2^{s-1}\prod_{i=1}^{t} x_i^{w_i+1} & \text{if } t < s \text{ and } w_j = 0 \text{ for all } j > t, \\ 0 & \text{otherwise,} \end{cases}$$

where w is the unique element of $\mathbb{F}_2^s$ for which $\phi = \phi_w$ and $w_s = 0$.

Proof. Embed $\mathbb{F}_2^t$ in $\mathbb{F}_2^s$ via the first t coordinates. Then

$$\prod_{i=1}^{t} (x_i + (-1)^{v_i}) = \sum_{u\in\mathbb{F}_2^t} \prod_{i=1}^{t} (x_i^{u_i+1}((-1)^{v_i})^{u_i}) = \sum_{u\in\mathbb{F}_2^t} \left(\prod_{i=1}^{t} x_i^{u_i+1}\right)(-1)^{\sum v_i u_i}$$
$$= \sum_{u\in\mathbb{F}_2^t} \prod_{i=1}^{t} x_i^{u_i+1}\cdot(-1)^{\langle v,u\rangle} = \sum_{u\in\mathbb{F}_2^t} \left(\prod_{i=1}^{t} x_i^{u_i+1}\right)\phi_u(v).$$

Writing $\mathbf{x}^{u+\mathbf{1}}$ for $\prod_{i=1}^{t} x_i^{u_i+1}$ and using the identity above and switching the order of summation gives

$$\sum_{v\in K} \phi_w(v) \prod_{i=1}^{t} (x_i + (-1)^{v_i}) = \sum_{u\in\mathbb{F}_2^t} \mathbf{x}^{u+\mathbf{1}} \sum_{v\in K} \phi_w(v)\phi_u(v).$$

The character theory for K implies that the inner sum equals zero unless $\phi_w = \phi_u$ on K, in which case it equals $|K| = 2^{s-1}$. Now the equality $\phi_w = \phi_u$ holds on all of K if and only if

$$w_1 + w_2 = u_1 + u_2, w_2 + w_3 = u_2 + u_3, \ldots, w_{s-1} + w_s = u_{s-1} + u_s.$$

If $t = s$, this happens if and only if $u = w$ or $u = w + (1, 1, \ldots, 1)$, giving the first part of the result.

Next consider the case where $t < s$. Since $u_{t+1} = u_{t+2} = \ldots = u_s = 0$, a necessary condition for $\phi_w = \phi_u$ is that $w_{t+1} + w_{t+2} = 0$, $w_{t+2} + w_{t+3} = 0, \ldots, w_{s-1} + w_s = 0$. Since $w_s = 0$ by the hypothesis, this means that $w_{t+1} = \cdots = w_s = 0$ for $\phi_w = \phi_u$ to hold, giving the third part of the result. Moreover, if $\phi_w = \phi_u$, then also $u_t = w_t$ since both $w_{t+1} = u_{t+1} = 0$. Continuing in this fashion $u_{t-1} = w_{t-1}, \ldots, u_1 = w_1$. So the unique solution for u is $u = w$, which is the second part of the result. □

3.3. Computing the Summation

Here, again we abbreviate $a \equiv b \bmod 2$ as "$a \equiv b$". For $\epsilon \in \{0, 1\}$, let

$$S_\epsilon^+ := \{j \in \mathbb{N} \mid j \equiv \epsilon, \mu_j \equiv 0, \mu_j \neq 0\} \text{ and}$$
$$S_\epsilon^- := \{j \in \mathbb{N} \mid j \equiv \epsilon, \mu_j \equiv 1\} \text{ and}$$
$$S_\epsilon := S_\epsilon^- \cup S_\epsilon^+ = \{j \in \mathbb{N} \mid j \equiv \epsilon, \mu_j > 0\}.$$

For types B, C, D, we set q to be a power of an odd prime.

Type C. For $\lambda \in \mathcal{P}_C(2n)$, pick $e \in \mathcal{O}_\lambda$. Then working in $G = \mathrm{Sp}_{2n}(\bar{\mathbb{F}}_q)$, a maximal reductive part of the centralizer $Z_G(e)$ is isomorphic to

$$\prod_{j \equiv 1} \mathrm{Sp}_{\mu_j}(\bar{\mathbb{F}}_q) \times \prod_{j \in S_0} \mathrm{O}_{\mu_j}(\bar{\mathbb{F}}_q). \tag{3.5}$$

Let A be the elementary abelian 2-group with basis $S_0 = S_0^- \cup S_0^+$. For $c \in A$, we write

$$c = (c_j)_{j \in S_0} \quad \text{with } c_j \in \mathbb{F}_2.$$

We identify A with the component group $A(e)$, where $(c_j) \in A$ corresponds to taking an element of determinant $(-1)^{c_j}$ in the orthogonal group in (3.5) indexed by j, for each $j \in S_0$.

Now, we assume that $e \in \mathfrak{g}^F$ and has split centralizer. For $c \in A(e)$, we twist e by c to get another rational element $e_c \in \mathcal{O}_e$. In this way, we obtain representatives from all the G^F-orbits on $\mathcal{O}_e^F$. Under our identification of A and $A(e)$, the group of rational points in a maximal reductive subgroup of $Z_G(e_c)$ is isomorphic to

$$\prod_{j \equiv 1} \mathrm{Sp}_{\mu_j}(\mathbb{F}_q) \times \prod_{j \in S_0^-} \mathrm{O}_{\mu_j}(\mathbb{F}_q) \times \prod_{j \in S_0^+} \mathrm{O}_{\mu_j}^{c_j}(\mathbb{F}_q),$$

where the groups in the last product are either split or twisted orthogonal groups of type D depending on whether c_j is equal to 0 or 1, respectively; see Shoji [40, Sect. 1].

At this stage another q-analogue notation is helpful: for a nonnegative integer n, let

$$\begin{aligned}\eta(N) &:= (q^2-1)(q^4-1)\cdots(q^{2\lfloor \frac{N}{2} \rfloor}-1) \\ &= \begin{cases} (q^2-1)(q^4-1)\cdots(q^N-1) & \text{if } N \text{ is even,} \\ (q^2-1)(q^4-1)\cdots(q^{N-1}-1) & \text{if } N \text{ is odd.} \end{cases}\end{aligned}$$

or in other words,

$$\eta(2m+1) = \eta(2m) = \prod_{i=1}^{m}(q^{2i}-1).$$

The cardinality of $Z_{G^F}(e_c)$ is therefore (see, e.g., Carter [14, Sect. 2.9, p. 75])

$$q^{d^u} \cdot |A(e)| \prod_{j \equiv 1} \eta(\mu_j) \prod_{j \in S_0^-} \eta(\mu_j) \prod_{j \in S_0^+} (q^{\frac{\mu_j}{2}} - (-1)^{c_j}) \cdot \eta(\mu_j - 2). \tag{3.6}$$

Getting a common denominator over all the conjugacy classes in $A(e)$, we obtain

$$\sum_c \frac{\phi(c)}{|Z_{G^F}(e_c)|} = \frac{\sum_c \phi(c) \prod_{j \in S_0^+}(q^{\frac{\mu_j}{2}} + (-1)^{c_j})}{q^{d^u}|A(e)| \prod_j \eta(\mu_j)}. \tag{3.7}$$

To evaluate this sum we use Lemma 3.2 with $x_i = q^{\frac{\mu_i}{2}}$, $s = |S_0|$, and $t = |S_0^+|$. Choose $w \in A$ to be the unique element so that $\phi = \phi_w$. Then by the lemma the expression in (3.7) equals 0 if $w_j \neq 0$ for any $j \in S_0^-$, and

$$\sum_c \frac{\phi(c)}{|Z_{G^F}(e_c)|} = \frac{q^{-d^u + \sum_{j \in S_0^+, w_j = 0} \frac{\mu_j}{2}}}{\prod_j \eta(\mu_j)} \tag{3.8}$$

if $w_j = 0$ for all $j \in S_0^-$. The value of d^u is computed in Sect. 3.4.

Type B. An important feature is that S_1^- is always non-empty, since $|\lambda|$ is odd. In type B_n we work with $G = \mathrm{SO}_{2n+1}(\bar{\mathbb{F}}_q)$, and thus a maximal reductive subgroup of $Z_G(e)$ is isomorphic to the determinant one elements in

$$\prod_{j \equiv 0} \mathrm{Sp}_{\mu_j}(\bar{\mathbb{F}}_q) \times \prod_{j \in S_1} \mathrm{O}_{\mu_j}(\bar{\mathbb{F}}_q). \tag{3.9}$$

Here, we define A to be the elementary abelian 2-group with basis S_1, with elements written as

$$c = (c_j)_{j \in S_1},$$

where $c_j \in \mathbb{F}_2$. Let K be the subgroup of A consisting of elements (c_j) with $\sum c_j = 0$. We identify K with the component group $A(e)$, where $(c_j) \in K$ corresponds to taking an element of determinant $(-1)^{c_j}$ in the orthogonal group in (3.9) indexed by j for each $j \in S_1$.

Keeping the same notation as in type C, the group of rational points in a maximal reductive subgroup of $Z_G(e_c)$ is isomorphic to

$$\prod_{j \equiv 0} \mathrm{Sp}_{\mu_j}(\mathbb{F}_q) \times \prod_{j \in S_1^-} \mathrm{O}_{\mu_j}(\mathbb{F}_q) \times \prod_{j \in S_1^+} \mathrm{O}^{c_j}_{\mu_j}(\mathbb{F}_q).$$

The cardinality of $Z_{G^F}(e_c)$ is, therefore,

$$q^{d^u} \cdot |A(e)| \prod_{j \equiv 0} \eta(\mu_j) \prod_{j \in S_1^-} \eta(\mu_j) \prod_{j \in S_1^+} (q^{\frac{\mu_j}{2}} - (-1)^{c_j}) \cdot \eta(\mu_j - 2). \tag{3.10}$$

Getting a common denominator over all the conjugacy classes in $A(e)$, we obtain

$$\sum_c \frac{\phi(c)}{|Z_{G^F}(e_c)|} = \frac{\sum_c \phi(c) \prod_{j \in S_1^+} (q^{\frac{\mu_j}{2}} + (-1)^{c_j})}{q^{d^u} |A(e)| \prod_j \eta(\mu_j)}. \tag{3.11}$$

To evaluate the sum we use Lemma 3.3 with $x_i = q^{\frac{\mu_i}{2}}$, $s = |S_1|$, and $t = |S_1^+|$. Note that $t < s$ since S_1^- is non-empty, so the first scenario in the lemma never occurs. Write ϕ as ϕ_w for $w \in A$ with $w_i = 0$ for some $i \in S_1^-$; such a w is unique. Now if $w_j \neq 0$ for any $j \in S_1^-$, then the third scenario in the lemma applies and the expression in (3.11) equals zero. On the other hand, if $w_j = 0$ for all $j \in S_1^-$, then since $t < s$, the second scenario of the lemma gives

$$\sum_c \frac{\phi(c)}{|Z_{G^F}(e_c)|} = \frac{q^{-d^u + \sum_{j \in S_1^+, w_j = 0} \frac{\mu_j}{2}}}{\prod_j \eta(\mu_j)}. \tag{3.12}$$

The value of d^u is computed in Sect. 3.4.

Type D. We proceed as in type B and write $\phi = \phi_w$ for the unique $w \in A$ with $w_i = 0$ for some $i \in S_1^-$ when $|S_1^-| > 0$. Now, $|\lambda|$ is even and so $|S_1^-|$ is even, but it could be zero. Therefore, in evaluating the sum $\sum_c \frac{\phi(c)}{|Z_{G^F}(e_c)|}$ all three scenarios in Lemma 3.3 can occur. When $s > 0$, we, therefore, have

$$\sum_c \frac{\phi(c)}{|Z_{G^F}(e_c)|} = \frac{q^{-d^u}}{\prod_j \eta(\mu_j)} \cdot \begin{cases} q^{\sum_{j\in S_1^+, w_j=1} \frac{\mu_j}{2}} + q^{\sum_{j\in S_1^+, w_j=0} \frac{\mu_j}{2}} & \text{if } |S_1^-| = 0, \\ q^{\sum_{j\in S_1^+, w_j=0} \frac{\mu_j}{2}} & \text{if } |S_1^-| > 0, w_j = 0 \text{ for all } j \in S_1^-, \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

Remark 3.4. The case $s = 0$ is equivalent to the partition λ having only even parts. In such a case, there are two $\mathrm{SO}_{2n}(\bar{\mathbb{F}}_q)$-orbits corresponding to the same λ. In each of these cases, $A(e)$ is trivial and the formula for $|S_1^-| = 0$ above is correct if we interpret it as the sum over two elements, e_1 and e_2, one in each of the two $\mathrm{SO}_{2n}(\bar{\mathbb{F}}_q)$-orbits: $\frac{1}{|Z_{G^F}(e_1)|} + \frac{1}{|Z_{G^F}(e_2)|}$. In other words, the value, when multiplied by $|G|$, gives the number of points in the $\mathrm{O}_{2n}(\bar{\mathbb{F}}_q)$-orbit through either element.

3.4. Value of d^u

Recall that d^u is the dimension of a maximal unipotent subgroup of $Z_G(e)$. Let d_1^u be the dimension of the unipotent radical of $Z_G(e)$ and d_2^u the dimension of a maximal unipotent subgroup of the reductive part Z_{red} of $Z_G(e)$. Then $d^u = d_1^u + d_2^u$. Since d_2^u is the number of positive roots for Z_{red}, we can compute its value from the known type of Z_{red} given previously. The value of d_1^u can be found in [14, pp. 398-9]. Recall that λ' is the dual partition for λ and that $c(\lambda) := \sum_j \lambda'_j \lambda'_{j+1}$.

Type C. We have

$$d_1^u = \frac{1}{2}\left(\sum (\lambda'_j)^2 - \sum \mu_j^2 + \sum_{j\equiv 0} \mu_j\right) \text{ and}$$

$$\begin{aligned} d_2^u &= \sum_{j\equiv 1} \frac{\mu_j^2}{4} + \sum_{\substack{j\equiv 0\\ \mu_j\equiv 1}} \frac{(\mu_j-1)^2}{4} + \sum_{\substack{j\equiv 0\\ \mu_j\equiv 0}} \left(\frac{\mu_j^2}{4} - \frac{\mu_j}{2}\right) \\ &= \frac{1}{4}\sum \mu_j^2 - \frac{1}{2}\sum_{j\equiv 0}\mu_j + \frac{L(\lambda)}{4}, \end{aligned}$$

where $L(\lambda)$ is the number of μ_j that are odd. Hence, since $\lambda'_1 = \ell(\lambda)$,

$$\begin{aligned} d^u = d_1^u + d_2^u &= \frac{1}{2}\sum (\lambda'_j)^2 - \frac{1}{4}\sum \mu_j^2 + \frac{L(\lambda)}{4} \\ &= \frac{1}{2}\sum (\lambda'_j)^2 - \frac{1}{4}\sum (\lambda'_j - \lambda'_{j+1})^2 + \frac{L(\lambda)}{4} \end{aligned}$$

$$= \frac{\ell(\lambda)^2}{4} + \frac{c(\lambda)}{2} + \frac{L(\lambda)}{4}. \tag{3.14}$$

Types B and D. We have

$$d_1^u = \frac{1}{2}\left(\sum(\lambda_j')^2 - \sum \mu_j^2 - \sum_{j\equiv 0}\mu_j\right) \text{ and } d_2^u = \frac{1}{4}\sum\mu_j^2 - \frac{1}{2}\sum_{j\equiv 1}\mu_j + \frac{L(\lambda)}{4}.$$

Hence, since $\sum \mu_j = \ell(\lambda)$,

$$\begin{aligned} d^u = d_1^u + d_2^u &= \frac{1}{2}\sum(\lambda_j')^2 - \frac{1}{4}\sum\mu_j^2 + \frac{L(\lambda)}{4} - \frac{\ell(\lambda)}{2} \\ &= \frac{\ell(\lambda)^2}{4} - \frac{\ell(\lambda)}{2} + \frac{c(\lambda)}{2} + \frac{L(\lambda)}{4}. \end{aligned} \tag{3.15}$$

3.5. Finishing the $f_{e,\phi}$ Calculation in Types B, C, D

We first handle types B_n and C_n. By [29] and [44], we have

$$\begin{aligned} \kappa &= \left\lfloor \frac{\ell(\lambda)}{2} \right\rfloor \text{ and} \\ (m_1, \ldots, m_\kappa) &= (1, 3, \ldots, 2\kappa - 1). \end{aligned} \tag{3.16}$$

Moreover, π_κ is always trivial (see [43]), so $f_{e,\phi}$ is computed by (2.4), which becomes

$$f_{e,\phi} = q^{m(n - \lfloor \frac{\ell(\lambda)}{2} \rfloor) + \lfloor \frac{\ell(\lambda)}{2} \rfloor^2} \prod_{j=1}^{\lfloor \frac{\ell(\lambda)}{2} \rfloor} \left(q^{m-(2j-1)} - 1\right)\left(\sum_c \frac{\phi(c)}{|Z_{G^F}(e_c)|}\right). \tag{3.17}$$

We introduce the following notation

$$\beta_\epsilon(\lambda, w) := \sum_{\substack{j\equiv\epsilon, \mu_j\equiv 0 \\ w_j = 0}} \frac{\mu_j}{2} \tag{3.18}$$

and recall from the introduction that

$$\delta(\lambda) := \left\lfloor \frac{\ell(\lambda)}{2} \right\rfloor^2 - \frac{\ell(\lambda)^2}{4} = \begin{cases} 0 & \text{for } \ell(\lambda) \text{ even}, \\ \frac{1}{4} - \frac{\ell(\lambda)}{2} & \text{for } \ell(\lambda) \text{ odd}, \end{cases}$$

$\hat{N} := \lfloor N/2 \rfloor$, and for $\nu = (\nu_1, \nu_2, \ldots)$, $\hat{\nu} := (\hat{\nu}_1, \hat{\nu}_2, \ldots)$.

Proposition 3.5. *Write $\phi = \phi_w$ as in Sect. 3.3. For $\mathfrak{g}$ of type B_n or C_n, $f_{e,\phi}$ equals zero unless $w_j = 0$ for all $j \in S_\epsilon^-$, where $\epsilon = 0$ for type C and $\epsilon = 1$ for type B, in which case $f_{e,\phi}$ equals*

$$q^{m\left(n - \hat{\ell}(\lambda)\right) + \frac{1}{4} - \frac{c(\lambda)}{2} - \frac{L(\lambda)}{4} + \beta_1(\lambda, w)} \cdot \prod_{i=1}^{\hat{L}(\lambda)} (q^{m-2i+1} - 1) \begin{bmatrix} \hat{m} - \hat{L}(\lambda) \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2} \quad (\textit{Type } B_n)$$

$$q^{m\left(n - \hat{\ell}(\lambda)\right) + \delta(\lambda) - \frac{c(\lambda)}{2} - \frac{L(\lambda)}{4} + \beta_0(\lambda, w)} \cdot \prod_{i=1}^{\hat{L}(\lambda)} (q^{m-2i+1} - 1) \begin{bmatrix} \hat{m} - \hat{L}(\lambda) \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2} \quad (\textit{Type } C_n).$$

Proof. Using (3.8) and (3.12) and the fact that

$$\prod_{j=1}^{\hat{\ell}(\lambda)} \left(q^{m-(2j-1)} - 1\right) = \frac{\eta(m)}{\eta\left(m - \ell(\lambda)\right)},$$

the expression in (3.17), when nonzero, equals

$$f_{e,\phi} = q^{m(n-\hat{\ell}(\lambda))+\hat{\ell}(\lambda)^2-d^u+\beta_\epsilon(\lambda,w)} \cdot \frac{\eta(m)}{\eta\left(m-\ell(\lambda)\right)\cdot\eta(\mu_1)\cdot\eta(\mu_2)\cdots}. \tag{3.19}$$

Next, we have

$$\frac{\eta(m)}{\eta\left(m-\ell(\lambda)\right)\cdot\eta(\mu_1)\cdot\eta(\mu_2)\cdots} = \prod_{i=1}^{\hat{L}(\lambda)} \left(q^{m-2i+1}-1\right) \begin{bmatrix} \hat{m}-\hat{L}(\lambda) \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2},$$

since $\hat{\ell}(\lambda) = \hat{L}(\lambda) + |\hat{\mu}(\lambda)|$. The results follow after substituting in the appropriate value of d^u from Sect. 3.4. □

Recall from the introduction that

$$\tau_\epsilon(\lambda) = \sum_{j\equiv\epsilon,\mu_j\equiv 0} \frac{\mu_j}{2},$$

which is the value of $\beta_\epsilon(\lambda,0)$ when $w=0$, which corresponds to the trivial character of $A(e)$. Thus the results in the Proposition simplify to those in Theorem 1.5 (Types B_n and C_n).

Type D_n. We now turn to type D_n. Here, the values of $m_1, m_2, \ldots, m_\kappa$ depend on both $\ell(\lambda)$ and the parity of μ_1 [44]:

- When μ_1 is odd, $\kappa = \frac{\ell(\lambda)}{2} - 1$ and
$$(m_1, m_2, \ldots, m_\kappa) = (1, 3, \ldots, 2\kappa - 1).$$
- When μ_1 is even, $\kappa = \frac{\ell(\lambda)}{2}$ and
$$(m_1, m_2, \ldots, m_\kappa) = \left(1, 3, \ldots, 2\kappa-3, \ell(\lambda) - \frac{\mu_1}{2} - 1\right).$$

What complicates the type D_n picture is that π_κ may be non-trivial when μ_1 is even. It is always trivial when μ_1 is odd. Let us now describe when this happens and what π_κ is. To that end, we define a subgroup $H \subset A(e)$. Suppose that μ_1 is even and nonzero. Then e lies in a proper Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ of type D of semisimple rank $n - \frac{\mu_1}{2}$. Now if λ contains an odd part different from 1, then $A(e)$ will be nontrivial and, moreover, the component group of e relative to $\mathfrak{l}$ defines an index two subgroup of $A(e)$, which we denote by H. We can now recall

Proposition 3.6 [43, Proposition 10]. *If μ_1 is odd or $\mu_1 = 0$ or $\mu_j = 0$ for all odd $j > 1$, then π_κ is trivial. Otherwise, π_κ is the nontrivial representation of $A(e)$ which is trivial on the subgroup H.*

We can now give the formula for $f_{e,\phi}$.

Proposition 3.7. *Write $\phi = \phi_w$ as in the Type D subsection of Sect. 3.3. Then $f_{e,\phi} = 0$ unless $w_j = 0$ for all $j \in S_1^-$. Otherwise,*

$$f_{e,\phi} = q^{\psi(n,m,\lambda)+\beta_1(\lambda,w)} \text{multiplied by}$$

$$\begin{cases} q^{m+1-\hat{\ell}(\lambda)} \cdot \prod_{j=1}^{\hat{L}(\lambda)-1} (q^{m-2j+1}-1) \cdot \begin{bmatrix} \hat{m}+1-\hat{L}(\lambda) \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2} & \text{if } \mu_1 \text{ is odd} \\ q^{\hat{\ell}(\lambda)-\mu_1} \prod_{j=1}^{\hat{L}(\lambda)} (q^{m-2j+1}-1) \cdot \begin{bmatrix} \hat{m}-\hat{L}(\lambda) \\ \hat{\mu}_{\geq 2}(\lambda) \end{bmatrix}_{q^2} & \\ \quad \times \begin{bmatrix} \hat{m}+1-\hat{L}(\lambda)-|\hat{\mu}_{\geq 2}(\lambda)| \\ \hat{\mu}_1(\lambda) \end{bmatrix}_{q^2} & \text{if } \mu_1 \text{ is even, } w_1{=}0, \text{ and } \hat{L}(\lambda){\geq}1 \\ q^{\hat{\ell}(\lambda)} \prod_{j=1}^{\hat{L}(\lambda)} (q^{m-2j+1}-1) \cdot \begin{bmatrix} \hat{m}-\hat{L}(\lambda) \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2} & \text{if } \mu_1 \text{ is even, } w_1{=}1, \text{ and } \hat{L}(\lambda){\geq}1 \\ q^{\hat{\ell}(\lambda)+\tau_1(\lambda)-2\beta_1(\lambda,w)} \begin{bmatrix} \hat{m} \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2} + q^{\hat{\ell}(\lambda)-\mu_1} & \\ \quad \times \begin{bmatrix} \hat{m} \\ \hat{\mu}_{\geq 2}(\lambda) \end{bmatrix}_{q^2} \begin{bmatrix} \hat{m}+1-|\hat{\mu}_{\geq 2}(\lambda)| \\ \hat{\mu}_1(\lambda) \end{bmatrix}_{q^2} & \text{if } w_1 = 0 \text{ and } \hat{L}(\lambda) = 0 \\ q^{\hat{\ell}(\lambda)} \begin{bmatrix} \hat{m} \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2} + q^{\hat{\ell}(\lambda)-\mu_1+\tau_1(\lambda)-2\beta_1(\lambda,w)} & \\ \quad \times \begin{bmatrix} \hat{m} \\ \hat{\mu}_{\geq 2}(\lambda) \end{bmatrix}_{q^2} \begin{bmatrix} \hat{m}+1-|\hat{\mu}_{\geq 2}(\lambda)| \\ \hat{\mu}_1(\lambda) \end{bmatrix}_{q^2} & \text{if } w_1 = 1 \text{ and } \hat{L}(\lambda) = 0. \end{cases}$$

In fact, since in the last case w can always be taken to have $w_1 = 0$, we only need the first formula.

Proof. When μ_1 is odd, then π_κ is trivial, so formula (2.3) becomes

$$\begin{aligned} f_{e,\phi} &= q^{m(n-\frac{\ell(\lambda)}{2}+1)+(\frac{\ell(\lambda)}{2}-1)^2} \prod_{j=1}^{\hat{\ell}(\lambda)-1} (q^{m-(2j-1)}-1) \sum_c \frac{\phi(c)}{|Z_{G^F}(e_c)|} \\ &= q^{m(n-\frac{\ell(\lambda)}{2}+1)+(\frac{\ell(\lambda)}{2}-1)^2} \frac{\eta(m)}{\eta\left(m+1-\ell(\lambda)\right)} \cdot \frac{q^{-d^u+\beta_1(\lambda,w)}}{\eta(\mu_1)\cdot\eta(\mu_2)\cdots} \end{aligned}$$

using the second part of (3.13) since $|S_1^-| > 0$. The result follows from the formula for d^u in (3.15) and the identity

$$\frac{\eta(m)}{\eta\left(m+1-\ell(\lambda)\right)\cdot\eta(\mu_1)\cdot\eta(\mu_2)\cdots} = \begin{cases} \prod_{j=1}^{\hat{L}(\lambda)-1} (q^{m-2j+1}-1) \cdot \begin{bmatrix} \hat{m}-(\hat{L}(\lambda)-1) \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2} & \text{if } \hat{L}(\lambda) \geq 1 \\ \frac{1}{[\hat{m}+1]_{q^2}} \cdot \begin{bmatrix} \hat{m}+1 \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2} & \text{if } \hat{L}(\lambda) = 0, \end{cases} \tag{3.20}$$

which uses the equality $\hat{\ell}(\lambda) = \hat{L}(\lambda) + |\hat{\mu}(\lambda)|$.

Next when μ_1 is even, formula (2.3) becomes

$$f_{e,\phi} = q^{m(n-\frac{\ell(\lambda)}{2})+\frac{\ell(\lambda)^2}{4}-\frac{\mu_1}{2}} \prod_{j=1}^{\hat{\ell}(\lambda)-1} (q^{m-(2j-1)}-1)$$
$$\cdot \left(q^{m-(\ell(\lambda)-\frac{\mu_1}{2}-1)} \sum_c \frac{\phi(c)}{|Z_{G^F}(e_c)|} - \sum_c \frac{\pi_\kappa(c)\phi(c)}{|Z_{G^F}(e_c)|} \right), \qquad (3.21)$$

since π_κ is always one-dimensional.

When μ_1 is even and $\hat{L}(\lambda) \geq 1$, since $|S_1^-| > 0$, the second part of (3.13) is used to evaluate each sum in (3.21). In this case, π_k is nontrivial if and only if μ_1 is nonzero. For $\phi = \phi_w$, the vector corresponding to $\pi_k\phi_w$ is the same as w but with the parity of w_1 changed. Thus as in the previous case, (3.21) becomes

$$q^{m(n-\frac{\ell(\lambda)}{2})+\frac{\ell(\lambda)^2}{4}-\frac{\mu_1}{2}} \prod_{j=1}^{\hat{L}(\lambda)-1} (q^{m-2j+1}-1) \cdot \begin{bmatrix} \hat{m}+1-\hat{L}(\lambda) \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2} q^{-d^u+\beta_1(\lambda,w)}$$
$$\times \begin{cases} q^{m+1-\ell(\lambda)+\frac{\mu_1}{2}} - q^{-\frac{\mu_1}{2}} & \text{if } w_1 = 0 \\ q^{m+1-\ell(\lambda)+\frac{\mu_1}{2}} - q^{\frac{\mu_1}{2}} & \text{if } w_1 = 1. \end{cases} \qquad (3.22)$$

The following two identities are easy to verify, for $A \in \mathbb{N}$ and ν a partition with $|\nu| \leq A$:

$$\begin{bmatrix} A \\ \nu \end{bmatrix}_q = \begin{bmatrix} A \\ \nu_{\geq 2} \end{bmatrix}_q \begin{bmatrix} A-|\nu_{\geq 2}| \\ \nu_1 \end{bmatrix}_q \text{ and } (q^{A+1-|\nu|}-1) \begin{bmatrix} A+1 \\ \nu \end{bmatrix}_q = (q^{A+1}-1) \begin{bmatrix} A \\ \nu \end{bmatrix}_q.$$

Using these identities and the identity $\hat{L}(\lambda)+|\hat{\mu}_{\geq 2}(\lambda)| = \hat{\ell}(\lambda)-\hat{\mu}_1(\lambda)$, we have

$$\begin{aligned}
&\begin{bmatrix} \hat{m}+1-\hat{L}(\lambda) \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2} \left(q^{m+1-\ell(\lambda)+\frac{\mu_1}{2}} - q^{-\frac{\mu_1}{2}} \right) \\
&= q^{-\hat{\mu}_1}((q^2)^{\hat{m}+1-\hat{\ell}(\lambda)+\hat{\mu}_1}-1) \begin{bmatrix} \hat{m}+1-\hat{L}(\lambda) \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2} \\
&= q^{-\hat{\mu}_1}((q^2)^{\hat{m}+1-\hat{\ell}(\lambda)+\hat{\mu}_1}-1) \begin{bmatrix} \hat{m}+1-\hat{L}(\lambda) \\ \hat{\mu}_{\geq 2}(\lambda) \end{bmatrix}_{q^2} \begin{bmatrix} \hat{m}+1-\hat{L}(\lambda)-|\hat{\mu}_{\geq 2}(\lambda)| \\ \hat{\mu}_1(\lambda) \end{bmatrix}_{q^2} \\
&= q^{-\hat{\mu}_1}((q^2)^{\hat{m}+1-\hat{L}(\lambda)}-1) \begin{bmatrix} \hat{m}-\hat{L}(\lambda) \\ \hat{\mu}_{\geq 2}(\lambda) \end{bmatrix}_{q^2} \begin{bmatrix} \hat{m}+1-\hat{L}(\lambda)-|\hat{\mu}_{\geq 2}(\lambda)| \\ \hat{\mu}_1(\lambda) \end{bmatrix}_{q^2}
\end{aligned}$$

and

$$\begin{aligned}
&\begin{bmatrix} \hat{m}+1-\hat{L}(\lambda) \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2} \left(q^{m+1-\ell(\lambda)+\frac{\mu_1}{2}} - q^{\frac{\mu_1}{2}} \right) \\
&= q^{\hat{\mu}_1}((q^2)^{\hat{m}+1-\hat{\ell}(\lambda)}-1) \begin{bmatrix} \hat{m}+1-\hat{L}(\lambda) \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2}
\end{aligned}$$

$$= q^{\hat{\mu}_1}((q^2)^{\hat{m}+1-\hat{L}(\lambda)-|\hat{\mu}(\lambda)|} - 1) \begin{bmatrix} \hat{m}+1-\hat{L}(\lambda) \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2}$$
$$= q^{\hat{\mu}_1}((q^2)^{\hat{m}+1-\hat{L}(\lambda)} - 1) \begin{bmatrix} \hat{m}-\hat{L}(\lambda) \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2}$$

and the result follows for this case by inserting these values into (3.22).

Finally, when μ_1 is even and $\hat{L}(\lambda) = 0$, we have $S_1^- = \emptyset$. Thus there are two possible choices for w: w and $w + (1, 1, \ldots, 1)$ will both give the same ϕ_w. The formulas will not depend on the choice. The two summations in (3.21) will both make use of the first part of (3.13). The calculation is similar to the previous case after noting that $\tau_1(\lambda) - \beta_1(\lambda, w) = \sum_{j \in S_1^+, w_j=1} \frac{\mu_j}{2}$. □

At $w = 0$, Proposition 3.7 is Theorem 1.5 (Type D_n), since $\beta_1(\lambda, w) = \tau_1(\lambda)$ and $\tau_1(\lambda) - 2\beta_1(\lambda, w) = -\tau_1(\lambda)$.

Remark 3.8. When λ has only even parts, then in particular $\mu_1 = 0$ and $\hat{L}(\lambda) = 0$ and $\tau_1(\lambda) = 0$. We are in the last case. Then the expression in the Corollary simplifies to $2q^{\psi(n,m,\lambda)+\hat{\ell}(\lambda)} \begin{bmatrix} \hat{m} \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2}$. This value is actually twice the value of $f_{e,1}$ for e in either nilpotent orbit associated to λ. See Remark 3.4.

3.6. The $f_{e,\phi}$ for the Exceptional Groups

The polynomials $f_{e,\phi}$ are listed in the third column of the following tables. The first column is the Bala-Carter notation for the nilpotent orbit $\mathcal{O}_e$ together with ϕ, if non-trivial. All $A(e)$ are symmetric groups and we denote ϕ by the corresponding partition for an irreducible representation of a symmetric group, where $[1^k]$ is the sign representation. Recall that an orbit is principal in a Levi subalgebra when there are no parentheses in the Bala-Carter notation. The letters in the notation denote the semisimple part of the Levi subalgebra. In the second column are the representation exponents m_i. Exponents are listed according to the value of π_i, so that if V occurs in $Q_{e,\phi}$, it is listed in the row of (e, ϕ); a dash indicates that V does not occur within $Q_{e,\phi}$. We abbreviate $[a]_q$ by $[a]$ in the last column of the tables.

G_2

(e,ϕ)	m_i	$f_{e,\phi}$
0	$1, 5$	$\frac{[m-1][m-5]}{[2][6]}$
A_1	1	$q^{m-5}\frac{[m-1]}{[2]}$
$\tilde{A}_1$	1	$q^{m-3}\frac{[m-1]}{[2]}$
$G_2(a_1)$	1	$(q^{m-1}-1) \cdot q^{m-3}$
$G_2(a_1), [2,1]$	–	0
$G_2(a_1), [1^3]$	–	0
G_2	–	q^{2m-2}

F_4		
(e,ϕ)	m_i	$f_{e,\phi}$
0	$1,5,7,11$	$\frac{[m-1][m-5][m-7][m-11]}{[2][6][8][12]}$
A_1	$1,5,7$	$q^{m-11}\frac{[m-1][m-5][m-7]}{[2][4][6]}$
$\tilde{A}_1$	$1,5,7$	$q^{m-5}\frac{[m-1][m-5][m-7]}{[2][4][6]}$
$\tilde{A}_1$, $[1^2]$	$-$	$q^{m-8}\frac{[m-1][m-5][m-7]}{[2][4][6]}$
$A_1+\tilde{A}_1$	$1,5$	$q^{2m-14}\frac{[m-1][m-5]}{[2][2]}$
A_2	$1,5$	$q^{2m-8}\frac{[m-1][m-5]}{[2][6]}$
A_2, $[1^2]$	$-$	$q^{2m-11}\frac{[m-1][m-5]}{[2][6]}$
$\tilde{A}_2$	$1,5$	$q^{2m-8}\frac{[m-1][m-5]}{[2][6]}$
$A_2+\tilde{A}_1$	1	$q^{3m-15}\frac{[m-1]}{[2]}$
B_2	$1,3$	$q^{2m-6}\frac{[m-1][m-3]}{[2][4]}$
B_2, $[1^2]$	$-$	$q^{2m-8}\frac{[m-1][m-3]}{[2][4]}$
$\tilde{A}_2+A_1$	1	$q^{3m-13}\frac{[m-1]}{[2]}$
$C_3(a_1)$	$1,3$	$(q^{m-1}-1)\cdot q^{2m-8}\frac{[m-3]}{[2]}$
$C_3(a_1)$, $[1^2]$	$-$	0
$F_4(a_3)$	1	$(q^{m-1}-1)\cdot q^{3m-11}$
$F_4(a_3),[3,1]$	$-$	0
$F_4(a_3),[2,2]$	3	$(q^{m-1}-1)\cdot(-q^{2m-8})$
$F_4(a_3),[2,1^2]$	$-$	0
B_3	1	$q^{3m-7}\frac{[m-1]}{[2]}$
C_3	1	$q^{3m-7}\frac{[m-1]}{[2]}$
$F_4(a_2)$	1	$(q^{m-1}-1)\cdot q^{3m-7}$
$F_4(a_2)$, $[1^2]$	$-$	0
$F_4(a_1)$	1	$(q^{m-1}-1)\cdot q^{3m-5}$
$F_4(a_1)$, $[1^2]$	$-$	0
F_4	$-$	q^{4m-4}

E_6		
(e,ϕ)	m_i	$f_{e,\phi}$
0	$1,4,5,7,8,11$	$\frac{[m-1][m-4][m-5][m-7][m-8][m-11]}{[2][5][6][8][9][12]}$
A_1	$1,4,5,7,8$	$q^{m-11}\frac{[m-1][m-4][m-5][m-7][m-8]}{[2][3][4][5][6]}$
$2A_1$	$1,4,5,7$	$q^{2m-16}\frac{[m-1][m-4][m-5][m-7]}{[1][2][4][6]}$
$3A_1$	$1,4,5$	$q^{3m-21}\frac{[m-1][m-4][m-5]}{[2][2][3]}$
A_2	$1,4,5$	$q^{2m-9}\frac{[m-1][m-4][m-5]([m-7]+q[m-3])}{[2][3][4][6]}$
$A_2,\ [1^2]$	5	$q^{2m-11}\frac{[m-1][m-4][m-5]([m-3]+q^5[m-7])}{[2][3][4][6]}$
A_2+A_1	$1,4,5$	$q^{3m-16}\frac{[m-1][m-4][m-5]}{[1][2][3]}$
$2A_2$	$1,5$	$q^{4m-16}\frac{[m-1][m-5]}{[2][6]}$
A_2+2A_1	$1,4$	$q^{4m-20}\frac{[m-1][m-4]}{[1][2]}$
A_3	$1,3,4$	$q^{3m-11}\frac{[m-1][m-3][m-4]}{[1][2][4]}$
$2A_2+A_1$	1	$q^{5m-21}\frac{[m-1]}{[2]}$
A_3+A_1	$1,3$	$q^{4m-15}\frac{[m-1][m-3]}{[1][2]}$
$D_4(a_1)$	1	$(q^{m-1}-1)\cdot q^{3m-11}\frac{[m-1][m-2]}{[2][3]}$
$D_4(a_1),[2,1]$	3	$(q^{m-1}-1)\cdot q^{3m-10}\frac{[m-2][m-4]}{[1][3]}$
$D_4(a_1),[1^3]$	$-$	$(q^{m-1}-1)\cdot q^{3m-8}\frac{[m-4][m-5]}{[2][3]}$
A_4	$1,3$	$q^{4m-11}\frac{[m-1][m-3]}{[1][2]}$
D_4	$1,2$	$q^{4m-10}\frac{[m-1][m-2]}{[2][3]}$
A_4+A_1	1	$q^{5m-14}\frac{[m-1]}{[1]}$
A_5	1	$q^{5m-11}\frac{[m-1]}{[2]}$
$D_5(a_1)$	$1,2$	$(q^{m-1}-1)\cdot q^{4m-10}\frac{[m-2]}{[1]}$
$E_6(a_3)$	1	$(q^{m-1}-1)\cdot q^{5m-11}$
$E_6(a_3),\ [1^2]$	2	$(q^{m-1}-1)\cdot(-q^{4m-9})$
D_5	1	$q^{5m-8}\frac{[m-1]}{[1]}$
$E_6(a_1)$	1	$(q^{m-1}-1)\cdot q^{5m-7}$
E_6	$-$	q^{6m-6}

E_7		
(e,ϕ)	m_i	$f_{e,\phi}$
0	1,5,7,9, 11,13,17	$\prod_{i=1}^{7} \frac{[m-m_i]}{[m_i+1]}$
A_1	1,5,7,9,11,13	$q^{m-17}\frac{[m-1][m-5][m-7][m-9][m-11][m-13]}{[2][4][6][6][8][10]}$
$2A_1$	$1,5,7,9,11$	$q^{2m-26}\frac{[m-1][m-5][m-7][m-9][m-11]}{[2][2][4][6][8]}$
$(3A_1)''$	$1,5,7,11$	$q^{3m-27}\frac{[m-1][m-5][m-7][m-11]}{[2][6][8][12]}$
$(3A_1)'$	$1,5,7,9$	$q^{3m-33}\frac{[m-1][m-5][m-7][m-9]}{[2][2][4][6]}$
A_2	$1,5,7,9$	$q^{2m-14}\frac{[m-1][m-5][m-7][m-9]([m-3]+q^2[m-13])}{[2][4][6][6][10]}$
$A_2,\ [1^2]$	8	$q^{2m-17}\frac{[m-1][m-5][m-7][m-9]([m-3]+q^8[m-13])}{[2][4][6][6][10]}$
$4A_1$	$1,5,7$	$q^{4m-38}\frac{[m-1][m-5][m-7]}{[2][4][6]}$
A_2+A_1	$1,5,7$	$q^{3m-25}\frac{[m-1][m-5][m-7](q^2[m-7]+[m-9])}{[2][2][4][6]}$
$A_2+A_1,\ [1^2]$	8	$q^{3m-26}\frac{[m-1][m-5][m-7]([m-7]+q^4[m-9])}{[2][2][4][6]}$
A_2+2A_1	$1,5,7$	$q^{4m-32}\frac{[m-1][m-5][m-7]}{[2]^3}$
$2A_2$	$1,5,7$	$q^{4m-26}\frac{[m-1][m-5][m-7]}{[2]^2[6]}$
A_2+3A_1	$1,5$	$q^{5m-35}\frac{[m-1][m-5]}{[2][6]}$
A_3	$1,5,5,7$	$q^{3m-17}\frac{[m-1][m-5][m-5][m-7]}{[2]^2[4][6]}$
$(A_3+A_1)''$	$1,5,7$	$q^{4m-22}\frac{[m-1][m-5][m-7]}{[2][4][6]}$
$2A_2+A_1$	$1,5$	$q^{5m-33}\frac{[m-1][m-5]}{[2]^2}$
$(A_3+A_1)'$	$1,5,5$	$q^{4m-24}\frac{[m-1][m-5][m-5]}{[2]^3}$
$D_4(a_1)$	$1,5$	$(q^{m-1}-1)\cdot q^{3m-17}\frac{[m-1][m-3][m-5]}{[2][4][6]}$
$D_4(a_1),\ [2,1]$	5	$(q^{m-1}-1)\cdot q^{3m-15}\frac{[m-3][m-5][m-7]}{[2]^2[6]}$
$D_4(a_1),\ [1^3]$	$-$	$(q^{m-1}-1)\cdot q^{3m-11}\frac{[m-5][m-7][m-9]}{[2][4][6]}$
A_3+2A_1	$1,5$	$q^{5m-29}\frac{[m-1][m-5]}{[2]^2}$
$D_4(a_1)+A_1$	$1,5$	$(q^{m-1}-1)\cdot q^{4m-22}\frac{[m-3][m-5]}{[2][4]}$
$D_4(a_1)+A_1,\ [1^2]$	5	$(q^{m-1}-1)\cdot q^{4m-20}\frac{[m-5][m-7]}{[2][4]}$
D_4	$1,3,5$	$q^{4m-16}\frac{[m-1][m-3][m-5]}{[2][4][6]}$
A_3+A_2	$1,5$	$q^{5m-25}\frac{[m-1][m-5]}{[2]^2}$
$A_3+A_2,\ [1^2]$	$-$	$q^{5m-26}\frac{[m-1][m-5]}{[2]^2}$
A_4	$1,5$	$q^{4m-16}\frac{[m-1][m-5](q^2[m-3]+[m-5])}{[2]^2[6]}$
$A_4,\ [1^2]$	4	$q^{4m-17}\frac{[m-1][m-5]([m-3]+q^4[m-5])}{[2]^2[6]}$
$A_3+A_2+A_1$	1	$q^{6m-30}\frac{[m-1]}{[2]}$
A_5''	$1,5$	$q^{5m-17}\frac{[m-1][m-5]}{[2][6]}$
D_4+A_1	$1,3$	$q^{5m-21}\frac{[m-1][m-3]}{[2][4]}$

E_7, part 2		
(e,ϕ)	m_i	$f_{e,\phi}$
A_4+A_1	1	$q^{5m-21}\frac{[m-1]}{[2]}\cdot\frac{[m-3]+[m-5]}{[2]}$
A_4+A_1, $[1^2]$	4	$q^{5m-22}\frac{[m-1]}{[2]}\cdot\frac{[m-3]+q^2[m-5]}{[2]}$
$D_5(a_1)$	$1,3$	$(q^{m-1}-1)\cdot q^{4m-16}\frac{[m-3]^2}{[2]^2}$
$D_5(a_1)$, $[1^2]$	4	$(q^{m-1}-1)\cdot q^{4m-15}\frac{[m-3][m-5]}{[2]^2}$
A_4+A_2	1	$q^{6m-24}\frac{[m-1]}{[2]}$
A_5'	$1,3$	$q^{5m-17}\frac{[m-1][m-3]}{[2]^2}$
$D_5(a_1)+A_1$	$1,3$	$(q^{m-1}-1)\cdot q^{5m-19}\frac{[m-3]}{[2]}$
A_5+A_1	1	$q^{6m-22}\frac{[m-1]}{[2]}$
$D_6(a_2)$	$1,3$	$(q^{m-1}-1)\cdot q^{5m-17}\frac{[m-3]}{[2]}$
$E_6(a_3)$	$1,3$	$(q^{m-1}-1)\cdot q^{5m-17}\frac{[m-3]}{[2]}$
$E_6(a_3)$, $[1^2]$	3	$(q^{m-1}-1)(-q^{4m-14})\frac{[m-3]}{[2]}$
$E_7(a_5)$	1	$(q^{m-1}-1)\cdot q^{6m-20}$
$E_7(a_5)$, $[2,1]$	3	$(q^{m-1}-1)\cdot(-q^{5m-17})$
$E_7(a_5)$, $[1^3]$	$-$	$(q^{m-1}-1)\cdot q^{4m-14}$
D_5	$1,3$	$q^{5m-13}\frac{[m-1][m-3]}{[2][2]}$
A_6	1	$q^{6m-16}\frac{[m-1]}{[2]}$
D_5+A_1	1	$q^{6m-16}\frac{[m-1]}{[2]}$
$D_6(a_1)$	$1,3$	$(q^{m-1}-1)\cdot q^{5m-13}\frac{[m-3]}{[2]}$
$E_7(a_4)$	1	$(q^{m-1}-1)\cdot q^{6m-16}$
$E_7(a_4)$, $[1^2]$	$-$	0
$E_6(a_1)$	1	$(q^{m-1}-1)\cdot q^{5m-11}\frac{[m-1]}{[2]}$
$E_6(a_1)$, $[1^2]$	2	$(q^{m-1}-1)\cdot q^{5m-10}\frac{[m-3]}{[2]}$
D_6	1	$q^{6m-12}\frac{[m-1]}{[2]}$
$E_7(a_3)$	1	$(q^{m-1}-1)\cdot q^{6m-12}$
$E_7(a_3)$, $[1^2]$	2	$(q^{m-1}-1)\cdot(-q^{5m-10})$
E_6	1	$q^{6m-10}\frac{[m-1]}{[2]}$
$E_7(a_2)$	1	$(q^{m-1}-1)\cdot q^{6m-10}$
$E_7(a_1)$	1	$(q^{m-1}-1)\cdot q^{6m-8}$
E_7	$-$	q^{7m-7}

E_8		
(e,ϕ)	m_i	$f_{e,\phi}$
0	1,7,11, 13,17,19, 23,29	$\prod_{i=1}^{8} \frac{[m-m_i]}{[m_i+1]}$
A_1	1,7,11, 13,17, 19,23	$q^{m-29}\frac{[m-1][m-7][m-11][m-13][m-17][m-19][m-23]}{[2][6][8][10][12][14][18]}$
$2A_1$	1,7,11, 13, 17,19	$q^{2m-46}\frac{[m-1][m-7][m-11][m-13][m-17][m-19]}{[2][4][6][8][10][12]}$
$3A_1$	1,7,11, 13,17	$q^{3m-57}\frac{[m-1][m-7][m-11][m-13][m-17]}{[2]^2[6][8][12]}$
A_2	1,7,11, 13,17	$q^{2m-24}\frac{[m-1][m-7][m-11][m-13][m-17]([m-5]+q^4[m-23])}{[2][6][8][10][12][18]}$
A_2, $[1^2]$	14	$q^{2m-29}\frac{[m-1][m-7][m-11][m-13][m-17]([m-5]+q^{14}[m-23])}{[2][6][8][10][12][18]}$
$4A_1$	$1,7,11,13$	$q^{4m-68}\frac{[m-1][m-7][m-11][m-13]}{[2][4][6][8]}$
A_2+A_1	$1,7,11,13$	$q^{3m-43}\frac{[m-1][m-7][m-11][m-13](q^2[m-11]+[m-17])}{[2][4][6]^2[10]}$
A_2+A_1, $[1^2]$	14	$q^{3m-46}\frac{[m-1][m-7][m-11][m-13]([m-11]+q^8[m-17])}{[2][4][6]^2[10]}$
A_2+2A_1	$1,7,11,13$	$q^{4m-56}\frac{[m-1][m-7][m-11][m-13]}{[2]^2[4][6]}$
A_3	$1,7,9,11,13$	$q^{3m-29}\frac{[m-1][m-7][m-9][m-11][m-13]}{[2][4][6][8][10]}$
A_2+3A_1	$1,7,11$	$q^{5m-65}\frac{[m-1][m-7][m-11]}{[2]^2[6]}$
$2A_2$	$1,7,11$	$q^{4m-44}\frac{[m-1][m-7][m-11]([m-5]+q^4[m-17])}{[2][4][6][12]}$
$2A_2$, $[1^2]$	11	$q^{4m-46}\frac{[m-1][m-7][m-11]([m-5]+q^8[m-17])}{[2][4][6][12]}$
$2A_2+A_1$	$1,7,11$	$q^{5m-57}\frac{[m-1][m-7][m-11]}{[2]^2[6]}$
A_3+A_1	$1,7,9,11$	$q^{4m-42}\frac{[m-1][m-7][m-9][m-11]}{[2]^2[4][6]}$
$D_4(a_1)$	$1,7,11$	$(q^{m-1}-1)\cdot q^{3m-29}\frac{[m-1][m-5][m-7][m-11]}{[2][6][8][12]}$
$D_4(a_1)$, $[2,1]$	9	$(q^{m-1}-1)\cdot q^{3m-25}\frac{[m-5][m-7][m-11][m-13]}{[2][4][6][12]}$
$D_4(a_1)$, $[1^3]$	–	$(q^{m-1}-1)\cdot q^{3m-17}\frac{[m-7][m-11][m-13][m-17]}{[2][6][8][12]}$
D_4	$1,5,7,11$	$q^{4m-28}\frac{[m-1][m-5][m-7][m-11]}{[2][6][8][12]}$
$2A_2+2A_1$	$1,7$	$q^{6m-66}\frac{[m-1][m-7]}{[2][4]}$
A_3+2A_1	$1,7,9$	$q^{5m-51}\frac{[m-1][m-7][m-9]}{[2]^2[4]}$
$D_4(a_1)+A_1$	$1,7$	$(q^{m-1}-1)\cdot q^{4m-40}\frac{[m-5][m-7]^2}{[2][4][6]}$
$D_4(a_1)+A_1,[2,1]$	9	$(q^{m-1}-1)\cdot q^{4m-38}\frac{[m-5][m-7][m-11]}{[2]^2[6]}$
$D_4(a_1)+A_1,[1^3]$	–	$(q^{m-1}-1)\cdot q^{4m-34}\frac{[m-7][m-11][m-13]}{[2][4][6]}$
A_3+A_2	$1,7,9$	$q^{5m-45}\frac{[m-1][m-7][m-9]}{[2]^2[4]}$
A_3+A_2, $[1^2]$	–	$q^{5m-46}\frac{[m-1][m-7][m-9]}{[2]^2[4]}$
A_4	$1,7,9$	$q^{4m-26}\frac{[m-1][m-7][m-9](q^2[m-5]+[m-11])}{[2][4][6][10]}$
A_4, $[1^2]$	8	$q^{4m-29}\frac{[m-1][m-7][m-9]([m-5]+q^8[m-11])}{[2][4][6][10]}$
$A_3+A_2+A_1$	$1,7$	$q^{6m-54}\frac{[m-1][m-7]}{[2]^2}$
D_4+A_1	$1,5,7$	$q^{5m-39}\frac{[m-1][m-5][m-7]}{[2][4][6]}$
$D_4(a_1)+A_2$	$1,7$	$(q^{m-1}-1)\cdot q^{5m-43}\frac{[m-5][m-7]}{[2][6]}$
$D_4(a_1)+A_2$, $[1^2]$	8	$(q^{m-1}-1)\cdot q^{5m-40}\frac{[m-7][m-11]}{[2][6]}$

E_8, part 2

(e,ϕ)	m_i	$f_{e,\phi}$
A_4+A_1	$1,7$	$q^{5m-37}\frac{[m-1][m-7]([m-5]+q^2[m-11])}{[2]^2[6]}$
$A_4+A_1,\ [1^2]$	8	$q^{5m-38}\frac{[m-1][m-7]([m-5]+q^4[m-11])}{[2]^2[6]}$
$2A_3$	$1,7$	$q^{6m-46}\frac{[m-1][m-7]}{[2][4]}$
$D_5(a_1)$	$1,5,7$	$(q^{m-1}-1)\cdot q^{4m-28}\frac{[m-5]^2[m-7]}{[2][4][6]}$
$D_5(a_1),\ [1^2]$	8	$(q^{m-1}-1)\cdot q^{4m-25}\frac{[m-5][m-7][m-11]}{[2][4][6]}$
A_4+2A_1	$1,7$	$q^{6m-44}\frac{[m-1][m-7]}{[2]^2}$
$A_4+2A_1,\ [1^2]$	–	$q^{6m-45}\frac{[m-1][m-7]}{[2]^2}$
A_4+A_2	$1,7$	$q^{6m-42}\frac{[m-1][m-7]}{[2]^2}$
A_5	$1,5,7$	$q^{5m-29}\frac{[m-1][m-5][m-7]}{[2]^2[6]}$
$D_5(a_1)+A_1$	$1,5,7$	$(q^{m-1}-1)\cdot q^{5m-35}\frac{[m-5][m-7]}{[2]^2}$
$A_4+A_2+A_1$	1	$q^{7m-49}\frac{[m-1]}{[2]}$
D_4+A_2	$1,5$	$q^{6m-36}\frac{[m-1][m-5]}{[2][6]}$
$D_4+A_2,\ [1^2]$	–	$q^{6m-39}\frac{[m-1][m-5]}{[2][6]}$
$E_6(a_3)$	$1,5,7$	$(q^{m-1}-1)\cdot q^{5m-29}\frac{[m-5][m-7]}{[2][6]}$
$E_6(a_3),\ [1^2]$	5	$(q^{m-1}-1)\cdot(-q^{4m-24})\frac{[m-5][m-7]}{[2][6]}$
D_5	$1,5,7$	$q^{5m-23}\frac{[m-1][m-5][m-7]}{[2][4][6]}$
A_4+A_3	1	$q^{7m-45}\frac{[m-1]}{[2]}$
A_5+A_1	$1,5$	$q^{6m-36}\frac{[m-1][m-5]}{[2]^2}$
$D_5(a_1)+A_2$	$1,5$	$(q^{m-1}-1)\cdot q^{6m-38}\frac{[m-5]}{[2]}$
$D_6(a_2)$	$1,5$	$(q^{m-1}-1)\cdot q^{5m-29}\frac{[m-3][m-5]}{[2][4]}$
$D_6(a_2),\ [1^2]$	5	$(q^{m-1}-1)\cdot q^{5m-27}\frac{[m-5][m-7]}{[2][4]}$
$E_6(a_3)+A_1$	$1,5$	$(q^{m-1}-1)\cdot q^{6m-36}\frac{[m-5]}{[2]}$
$E_6(a_3)+A_1,\ [1^2]$	5	$(q^{m-1}-1)\cdot(-q^{5m-31})\frac{[m-5]}{[2]}$
$E_7(a_5)$	$1,5$	$(q^{m-1}-1)\cdot q^{6m-34}\frac{[m-5]}{[2]}$
$E_7(a_5),\ [2,1]$	5	$(q^{m-1}-1)\cdot(-q^{5m-29})\frac{[m-5]}{[2]}$
$E_7(a_5),\ [1^3]$	–	$(q^{m-1}-1)\cdot q^{4m-24}\frac{[m-5]}{[2]}$
D_5+A_1	$1,5$	$q^{6m-30}\frac{[m-1][m-5]}{[2]^2}$
$E_8(a_7)$	1	$(q^{m-1}-1)\cdot q^{7m-39}$
$E_8(a_7),[4,1]$	5	$(q^{m-1}-1)\cdot(-q^{6m-34})$
$E_8(a_7),[3,2]$	–	0
$E_8(a_7),[3,1^2]$	–	$(q^{m-1}-1)\cdot q^{5m-29}$
$E_8(a_7),[2^2,1]$	–	0
$E_8(a_7),[2,1^3]$	–	$(q^{m-1}-1)\cdot(-q^{4m-24})$
A_6	$1,5$	$q^{6m-28}\frac{[m-1][m-5]}{[2]^2}$

E_8, part 3		
(e,ϕ)	m_i	$f_{e,\phi}$
$D_6(a_1)$	$1,5$	$(q^{m-1}-1)\cdot q^{5m-23}\frac{[m-3][m-5]}{[2][4]}$
$D_6(a_1)$, $[1^2]$	5	$(q^{m-1}-1)\cdot q^{5m-21}\frac{[m-5][m-7]}{[2][4]}$
A_6+A_1	1	$q^{7m-33}\frac{[m-1]}{[2]}$
$E_7(a_4)$	$1,5$	$(q^{m-1}-1)\cdot q^{6m-28}\frac{[m-5]}{[2]}$
$E_7(a_4)$, $[1^2]$	$-$	0
$E_6(a_1)$	$1,5$	$(q^{m-1}-1)\cdot q^{5m-19}\frac{[m-1][m-5]}{[2][6]}$
$E_6(a_1)$, $[1^2]$	4	$(q^{m-1}-1)\cdot q^{5m-16}\frac{[m-5][m-7]}{[2][6]}$
D_5+A_2	1	$q^{7m-31}\frac{[m-1]}{[2]}$
D_5+A_2, $[1^2]$	$-$	$q^{7m-32}\frac{[m-1]}{[2]}$
D_6	$1,3$	$q^{6m-22}\frac{[m-1][m-3]}{[2][4]}$
E_6	$1,5$	$q^{6m-18}\frac{[m-1][m-5]}{[2][6]}$
$D_7(a_2)$	1	$(q^{m-1}-1)\cdot q^{6m-26}\frac{[m-3]}{[2]}$
$D_7(a_2)$, $[1^2]$	4	$(q^{m-1}-1)\cdot q^{6m-25}\frac{[m-5]}{[2]}$
A_7	1	$q^{7m-27}\frac{[m-1]}{[2]}$
$E_6(a_1)+A_1$	1	$(q^{m-1}-1)\cdot q^{6m-24}\frac{[m-3]}{[2]}$
$E_6(a_1)+A_1$, $[1^2]$	4	$(q^{m-1}-1)\cdot q^{6m-23}\frac{[m-5]}{[2]}$
$E_7(a_3)$	$1,3$	$(q^{m-1}-1)\cdot q^{6m-22}\frac{[m-3]}{[2]}$
$E_7(a_3)$, $[1^2]$	4	$(q^{m-1}-1)\cdot(-q^{5m-18})\frac{[m-3]}{[2]}$
$E_8(b_6)$	1	$(q^{m-1}-1)\cdot q^{7m-27}$
$E_8(b_6)$, $[2,1]$	$-$	0
$E_8(b_6)$, $[1^3]$	$-$	0
$D_7(a_1)$	$1,3$	$(q^{m-1}-1)\cdot q^{6m-20}\frac{[m-3]}{[2]}$
$D_7(a_1)$, $[1^2]$	$-$	$(q^{m-1}-1)\cdot q^{6m-21}\frac{[m-3]}{[2]}$
E_6+A_1	1	$q^{7m-23}\frac{[m-1]}{[2]}$
$E_7(a_2)$	$1,3$	$(q^{m-1}-1)\cdot q^{6m-18}\frac{[m-3]}{[2]}$
$E_8(a_6)$	1	$(q^{m-1}-1)\cdot q^{7m-23}$
$E_8(a_6)$, $[2,1]$	3	$(q^{m-1}-1)\cdot(-q^{6m-20})$
$E_8(a_6)$, $[1^3]$	$-$	$(q^{m-1}-1)\cdot q^{5m-17}$
D_7	1	$q^{7m-19}\frac{[m-1]}{[2]}$
$E_8(b_5)$	1	$(q^{m-1}-1)\cdot q^{7m-21}$
$E_8(b_5)$, $[2,1]$	3	$(q^{m-1}-1)\cdot(-q^{6m-18})$
$E_8(b_5)$, $[1^3]$	$-$	$(q^{m-1}-1)\cdot q^{5m-15}$
$E_7(a_1)$	$1,3$	$(q^{m-1}-1)\cdot q^{6m-14}\frac{[m-3]}{[2]}$
$E_8(a_5)$	1	$(q^{m-1}-1)\cdot q^{7m-19}$
$E_8(a_5)$, $[1^2]$	$-$	0
$E_8(b_4)$	1	$(q^{m-1}-1)q^{7m-17}$
$E_8(b_4)$, $[1^2]$	$-$	0

E_8, part 4		
(e,ϕ)	m_i	$f_{e,\phi}$
E_7	1	$q^{7m-13}\frac{[m-1]}{[2]}$
$E_8(a_4)$	1	$(q^{m-1}-1)\cdot q^{7m-15}$
$E_8(a_4)$, $[1^2]$	2	$(q^{m-1}-1)\cdot(-q^{6m-13})$
$E_8(a_3)$	1	$(q^{m-1}-1)\cdot q^{7m-13}$
$E_8(a_3)$, $[1^2]$	2	$(q^{m-1}-1)\cdot(-q^{6m-11})$
$E_8(a_2)$	1	$(q^{m-1}-1)\cdot q^{7m-11}$
$E_8(a_1)$	1	$(q^{m-1}-1)\cdot q^{7m-9}$
E_8	–	q^{8m-8}

4. Proof of Theorem 1.6

Before recalling here the statement of the theorem, and giving its proof, let us review some of the terminology. We let $R = \mathrm{rank}(Z_G(e))$, while $H^*(\mathcal{B}_e)$ denotes the cohomology of the Springer fiber for $e \in \mathcal{O}$, regarded as a W representation. Finally, the ill-behaved nilpotent orbits from (1.7) are

$$F_4(a_3),\ E_6(a_3),\ E_6(a_3)+A_1,\ E_7(a_5),\ E_7(a_3),\ E_8(a_7),\ E_8(a_6),\ E_8(b_5),\ E_8(a_4),\ E_8(a_3).$$

Theorem 1.6 *Let e be a nilpotent element* **not** *among the ill-behaved orbits from* (1.7), *and assume that $f_{e,\phi}$ is not identically zero. Then there exists $L, c \in \mathbb{N}$, independent of ϕ, such that*

$$f_{e,\phi}(m;q) = \prod_{j=1}^{L}(q^{m+1-2j}-1)\cdot q^{cm}\cdot g_\phi(m;q),$$

where $g_\phi(m;q)$ is the sum of at most two products of the form $q^{-z}\prod_{i=1}^{R}\frac{[m-a_i]_q}{[b_i]_q}$ for some $a_i, b_i, z \in \mathbb{N}$. *Moreover,* we have the following properties

(i) *For each very good m, the polynomial $q^{cm}\cdot g_\phi(m;q)$ lies in $\mathbb{N}[q]$.*
(ii) *The rank r of $\mathfrak{g}$ equals $L+c+R$.*
(iii) *The multiplicity of V in the W-representation $H^*(\mathcal{B}_e)$* is $r-c$.
(iv) *If e is principal-in-a-Levi, then $L=0$. In particular, $f_{e,\phi}(m;q)\in\mathbb{N}[q]$ for each very good m.*
(v) *If e is not principal-in-a-Levi, then $L\geq 1$. In the exceptional types it always happens that $L=1$.*

Even when e is one of the ill-behaved orbits from (1.7), *at least for the case when $\phi=1$, the polynomials $f_{e,1}(m;q)$ are always nonzero, and still have properties* (i), (ii), (iv), (v) *listed above.*

Let us embark on the proof. The fact that $f_{e,\phi}$ takes the form asserted in the theorem follows from inspection of the formulas for the $f_{e,\phi}$. The formula in part (ii) is a consequence of (2.3) and the fact that

$$\sum_x \frac{\phi(x)}{|Z_{G^F}(e_x)|} \neq 0,$$

whenever $\phi = 1$ or e does not belong to one of the orbits in (1.7). This also explains why $f_{e,1}$ is always nonzero. The formula in part (iii) is a consequence of (2.3) and the fact that

$$\sum_x \frac{\left(\wedge^{d_\kappa}\pi_\kappa\right)(x)\cdot\phi(x)}{|Z_{G^F}(e_x)|} \neq 0,$$

whenever e does not belong to one of the orbits in (1.7), so that $c = r - (\kappa - 1 + d_\kappa)$.

For (i), (iv), (v), it remains to show that $L = 0$ if and only if e is principal-in-a-Levi and that $g_\phi(m;q)$ has the desired positivity property. We do this case-by-case.

4.1. Type A

Since every nilpotent orbit in type A_n is principal-in-a-Levi, we need to show that for every $\lambda \in \mathcal{P}(n)$, when $\gcd(m,n) = 1$ one has

$$\frac{1}{[m]_q}\begin{bmatrix} m \\ \mu(\lambda)\end{bmatrix}_q \in \mathbb{N}[q].$$

According to [36, Corollary 10.4], this follows if all the $\mu_j(\lambda)$'s together with m have the trivial greatest common divisor. But this is true since a common divisor of all the $\mu_j(\lambda)$'s would also be a divisor of $n = |\lambda| = \sum_j j\mu_j(\lambda)$, and we assumed that $\gcd(m,n) = 1$.

4.2. Types B, C

For λ in $\mathcal{P}_B(2n+1)$ or $\mathcal{P}_C(2n)$, the orbit $\mathcal{O}_\lambda$ is principal-in-a-Levi if and only if $\hat{L}(\lambda) = 0$. Thus, whenever $\mathcal{O}_\lambda$ is not principal-in-a-Levi, so that $\hat{L}(\lambda) > 0$, the formula for $f_{e,\phi}$ in Proposition 3.5 contains as a factor the product $\prod_{i=1}^{\hat{L}(\lambda)}(q^{m-2i+1} - 1)$. On the other hand, if $\hat{L}(\lambda) = 0$, this product is empty, and $f_{e,\phi} \in \mathbb{N}[q]$ because it is a power of q times a q-multinomial. For the same reason in all cases, aside from this product, the remaining factor lies in $\mathbb{N}[q]$.

4.3. Type D

For $\lambda \in \mathcal{P}_D(2n)$, the orbit $\mathcal{O}_\lambda$ is principal-in-a-Levi if and only if $L(\lambda) = 0$ or $L(\lambda) = 2$ with μ_1 odd. Note that $|L(\lambda)|$ is always even since λ is partition of $2n$. We examine separately the three conditions on λ in Proposition 3.7.

- If μ_1 is odd, then $L(\lambda) \geq 2$, or equivalently $\hat{L}(\lambda) \geq 1$. Thus, in this case, $\mathcal{O}_\lambda$ is principal-in-a-Levi if and only if $\hat{L}(\lambda) = 1$. Thus, the product $\prod_{i=1}^{\hat{L}(\lambda)-1}(q^{m-2i+1} - 1)$ is non-empty exactly when $\mathcal{O}_\lambda$ is not principal-in-a-Levi. The remaining factors in $f_{e,\phi}$ all lie in $\mathbb{N}[q]$.
- When $L(\lambda) \geq 2$ and μ_1 is even, $\mathcal{O}_\lambda$ is never principal-in-a-Levi. Since $\hat{L}(\lambda) \geq 1$, the product $\prod_{i=1}^{\hat{L}(\lambda)}(q^{m-2i+1} - 1)$ is always non-empty. The other terms in $f_{e,\phi}$ lie in $\mathbb{N}[q]$.

- When $\hat{L}(\lambda) = 0$, $\mathcal{O}_\lambda$ is always principal-in-a-Levi, and $f_{e,\phi} \in \mathbb{N}[q]$ because it is a sum of a q-multinomial and a product of two q-multinomials, shifted by powers of q.

Remark 4.1. In types A, B, C, as well as in the first case of type D, those $\mathcal{O}_\lambda$ which are principal-in-a-Levi not only have $\mathrm{Krew}(\Phi, \mathcal{O}_\lambda, m; q)$ in $\mathbb{N}[q]$, but also have their coefficient sequence symmetric– this follows in type A from the same result [36, Corollary 10.4] quoted earlier, and follows in the other types because q-multinomials have this property.

However, this is *not* in general true for the third case in type D, even though they are always principal-in-a-Levi. For example, when $\lambda = (3,3,1,1)$ in $\mathcal{P}_D(8)$ one has

$$\mathrm{Krew}(D_4, \mathcal{O}_{(3,3,1,1)}, m; q) = q^{14} \left(\begin{bmatrix} \hat{m} \\ 1,1 \end{bmatrix}_{q^2} + \begin{bmatrix} \hat{m} \\ 1 \end{bmatrix}_{q^2}^2 \right)$$

which equals $2q^{14} + 4q^{16} + 6q^{18} + 7q^{20} + 5q^{22} + 3q^{24} + q^{26}$ when $m = 9$.

4.4. Exceptional Types

In the exceptional types many $f_{e,\phi}(m; q)$ can be related to $\mathrm{Cat}(W', m; q)$ for some Weyl group W', which has the desired positivity property.

Most of the remaining cases can be handled by writing

$$\prod_{i=1}^{R} \frac{[m - a_i]_q}{[b_i]_q} \tag{4.1}$$

as a product of polynomials in q with positive coefficients as in the paper of Krattenthaler–Müller [27]. This is accomplished by restricting m to a fixed congruence class modulo the least common multiple of the b_i's (with m also relatively prime to h). We wrote a program in Sage [38], posted on the second author's webpage, that accomplishes this task, except for a handful of cases, making use of [27, Corollary 6], which states that

$$\frac{[\gamma]_q [ab]_q}{[a]_q [b]_q} \tag{4.2}$$

is a polynomial in q with positive coefficients when $\gcd(a, b) = 1$ and $\gamma \geq (a-1)(b-1)$.

Example 4.2. Let e be of type A_1 in E_8. When $m \equiv 17$ modulo 2520, we find that

$$\frac{[m-1][m-7][m-11][m-13][m-17][m-19][m-23]}{[2][6][8][10][12][14][18]}$$

is equal to

$$\left[\frac{m-17}{504}\right]_{q^{504}} \left(\frac{[\frac{m-11}{6}]_{q^6} [84]_{q^6}}{[3]_{q^6} [28]_{q^6}} \right) \left(\frac{[\frac{m-19}{2}]_{q^2} [84]_{q^2}}{[7]_{q^2} [12]_{q^2}} \right) \left(\frac{[\frac{m-13}{4}]_{q^4} [6]_{q^4}}{[3]_{q^4} [2]_{q^4}} \right)$$
$$\times \left[\frac{m-7}{10}\right]_{q^{10}} \left[\frac{m-23}{6}\right]_{q^6} \left[\frac{m-1}{2}\right]_{q^2}.$$

Each term is a polynomial with positive coefficients, using (4.2) for the expressions in parentheses.

The remaining cases are a few of those where $g_\phi(m;q)$ is a sum of two terms of the form in (4.1). For some congruence classes, each expression of the form (4.1) alone will not even be polynomial, let alone positive. We give an example that illustrates how these cases are handled.

Example 4.3. Let e be of type A_2 in E_8. The expression for $f_{e,1}(m;q)$, up to a power of q, is

$$\frac{[m-1][m-7][m-11][m-13][m-17]\left([m-5]+q^4[m-23]\right)}{[2][6][8][10][12][18]}.$$

As long as $\gcd(m,30)=1$ and $m \not\equiv 29$ modulo 30, the program returns $f_{e,1}$ as a sum of polynomials with positive coefficients, possibly after rewriting $[m-5]+q^4[m-23]$ as $[m-19]+q^4[m-9]$. Otherwise, neither summand as in (4.1) is polynomial and this also holds even if we rewrite $[m-5]+q^4[m-23]$ as $[m-19]+q^4[m-9]$. In such cases, we need to deal with the full expression $[m-5]+q^4[m-23]$. For example, when $m \equiv 29$ modulo 360, we can write $[m-1]\left(\frac{[m-5]+q^4[m-23]}{[6][10]}\right)$ as

$$\left[\frac{m-1}{2}\right]_{q^2}\left((q^{10}+q^{24})\cdot\left[\frac{m-29}{30}\right]_{q^{30}}\cdot\frac{[15]_{q^2}}{[3]_{q^2}[5]_{q^2}}+\frac{[12]_{q^2}+q^4[3]_{q^2}}{[3]_{q^2}[5]_{q^2}}\right).$$

Then, $\frac{[12]_{q^2}+q^4[3]_{q^2}}{[3]_{q^2}[5]_{q^2}}=q^{10}-q^8+q^4-q^2+1$, so the product of this polynomial with $\left[\frac{m-1}{2}\right]_{q^2}$ has positive coefficients as in the proof of Corollary 6 in [27]. The remaining terms in $f_{e,1}$ are

$$\left[\frac{m-7}{2}\right]_{q^2}\left[\frac{m-11}{18}\right]_{q^{18}}\left[\frac{m-13}{8}\right]_{q^8}\left[\frac{m-17}{12}\right]_{q^{12}},$$

and so $f_{e,1}(m;q)$ has positive coefficients when $m \equiv 29$ modulo 360.

All cases in the exceptional groups can be handled by reducing to the Catalan case or the case of one of these two examples. It would be nice to have a uniform proof, or at least one that makes use of the fact that $f_{e,\phi}(m;q)$ is polynomial for all very good m.

This completes the proof of Theorem 1.6.

5. The q-Narayana Formulas

In this section, we prove, in types A, B, C, that the q-Kreweras numbers, when summed over nilpotent orbits $\mathcal{O}$ with a fixed value of the statistic $d(\mathcal{O})$ as in (1.8), give the q-Narayana formulas in Theorem 1.10.

In type A_{n-1} we have $d(\mathcal{O}_\lambda) = \ell(\lambda) - 1$ from the formula for $\mathrm{Krew}(A_{n-1}, \mathcal{O}_\lambda, m; q)$ since $r = n-1$. Thus, we want to show that for k

in the range $0 \leq k \leq n-1$ that

$$\sum_{\substack{\lambda \in \mathcal{P}(n):\\ \ell(\lambda)=k+1}} \mathrm{Krew}(A_{n-1}, \mathcal{O}_\lambda, m; q) = \frac{q^{(n-1-k)(m-1-k)}}{[k+1]_q} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} m-1 \\ k \end{bmatrix}_q . \quad (5.1)$$

In types B_n and C_n, we have $d(\mathcal{O}) = \hat{\ell}(\lambda)$, so we wish to show for k in the range $0 \leq k \leq n$ that

$$\sum_{\substack{\lambda \in \mathcal{P}_B(2n+1):\\ \hat{\ell}(\lambda)=k}} \mathrm{Krew}(B_n, \mathcal{O}_\lambda, m; q)$$

$$= \sum_{\substack{\lambda \in \mathcal{P}_C(2n):\\ \hat{\ell}(\lambda)=k}} \mathrm{Krew}(C_n, \mathcal{O}_\lambda, m; q) = (q^2)^{(n-k)(\hat{m}-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \begin{bmatrix} \hat{m} \\ k \end{bmatrix}_{q^2} . \quad (5.2)$$

We now give a proof that relies on counting the number of nilpotent elements (over a finite field) of certain prescribed rank. In a sequel paper we give some alternative proofs.

5.1. Type *A*

The sum on the left in (5.1) is over nilpotent orbits $\mathcal{O}_\lambda$ with $\ell(\lambda) = k+1$. In the formula (3.2) for $f_{e,1}$, all the terms depend only on $\ell(\lambda)$ except for $|Z_G^F(e)|$, where $e = e_\lambda \in \mathcal{O}_\lambda$. Now $\ell(\lambda) = k+1$ means that e_λ is of rank $n-k-1$ when viewed as an $n \times n$ matrix. It follows that the number of nilpotent $n \times n$ matrices of rank $n-k-1$ over $\mathbb{F}_q$ is given by

$$\sum_{\substack{\lambda \in \mathcal{P}(n):\\ \ell(\lambda)=k+1}} \frac{|G^F|}{|Z_{G^F}(e_\lambda)|},$$

and thus by (3.2) the sum in (5.1) becomes

$$q^{m(n-k-1)+\binom{k+1}{2}} \frac{(q-1)^{k+1}[m-1]!_q}{[m-k-1]!_q} \cdot \frac{\#\{\text{nilpotent } n \times n \text{ matrices of rank } n-k-1\}}{|G^F|}. \quad (5.3)$$

The number of nilpotent matrices of rank $n-k-1$ equals (see [16,30])

$$q^{\binom{n-k-1}{2}} \frac{(q-1)^{n-k-1}[n]!_q}{[k+1]!_q} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$$

and $|G^F| = q^{\binom{n}{2}}(q-1)^n[n]!_q$. Substituting these into (5.3) gives (5.1).

5.2. Types *B*, *C*

The sums in (5.2) are over orbits $\mathcal{O}_\lambda$ where $\hat{\ell}(\lambda)$ is fixed. As in type A_{n-1}, the formula for $f_{e,1}$ in (3.17) depends only on $\hat{\ell}(\lambda)$ except for $|Z_G^F(e)|$, where $e = e_\lambda \in \mathcal{O}_\lambda$. Now $\ell(\lambda)$ is the dimension of the kernel of e in the standard representation of $\mathfrak{g}$. Thus, the rank of e_λ is $2n+1-\ell(\lambda)$ in type B_n and $2n-\ell(\lambda)$ in type C_n. Therefore, the condition that $\hat{\ell}(\lambda) = k$ means that the rank of e_λ

is either $2n-2k$ or $2n-2k-1$. Now in type B_n, $\ell(\lambda)$ is always odd and so in particular there are no elements of odd rank. Using this interpretation and (3.17), the sums in (5.2) become

$$\begin{aligned}\sum_{\substack{\lambda\in\mathcal{P}_B(2n+1):\\ \hat\ell(\lambda)=k}} \operatorname{Krew}(B_n,e_\lambda,m;q) &= q^{m(n-k)+k^2}\prod_{j=1}^{k}(q^{m-(2j-1)}-1)\\ &\quad\cdot\frac{\#\left\{\begin{matrix}\text{nilpotent elts in } \mathfrak{g} \text{ of rank}\\ 2(n-k)\end{matrix}\right\}}{|G^F|}\\ \sum_{\substack{\lambda\in\mathcal{P}_C(2n):\\ \hat\ell(\lambda)=k}} \operatorname{Krew}(C_n,e_\lambda,m;q) &= q^{m(n-k)+k^2}\prod_{j=1}^{k}(q^{m-(2j-1)}-1)\\ &\quad\cdot\frac{\#\left\{\begin{matrix}\text{nilpotent elts in } \mathfrak{g} \text{ of rank}\\ 2(n-k) \text{ or } 2(n-k)-1\end{matrix}\right\}}{|G^F|}.\end{aligned}\tag{5.4}$$

Lemma 5.1. *The number of nilpotent elements of type B of rank $2s$ is equal to the number of nilpotent elements of type C of rank $2s$ or $2s-1$. Both are equal to*

$$q^{s^2-s}\frac{\eta(2n)}{\eta(2n-2s)}\begin{bmatrix}n\\ s\end{bmatrix}_{q^2}.$$

Proof. We use the formulas in [30, Theorems 3.1, 3.2]. The stated formula is exactly [30, Theorems 3.1] for the number of nilpotent elements of rank $2s$ in type B_n. The number of rank $2s$ and rank $2s-1$ elements in type C_n are, respectively,

$$q^{s^2+s}\begin{bmatrix}n\\ s\end{bmatrix}_{q^2}\frac{\eta(2n-2))(q^{2n-2s}-1)}{\eta(2n-2s)} \text{ and } q^{s^2-s}\begin{bmatrix}n\\ s\end{bmatrix}_{q^2}\frac{\eta(2n-2))(q^{2s}-1)}{\eta(2n-2s)}.$$

Adding these together gives

$$q^{s^2-s}\begin{bmatrix}n\\ s\end{bmatrix}_{q^2}\frac{\eta(2n-2)}{\eta(2n-2s)}\left(q^{2s}(q^{2n-2s)}-1)+q^{2s}-1\right) = q^{s^2-s}\begin{bmatrix}n\\ s\end{bmatrix}_{q^2}\frac{\eta(2n)}{\eta(2n-2s))}.$$

□

From the lemma, it is immediate that the two expressions in (5.4) are equal. To evaluate them, we write

$$\prod_{j=1}^{k}(q^{m-(2j-1)}-1) = \frac{\eta(m)}{\eta(m-2k)}.$$

With $s=n-k$, the expressions in (5.4) evaluate to

$$q^{m(n-k)+k^2}\frac{\eta(m)}{\eta(m-2k)}\cdot q^{(n-k)^2-(n-k)}\frac{\eta(2n)}{\eta(2k)}\begin{bmatrix}n\\ k\end{bmatrix}_{q^2}\cdot\frac{1}{|G^F|}.$$

Since $|G^F| = q^{n^2}\eta(2n)$ in both types, this becomes

$$q^{m(n-k)+k^2+(n-k)^2-(n-k)-n^2} \begin{bmatrix} \hat{m} \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2}$$

and the exponent of q simplifies to $2(n-k)(\hat{m}-k)$ as desired, where as before $\hat{m} = \frac{m-1}{2}$.

Remark 5.2. From the description of special nilpotent pieces in [32], special nilpotent pieces in types B and C consist of some nilpotent orbits whose elements are of rank $2s$ or $2s-1$ for some fixed s. It follows that the set of nilpotent elements of rank $2s$ and $2s-1$ is a union of special nilpotent pieces. Moreover, a special nilpotent piece in type B corresponds to a special nilpotent piece in type C for the same value of s. Therefore, the equality between the number of elements of rank $2s$ and $2s-1$ in type B and in type C also follows from Lusztig's work that the corresponding special pieces in B and C have the same cardinality [32, Sect. 6.9].

6. Proof of Theorem 1.7

Recall the statement of the theorem:

Theorem 6.6. *In types A, B, C, D, for $m \equiv 1$ mod h, say $m = sh+1$, and for each W-orbit $[X]$ of intersection subspaces of reflecting hyperplanes,* $\mathrm{Krew}(\Phi, \mathcal{O}_X, m; q = \omega_d)$ *counts those $w_1 \leq \cdots \leq w_s$ in $NC^{(s)}(W)$ which*

- *are fixed under the action of an element of order d in the $\mathbb{Z}/sh\mathbb{Z}$-action, and*
- *have the subspace V^{w_1} lying in the W-orbit $[X]$.*

Remark 6.1. We mention here how recent work has generalized the type A_{n-1} special case of Theorem 1.7 from the special case $m = sn+1$ to the case of *all* very good m in type A_{n-1}, that is, where $\gcd(m,n) = 1$. In [2], Armstrong, Rhoades and Williams introduced for all $m > n$ with $\gcd(m,n) = 1$ the set $NC(n,m)$ of *rational* or (n,m)-*noncrossing partitions.* These (n,m)-noncrossing partitions are a subset of $NC(m-1)$, specializing to $NC^{(s)}(W)$ when $m = sn+1$. One might ask whether the subset $NC(n,m) \subset NC(m-1)$ is closed under the natural dihedral symmetry group of order $2(m-1)$ acting on $NC(m-1)$; this was left open in [2], but later resolved affirmatively in work of Bodnar and Rhoades [9]. In fact, recent work of Bodnar[3] has shown how to define $NC(n,m)$ when $m < n$, again with the action of a dihedral group of order $2(m-1)$.

In particular, considering the cyclic $\mathbb{Z}/(m-1)\mathbb{Z}$-action via rotations, the Bodnar and Rhoades also proved a cyclic sieving phenomenon [9, Thm. 5.1] whose $m = sn+1$ special case is equivalent to the type A_{n-1} special case of Theorem 1.7. These results involve a q-Kreweras number that differs slightly from the one in Theorem 1.5, in that it is missing the factor of $q^{m(n-\ell(\lambda))-c(\lambda)}$.

[3]B. Rhoades, personal communication, 2016.

However, this power of q makes no difference in the proof of the cyclic sieving phenomenon, as it happens to equal 1 whenever the q-Kreweras number evaluated at an $(m-1)^{st}$ root-of-unity is nonvanishing– see Lemma 6.7 below.

The aforementioned work of Bodnar also extends this cyclic sieving phenomenon to the case $m < n$.

In proving Theorem 1.7, our plan will be to first compute the evaluations $\mathrm{Krew}(\Phi, \mathcal{O}_X, m; q = \omega_d)$, and then compare them with the combinatorial models for $NC^{(s)}(W)$ in the classical types A, B, C, D.

6.1. Root-of-Unity Evaluation Lemmas

We collect here a few well-known observations on evaluating certain polynomials in q at a primitive d^{th} root of unity ω_d. The proofs are straightforward and omitted.

Warning: For the remainder of the paper, we abandon the convention that "$a \equiv b$" means $a \equiv b \bmod 2$, as we will now need to often consider the equivalence modulo d for other moduli $d \neq 2$.

Lemma 6.2. *Let $\omega_d := e^{2\pi i d}$ or any other primitive d^{th} root-of-unity.*

(i) *The polynomial*

$$[m]_q := \frac{1-q^m}{1-q}$$

has ω_d as a root with multiplicity 1 or 0, depending on whether d divides m or not.

(ii) *For any positive integer m, the q-factorial*

$$[m]!_q := [1]_q[2]_q \cdots [m]_q$$

has ω_d as a root of multiplicity $\lfloor \frac{m}{d} \rfloor$.

(iii) *For d dividing N, the product*

$$[N]_q[N-1]_q \cdots [N-k+1]$$

has ω_d as a root of multiplicity $\lceil \frac{k}{d} \rceil$.

(iv) *If a, b are positive integers with $a \equiv b \mod d$, then*

$$\lim_{q \to \omega_d} \frac{[a]_q}{[b]_q} = \begin{cases} \frac{a}{b} & \text{if } a \equiv b \equiv 0 \mod d \\ 1 & \text{if } a \equiv b \not\equiv 0 \mod d. \end{cases}$$

(v) *For nonnegative n, k expressed uniquely as $n = d \cdot \hat{n} + \hat{\hat{n}}$ and $k = d \cdot \hat{k} + \hat{\hat{k}}$ with $0 \leq \hat{\hat{k}}, \hat{\hat{n}} < d$, one has*

$$\lim_{q \to \omega_d} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{\hat{n}}{\hat{k}} \cdot \lim_{q \to \omega_d} \begin{bmatrix} \hat{\hat{n}} \\ \hat{\hat{k}} \end{bmatrix}_q .$$

In types B, C, D, we will need to evaluate certain polynomials in q^2 at $q = \omega_d$. As notation, let

$$d^- := \frac{d}{\gcd(2,d)} = \left\{ \begin{array}{l} d \text{ for } d \text{ odd}, \\ \dfrac{d}{2} \text{ for } d \text{ even.} \end{array} \right\} \qquad \text{and}$$

$$d^+ := \operatorname{lcm}(2,d) = \left\{ \begin{matrix} d \text{ for } d \text{ even,} \\ 2d \text{ for } d \text{ odd} \end{matrix} \right\} = 2d^-.$$

Some facts that will be used frequently without mention are that, for an even integer $2N$,

$$d \text{ divides } 2N \quad \Leftrightarrow \quad d^+ \text{ divides } 2N \quad \Leftrightarrow \quad d^- \text{ divides } N,$$

and in this situation,

$$\frac{2N}{d^+} = \frac{N}{d^-}.$$

The proof of the following assertions are then straightforward.

Lemma 6.3. *Assume throughout that d* **divides** *$2N$.*

(i) *For a sequence of nonnegative integers $(\alpha_1, \ldots, \alpha_\ell)$, one has*

$$\lim_{q \to \omega_d} \begin{bmatrix} N \\ \alpha_1, \ldots, \alpha_\ell \end{bmatrix}_{q^2} = \begin{cases} \binom{\frac{2N}{d^+}}{\frac{2\alpha_1}{d^+}, \ldots, \frac{2\alpha_\ell}{d^+}} = \binom{\frac{N}{d^-}}{\frac{\alpha_1}{d^-}, \ldots, \frac{\alpha_\ell}{d^-}} & \text{if } d \text{ divides } 2\alpha_i \text{ for each } i, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *Given a nonnegative integer k, for $d = 1, 2$ one has*

$$\lim_{q \to \omega_d} \begin{bmatrix} N+1 \\ k \end{bmatrix}_{q^2} = \binom{N+1}{k}$$

but for $d \geq 3$ one has

$$\lim_{q \to \omega_d} \begin{bmatrix} N+1 \\ k \end{bmatrix}_{q^2} = \begin{cases} \binom{\frac{2N}{d^+}}{\lfloor \frac{2k}{d^+} \rfloor} = \binom{\frac{N}{d^-}}{\lfloor \frac{k}{d^-} \rfloor} & \text{if } k \equiv 0, 1 \bmod d^- \\ 0 & \text{otherwise.} \end{cases}$$

6.2. The q-Kreweras Numbers Evaluated at Roots of Unity

We next use Lemmas 6.2 and 6.3 together with our formulas for $\operatorname{Krew}(\Phi, \mathcal{O}_\lambda, m, q)$ to evaluate them at $q = \omega_d$ whenever $\mathcal{O}_\lambda$ is principal-in-a-Levi, and d divides $m - 1 = sh$ for some $s \geq 1$. We first compute only their complex modulus, ignoring multiplicative factors of powers of q (Proposition 6.4). Then we check they are correct on the nose, not just up to modulus (Proposition 6.6).

Proposition 6.4. *Let s be a positive integer, and assume that d divides $m-1 = sh$.*

(i) *In type A_{n-1} one has $h = n$, so that $m-1 = sn$. Every λ in $\mathcal{P}(n)$ has $\mathcal{O}_\lambda$ principal-in-a-Levi. Then $\operatorname{Krew}(A_{n-1}, \mathcal{O}_\lambda, m; q = \omega_d)$ is nonvanishing if and only if at most one $\mu_{j_0}(\lambda)$ is not divisible by d, and if such a j_0 exists, then $\mu_{j_0}(\lambda) \equiv 1 \bmod d$. Furthermore, in this situation*

$$\|\mathrm{Krew}(A_{n-1},\mathcal{O}_\lambda,m;q=\omega_d)\| = \left\| \lim_{q\to\omega_d} \frac{1}{[m]_q} \begin{bmatrix} m \\ \mu(\lambda) \end{bmatrix}_q \right\|$$
$$= \begin{cases} \frac{1}{m}\binom{m}{\mu(\lambda)} & \text{if } d=1, \\ \binom{\frac{sn}{d}}{\lfloor \frac{\mu(\lambda)}{d} \rfloor} & \text{if } d \geq 2. \end{cases}$$

(ii) *In type B_n, C_n one has $h = 2n$, so that $m-1 = 2sn$. Then λ in $\mathcal{P}_B(2n+1)$ or $\mathcal{P}_C(2n)$ has $\mathcal{O}_\lambda$ principal-in-a-Levi if and only if at most one part j_0 has $\mu_{j_0}(\lambda)$ odd, that is $\hat{L}(\lambda) = 0$. Then* $\mathrm{Krew}(\Phi,\mathcal{O}_\lambda,m;q=\omega_d)$ *is nonvanishing if and only if d divides $2\hat{\mu}_j(\lambda)$ for each j, in which case*

$$\|\mathrm{Krew}(\Phi,\mathcal{O}_\lambda,m;q=\omega_d)\| = \left\| \lim_{q\to\omega_d} \begin{bmatrix} sn \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2} \right\| = \binom{\frac{sn}{d^-}}{\frac{\hat{\mu}(\lambda)}{d^-}}.$$

(iii) *In type D_n one has $h = 2(n-1)$, so that $m-1 = 2s(n-1)$. Then λ in $\mathcal{P}_D(2n)$ has $\mathcal{O}_\lambda$ principal-in-a-Levi if and only if either*

- *there are two part sizes, namely 1 and some odd $j_0 \geq 3$, with odd multiplicity, so $\hat{L}(\lambda) = 1$, or*
- *there are no parts with odd multiplicity, that is, $\hat{L}(\lambda) = 0$.*

In the former $\hat{L}(\lambda) = 1$ case, $\mathrm{Krew}(D_n,\mathcal{O}_\lambda,m;q=\omega_d)$ *is nonvanishing if and only if d divides $2\hat{\mu}_j(\lambda)$ for all j, in which case*

$$\|\mathrm{Krew}(D_n,\mathcal{O}_\lambda,m;q=\omega_d)\| = \left\| \lim_{q\to\omega_d} \begin{bmatrix} s(n-1) \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2} \right\| = \binom{\frac{s(n-1)}{d^-}}{\frac{\hat{\mu}(\lambda)}{d^-}}.$$

In the latter $\hat{L}(\lambda) = 0$ case, $\mathrm{Krew}(D_n,\mathcal{O}_\lambda,m;q=\omega_d)$ *is nonvanishing if and only if both*

(a) $\mu_j(\lambda) \equiv 0 \bmod d^+$ *for all $j \geq 2$, and*
(b) $\mu_1(\lambda) \equiv 0$ *or* $2 \bmod d^+$,

in which case

$$\|\mathrm{Krew}(D_n,\mathcal{O}_\lambda,m;q=\omega_d)\|$$
$$= \left\| \lim_{q\to\omega_d} \left(\begin{bmatrix} s(n-1) \\ \hat{\mu}_{\geq 2}(\lambda) \end{bmatrix}_{q^2} \begin{bmatrix} s(n-1)+1-|\hat{\mu}_{\geq 2}(\lambda)| \\ \hat{\mu}_1(\lambda) \end{bmatrix}_{q^2} + q^{\mu_1(\lambda)-\tau_1(\lambda)} \begin{bmatrix} s(n-1) \\ \hat{\mu}(\lambda) \end{bmatrix}_{q^2} \right) \right\|$$
$$= \begin{cases} \binom{s(n-1)}{\hat{\mu}_{\geq 2}(\lambda)} \binom{s(n-1)+1-|\hat{\mu}_{\geq 2}(\lambda)|}{\hat{\mu}_1(\lambda)} + \binom{s(n-1)}{\hat{\mu}(\lambda)} & \text{if } d=1, \\ \binom{s(n-1)}{\hat{\mu}_{\geq 2}(\lambda)} \binom{s(n-1)+1-|\hat{\mu}_{\geq 2}(\lambda)|}{\hat{\mu}_1(\lambda)} + (-1)^n \binom{s(n-1)}{\hat{\mu}(\lambda)} & \text{if } d=2, \\ \left(1+(-1)^{\frac{2n}{d}}\right) \binom{\frac{2s(n-1)}{d^+}}{\frac{\mu(\lambda)}{d^+}} & \text{if } d\geq 3 \text{ and } \mu_1(\lambda)\equiv 0 \bmod d^+, \\ \binom{\frac{2s(n-1)}{d^+}}{\frac{\mu_1(\lambda)-2}{d^+}, \frac{\mu_{\geq 2}(\lambda)}{d^+}} & \text{if } d\geq 3 \text{ and } \mu_1(\lambda)\equiv 2 \bmod d^+. \end{cases}$$

Proof. Type A. The $d = 1$ case is clear, since one is setting $q = \omega = 1$. Thus, without loss of generality, we may assume that $d \geq 2$. We know from Theorem 1.6 that

$$\frac{1}{[m]_q}\begin{bmatrix} m \\ \mu(\lambda)\end{bmatrix}_q = \frac{[m-1]_q[m-2]_q\cdots[m-\ell(\lambda)+1]_q}{\prod_j [\mu_j(\lambda)]!_q}$$

is a polynomial in q. Lemma 6.2(ii,iii) tell us that it has ω_d as a root of multiplicity $\lfloor \frac{\ell(\lambda)-1}{d} \rfloor$ in the numerator, and of multiplicity $\sum_{j\geq 1} \lfloor \frac{\mu_j(\lambda)}{d} \rfloor$ in the denominator. Hence, one must always have the inequality

$$\left\lfloor \frac{\ell(\lambda)-1}{d} \right\rfloor \geq \sum_{j\geq 1} \left\lfloor \frac{\mu_j(\lambda)}{d} \right\rfloor \tag{6.1}$$

and this must be an equality whenever this polynomial is nonvanishing at $q = \omega_d$. Writing r_j for the remainder of $\mu_j(\lambda)$ on division by d with $0 \leq r_j \leq d-1$, equality in (6.1) would force

$$\frac{\ell-1}{d} \leq \left\lceil \frac{\ell-1}{d} \right\rceil = \sum_{j\geq 1} \left\lfloor \frac{\mu_j(\lambda)}{d} \right\rfloor = \sum_{j\geq 1} \frac{\mu_j - r_j}{d} = \frac{\ell - \sum_{j\geq 1} r_j}{d}.$$

Thus, $\sum_j r_j \leq 1$, or in other words, at most one of the $\mu_j(\lambda)$ is not divisible by d, and its remainder is 1. In this situation, one can use Lemma 6.2(iv) to match up the numerator and denominator factors yielding the asserted evaluation in (i).

Types B, C. This follows from Lemma 6.3(i) with $N = sn$.

Type D. The first case, where $\hat{L}(\lambda) = 1$, similarly to types B/C, follows from Lemma 6.3(i) with $N = s(n-1)$.

In the second case, where $\hat{L}(\lambda) = 0$, we must set $q = \omega_d$ in

$$\begin{bmatrix} s(n-1) \\ \hat{\mu}_{\geq 2}(\lambda)\end{bmatrix}_{q^2} \begin{bmatrix} s(n-1)+1-|\hat{\mu}_{\geq 2}(\lambda)| \\ \hat{\mu}_1(\lambda)\end{bmatrix}_{q^2} + q^{\mu_1(\lambda)-\tau_1(\lambda)} \begin{bmatrix} s(n-1) \\ \hat{\mu}(\lambda)\end{bmatrix}_{q^2}. \tag{6.2}$$

Note that Lemma 6.3(i) with $N = s(n-1)$ shows that, whenever condition (a) above fails, both summands in (6.2) vanish– the first summand vanishes because its first factor vanishes. Similarly, Lemma 6.3(ii) with $N = s(n-1) - |\mu_{\geq 2}(\lambda)|$ shows that whenever condition (b) above fails, both summands in (6.2) vanish– the second factor in the first summand vanishes unless $\mu_1(\lambda) \equiv 0$ or $2 \bmod d^+$. ☐

Hence without loss of generality we may assume that both conditions (a) and (b) hold, and we examine what happens when one evaluates (6.2) at $q = \omega_d$ for various values of d. Note that when $d = 1$ so that $q = 1$, it gives the asserted evaluation, so without loss of generality, we may assume that $d \geq 2$. For this case, we need a lemma.

Lemma 6.5. *A λ in $\mathcal{P}_D(2n)$ with $\mu_j(\lambda) \equiv 0 \bmod d^+$ has $2n \equiv 0 \bmod d^+$ and $\omega_d^{\mu_1(\lambda)-\tau_1(\lambda)} = (-1)^{\frac{2n}{d}}$.*

Proof. One has

$$2n = |\lambda| = \sum_j j\mu_j(\lambda) \equiv 0 \bmod d^+$$

and

$$\begin{aligned}\mu_1(\lambda) - \tau_1(\lambda) &= \mu_1(\lambda) - \frac{1}{2}\sum_{j \text{ odd}} \mu_j(\lambda) = \frac{1}{2}\left(\mu_1(\lambda) - \sum_{\text{odd } j\geq 3} \mu_j(\lambda)\right)\\ &= \frac{d}{2}\left(\frac{\mu_1(\lambda)}{d} - \sum_{\text{odd } j\geq 3} \frac{\mu_j(\lambda)}{d}\right)\end{aligned}$$

and, therefore,

$$\begin{aligned}\omega_d^{\mu_1(\lambda)-\tau_1(\lambda)} &= \left(\omega_{2d}^{d}\right)^{\frac{\mu_1(\lambda)}{d} - \sum_{\text{odd } j\geq 3}\frac{\mu_j(\lambda)}{d}} = (-1)^{\frac{\mu_1(\lambda)}{d} - \sum_{\text{odd } j\geq 3}\frac{\mu_j(\lambda)}{d}}\\ &= (-1)^{\sum_{\text{odd } j}\frac{\mu_j(\lambda)}{d}} = (-1)^{\sum_j \frac{j\cdot\mu_j(\lambda)}{d}} = (-1)^{\frac{2n}{d}}.\end{aligned}$$

□

Now continuing the proof of Proposition 6.4 in type D, together with Lemma 6.3(ii), Lemma 6.5 gives the asserted evaluation for $d = 2$, as in that case, $(-1)^{\frac{2n}{d}} = (-1)^n$.

Now assume $d \geq 3$ and both conditions (a) and (b) hold. If $\mu_1(\lambda) \equiv 0 \bmod d^+$, then setting $q = \omega_d$ in (6.2) gives, after applying Lemma 6.3(i) with $N = s(n-1) - |\hat{\mu}_{\geq 2}(\lambda)|$,

$$\begin{aligned}&\binom{\frac{2s(n-1)}{d^+}}{\frac{\mu_{\geq 2}(\lambda)}{d^+}}\binom{\frac{2s(n-1)-|\mu_{\geq 2}(\lambda)|}{d^+}}{\frac{\mu_1(\lambda)}{d^+}}\\ &\quad+(-1)^{\frac{2n}{d}}\binom{\frac{2s(n-1)}{d^+}}{\frac{\mu(\lambda)}{d^+}} = \left(1+(-1)^{\frac{2n}{d}}\right)\binom{\frac{2s(n-1)}{d^+}}{\frac{\mu(\lambda)}{d^+}},\end{aligned}$$

as desired. On the other hand, if $\mu_1(\lambda) \equiv 2 \bmod d^+$ then the first summand in (6.2) vanishes because its second factor is zero, and hence Lemma 6.3(i) gives (up to sign), the stated answer. □

Proposition 6.6. *In types A, B, C, D and for a positive integer s and a divisor d of $m-1 = sh$, whenever a principal-in-a-Levi nilpotent orbit $\mathcal{O}_\lambda$ has* $\mathrm{Krew}(\Phi, \mathcal{O}_\lambda, m; q = \omega_d)$ *nonvanishing, it equals its (nonnegative integer) complex modulus* $\|\mathrm{Krew}(\Phi, \mathcal{O}_\lambda, m; q = \omega_d)\|$ *given in Proposition 6.4.*

As preparation for the proof of this proposition, we recall that the power of q appearing as the factor in front of the q-Kreweras formula in Theorem 1.5 (Type A) involves the quantity $c(\lambda) = \sum_j \lambda'_j\lambda'_{j+1}$, while the corresponding powers of q in Theorem 1.5 (Types B, C, D) involve $c(\lambda)/2$. Thus, the following lemma on values modulo d, that is, in $\mathbb{Q}/d\mathbb{Z}$, will be helpful.

Lemma 6.7. *Assume $\mathcal{O}_\lambda$ is principal-in-a-Levi and* $\mathrm{Krew}(\Phi, \mathcal{O}_\lambda, m; q = \omega_d) \neq 0$.

- *In type* A_{n-1}*,*
  (i) *one either has* $\mu_j(\lambda) \equiv 0 \bmod d$ *for all* j*, in which case,*
$$c(\lambda) \equiv 0 \bmod d, \ \ \textit{or}$$
  (ii) *one* j_0 *has* $\mu_j(\lambda) \equiv 0 \bmod d$ *for all* $j \neq j_0$ *and* $\mu_{j_0}(\lambda) \equiv 1 \bmod d$*, in which case*
$$c(\lambda) \equiv j_0 - 1 \bmod d.$$
- *In types* B_n, C_n*,*
  (i) *one either has all* $\mu_j(\lambda)$ *even and divisible by* d*, in which case*
$$\frac{c(\lambda)}{2} \equiv 0 \bmod d, \ \ \textit{or}$$
  (ii) *one* j_0 *has* $\mu_{j_0}(\lambda)$ *odd,* $\mu_j(\lambda)$ *even for* $j \neq j_0$*, and* $2\hat{\mu}_j(\lambda) \equiv 0 \bmod d$ *for all* j*, in which case*
$$\frac{c(\lambda)}{2} \equiv \frac{j_0 - 1}{2} + \hat{\ell}(\lambda) - \hat{\mu}_{j_0}(\lambda) \bmod d.$$
- *In type* D_n*,*
  (i) *one either has all* $\mu_j(\lambda)$ *are even,* $2\hat{\mu}_j(\lambda) \equiv 0 \bmod d$ *for* $j \geq 2$*, and* $2\hat{\mu}_1(\lambda) \equiv 0$ *or* $2 \bmod d^+$*, in which case*
$$\frac{c(\lambda)}{2} \equiv 0 \bmod d, \ \ \textit{or}$$
  (ii) *one has* $\mu_1(\lambda)$ *and* $\mu_{j_0}(\lambda)$ *odd for a unique odd part size* $j_0 \geq 3$*, but* $\mu_j(\lambda)$ *even for all* $j \neq 1, j_0$*, and* $2\hat{\mu}_j(\lambda) \equiv 0 \bmod d$ *for all* j*, in which case*
$$\frac{c(\lambda)}{2} \equiv \frac{j_0}{2} + \hat{\mu}_1(\lambda) + \hat{\mu}_{j_0}(\lambda) \bmod d.$$

Proof of Proposition 6.7. The arguments are all similar, so we show only the last (hardest) case: the one in type D with a unique odd $j_0 \geq 3$ for which $\mu_1(\lambda), \mu_{j_0}(\lambda)$ are odd, $\mu_j(\lambda)$ is even for $j \neq 1, j_0$, and $2\hat{\mu}_j(\lambda) \equiv 0 \bmod d$ for all j. Note that
$$\mu_i(\lambda) = \begin{cases} 2\hat{\mu}_i(\lambda) & \text{if } i \neq 1, j_0, \\ 1 + 2\hat{\mu}_i(\lambda) & \text{if } i = 1, j_0, \end{cases} \quad \text{and hence} \quad \lambda'_j = \sum_{i \geq j} \mu_i(\lambda)$$
$$= \begin{cases} \sum_{i \geq j} 2\hat{\mu}_i(\lambda) & \text{if } j \geq j_0 + 1, \\ 1 + \sum_{i \geq j} 2\hat{\mu}_i(\lambda) & \text{if } 2 \leq j \leq j_0, \\ 2 + \sum_{i \geq j} 2\hat{\mu}_i(\lambda) & \text{if } j = 1. \end{cases}$$

Therefore, using "$\equiv$" to denote equivalence modulo $d\mathbb{Z}$ in $\mathbb{Q}/d\mathbb{Z}$, one has

$$\frac{\lambda'_j\lambda'_{j+1}}{2} = \begin{cases} \frac{1}{2}\left(\sum_{i\geq j} 2\hat{\mu}_i(\lambda)\right)\left(\sum_{i\geq j+1} 2\hat{\mu}_i(\lambda)\right) \equiv 0 & \text{if } j \geq j_0+1, \\ \frac{1}{2}\left(1+\sum_{i\geq j_0} 2\hat{\mu}_i(\lambda)\right) & \\ \quad \times \left(\sum_{i\geq j_0+1} 2\hat{\mu}_i(\lambda)\right) \equiv \sum_{i\geq j_0+1}\hat{\mu}(\lambda) & \text{if } j = j_0, \\ \frac{1}{2}\left(1+\sum_{i\geq j} 2\hat{\mu}_i(\lambda)\right)\left(1+\sum_{i\geq j+1} 2\hat{\mu}_i(\lambda)\right) & \\ \equiv \frac{1}{2}+\sum_{i\geq j+1}\hat{\mu}_i(\lambda)+\sum_{i\geq j}\hat{\mu}_i(\lambda) & \\ \equiv \frac{1}{2}+\hat{\mu}_j(\lambda) & \text{if } 2 \leq j \leq j_0-1, \\ \frac{1}{2}\left(2+\sum_{i\geq j} 2\hat{\mu}_i(\lambda)\right) & \\ \quad \times \left(1+\sum_{i\geq j+1} 2\hat{\mu}_i(\lambda)\right) \equiv 1+\sum_{i\geq 1}\hat{\mu}_i(\lambda) & \text{if } j = 1. \end{cases}$$

Consequently,

$$\begin{aligned} c(\lambda) &= \sum_j \frac{\lambda'_j\lambda'_{j+1}}{2} \\ &= \sum_{j\geq j_0+1} \frac{\lambda'_j\lambda'_{j+1}}{2} + \frac{\lambda'_{j_0}\lambda'_{j_0+1}}{2} + \sum_{j=2}^{j_0-1} \frac{\lambda'_j\lambda'_{j+1}}{2} + \frac{\lambda'_1\lambda'_2}{2} \\ &\equiv 0 + \sum_{i\geq j_0+1}\hat{\mu}_i(\lambda) + \sum_{j=2}^{j_0-1}\left(\frac{1}{2}+\hat{\mu}_j(\lambda)\right) + \left(1+\sum_{i\geq 1}\hat{\mu}_i(\lambda)\right) \\ &\equiv \hat{\mu}_1(\lambda)+\hat{\mu}_{j_0}(\lambda)+\frac{j_0}{2}. \end{aligned}$$

□

Proof of Proposition 6.6. We go through eight cases from Proposition 6.7. Here, "$\equiv$" is equivalence in $\mathbb{Q}/d\mathbb{Z}$.
Type A_{n-1}. Here, one needs to show that the exponent

$$E := m(n-\ell(\lambda)) - c(\lambda)$$

has $E \equiv 0$. In the case $A_{n-1}(i)$, since d divides $\mu_j(\lambda)$ for all j, it also divides $n = \sum_j j\mu_j(\lambda)$ and $\ell(\lambda) = \sum_j \mu_j(\lambda)$. Also $c(\lambda) \equiv 0$ by Proposition 6.7, so $E \equiv 0$, as desired. In the case $A_{n-1}(ii)$, one has

$$n-\ell(\lambda) = \sum_{j\geq 1}(j-1)\mu_j \equiv j_0-1$$

and Proposition 6.7 showed $c(\lambda) \equiv j_0-1$. But then $m \equiv 1$ gives

$$\begin{aligned} E = m(n-\ell(\lambda)) - c(\lambda) &\equiv (n-\ell(\lambda)) - c(\lambda) \\ &\equiv j_0-1-(j_0-1) \equiv 0. \end{aligned}$$

Types B, C. Here, one needs to show that the exponent $E = \psi(n,m,\lambda)+\sigma(\lambda) \equiv 0$, using the abbreviations

$$\psi(n,m,\lambda) := m(n-\hat{\ell}(\lambda)) - \frac{c(\lambda)}{2} - \frac{L(\lambda)}{4}$$

$$\sigma(\lambda) := \begin{cases} \tau_1(\lambda)+\frac{1}{4} & \text{in type } B, \\ \tau_0(\lambda) & \text{in type } C \text{ if } \ell(\lambda) \text{ even}, \\ \tau_0(\lambda)+\frac{1}{4}-\frac{\ell(\lambda)}{2} & \text{in type } C \text{ if } \ell(\lambda) \text{ odd.} \end{cases}$$

Note that since $m \equiv 1$, one has the congruence $E \equiv n - \hat{\ell}(\lambda) - \frac{c(\lambda)}{2} - \frac{L(\lambda)}{4} + \sigma(\lambda)$.

In case $B_n/C_n(i)$, Proposition 6.7 says that $\frac{c(\lambda)}{2} \equiv 0$. Furthermore, $L(\lambda) = 0$, and the assumption that $\mu_j(\lambda)$ are all even implies both $\ell(\lambda) = \sum_j \mu_j(\lambda)$ and $|\lambda| = \sum_j j \cdot \mu_j(\lambda)$ are even. Thus, one must be in type C_n, with $\sigma(\lambda) = \tau_0(\lambda)$, so

$$\begin{aligned} E &\equiv n - \hat{\ell}(\lambda) + \tau_0(\lambda) \\ &\equiv \sum_j j\hat{\mu}_j(\lambda) - \sum_j \hat{\mu}_j(\lambda) + \sum_{j \text{ even}} \hat{\mu}_j(\lambda) \\ &= \sum_{j \text{ odd}} (j-1)\hat{\mu}_j(\lambda) + \sum_{j \text{ even}} j\hat{\mu}_j(\lambda). \end{aligned}$$

These last two sums are both even because $2\hat{\mu}_j(\lambda) = \mu_j(\lambda) \equiv 0$ for all j.

In case $B_n/C_n(ii)$, Proposition 6.7 says that $\frac{c(\lambda)}{2} \equiv \frac{j_0-1}{2} + \hat{\ell}(\lambda) - \hat{\mu}_{j_0}(\lambda)$. Also $L(\lambda) = 1$, so one has

$$\begin{aligned} E &\equiv n - \hat{\ell}(\lambda) - \left(\frac{j_0-1}{2} + \hat{\ell}(\lambda) - \hat{\mu}_{j_0}(\lambda)\right) - \frac{1}{4} + \sigma(\lambda) \\ &\equiv n - \frac{j_0-1}{2} + \mu_{j_0}(\lambda) - \frac{1}{4} + \sigma(\lambda) \end{aligned}$$

since $2\hat{\ell}(\lambda) \equiv 0$. In addition, since λ is either in $\mathcal{P}_B(2n+1)$ or $\mathcal{P}_C(2n)$, one can rewrite n as

$$n = \left\lfloor \frac{|\lambda|}{2} \right\rfloor = \left\lfloor \frac{j_0}{2} \right\rfloor + \sum_j j\hat{\mu}_j(\lambda) \equiv \left\lfloor \frac{j_0}{2} \right\rfloor + \sum_{j \text{ odd}} \hat{\mu}_j(\lambda).$$

Therefore,

$$E \equiv \left\lfloor \frac{j_0}{2} \right\rfloor - \frac{j_0-1}{2} + \sum_{j \text{ odd}} \hat{\mu}_j(\lambda) + \hat{\mu}_{j_0}(\lambda) - \frac{1}{4} + \sigma(\lambda). \tag{6.3}$$

If j_0 is odd, so we are in type B_n, then $\left\lfloor \frac{j_0}{2} \right\rfloor = \frac{j_0-1}{2}$, and $\sigma(\lambda) = \tau_1(\lambda) + \frac{1}{4}$, so (6.3) becomes

$$E \equiv \sum_{j \text{ odd}} \hat{\mu}_j(\lambda) + \hat{\mu}_{j_0}(\lambda) + \tau_1(\lambda) = \sum_{j \text{ odd}} 2\hat{\mu}_j(\lambda) \equiv 0.$$

If j_0 is even, so we are in type C_n, then $\left\lfloor \frac{j_0}{2} \right\rfloor - \frac{j_0-1}{2} = \frac{1}{2}$, and $\sigma(\lambda) = \tau_0(\lambda) + \frac{1}{4} - \frac{\ell(\lambda)}{2}$, so (6.3) becomes

$$E \equiv \frac{1}{2} + \sum_{j \text{ odd}} \hat{\mu}_j(\lambda) + \hat{\mu}_{j_0}(\lambda) + \tau_0(\lambda) - \frac{\ell(\lambda)}{2} = \frac{1-\ell(\lambda)}{2} + \sum_j \hat{\mu}_j(\lambda) = 0.$$

Type D. In case $D_n(i)$, one needs to compare the powers of q in front of (6.2) and the q-Kreweras formula in Theorem 1.5(type D_n). Noting that in the case where $\mu_1(\lambda) \equiv 2 \bmod d^+$, the factor of $1+(-1)^{\frac{2n}{d}}$ vanishes unless $n \equiv 0 \bmod d$, one finds that here one needs to show that this exponent

$$\begin{aligned} E \; &:= \psi(n,m,\lambda) + \tau_1(\lambda) + \frac{\ell(\lambda)}{2} \\ &\qquad - \left\{ \begin{array}{ll} \tau_1(\lambda) & \text{if } \mu_1(\lambda) \equiv 0 \bmod d^+ \text{ and } n \equiv 0 \bmod d, \\ \mu_1(\lambda) & \text{if } \mu_1(\lambda) \equiv 2 \bmod d^+ \end{array} \right\} \end{aligned}$$

has $E \equiv 0 \bmod d$. Using $m \equiv 1, L(\lambda) = 0, \hat{\ell}(\lambda) = \frac{\ell(\lambda)}{2}$, and since Proposition 6.7 gives $\frac{c(\lambda)}{2} \equiv 0$, one has

$$E \equiv \begin{cases} n & \text{if } \mu_1(\lambda) \equiv 0 \bmod d^+ \text{ and } n \equiv 0 \bmod d, \\ n + \tau_1(\lambda) - \mu_1(\lambda) & \text{if } \mu_1(\lambda) \equiv 2 \bmod d^+. \end{cases}$$

In the first case, the assumption of case $D_n(i)$ implies $n \equiv 0$. In the second case, one can compute

$$\begin{aligned} E &\equiv n + \tau_1(\lambda) - \mu_1(\lambda) \\ &= \sum_j j\hat{\mu}_j(\lambda) + \sum_{j \text{ odd}} \hat{\mu}_j(\lambda) - \mu_1(\lambda) \\ &= \sum_{j \text{ even}} j\hat{\mu}_j(\lambda) + \sum_{\text{odd } j \geq 3} (j+1)\hat{\mu}_j(\lambda) \; + \; (2\hat{\mu}_1(\lambda) - \mu_1(\lambda)) \\ &\equiv 0 + 0 + 0 = 0. \end{aligned}$$

In case $D_n(ii)$, one needs to show that the exponent

$$\begin{aligned} E &:= \psi(n,m,\lambda) + \tau_1(\lambda) + m - \frac{\ell(\lambda)}{2} + 1 \\ &= \left(m(n - \hat{\ell}(\lambda)) - \frac{c(\lambda)}{2} - \frac{L(\lambda)}{4} \right) + \tau_1(\lambda) + m - \frac{\ell(\lambda)}{2} + 1 \end{aligned}$$

has $E \equiv 0 \bmod d$. Using the facts that $m \equiv 1, L(\lambda) = 2, \hat{\ell}(\lambda) = \frac{\ell(\lambda)}{2}$, and since Proposition 6.7 gives $\frac{c(\lambda)}{2} \equiv \frac{j_0}{2} + \hat{\mu}_1(\lambda) + \hat{\mu}_{j_0}(\lambda)$, one has

$$E \equiv n - \ell(\lambda) + \tau_1(\lambda) + \frac{3}{2} - \left(\frac{j_0}{2} + \hat{\mu}_1(\lambda) + \hat{\mu}_{j_0}(\lambda) \right). \tag{6.4}$$

Note that since all $\mu_j(\lambda)$ are even except for $j = 1, j_0$, one has

$$\begin{aligned} n \; &= \frac{|\lambda|}{2} = \frac{1}{2} + \frac{j_0}{2} + \sum_j j\hat{\mu}_j(\lambda) \equiv \frac{j_0+1}{2} + \sum_{j \text{ odd}} \hat{\mu}_j(\lambda), \\ \ell(\lambda) \; &= \sum_j \mu_j(\lambda) = 2 + \sum_j 2\hat{\mu}_j(\lambda) \equiv 2, \\ \tau_1(\lambda) \; &= \sum_{\substack{j \text{ odd}: \\ \mu_j(\lambda) \text{ even}}} \frac{\mu_j(\lambda)}{2} = \sum_{\substack{j \text{ odd}: \\ j \neq 1, j_0}} \hat{\mu}_j(\lambda). \end{aligned}$$

Thus, one can rewrite (6.4) as

$$E \equiv \frac{j_0+1}{2} + \sum_{j \text{ odd}} \hat{\mu}_j(\lambda) - 2 + \sum_{\substack{j \text{ odd}:\\ j\neq 1, j_0}} \hat{\mu}_j(\lambda) + \frac{3}{2} - \left(\frac{j_0}{2} + \hat{\mu}_1(\lambda) + \hat{\mu}_{j_0}(\lambda) \right)$$
$$\equiv \sum_{\substack{j \text{ odd}:\\ j\neq 1, j_0}} 2\hat{\mu}_j(\lambda) \equiv 0.$$

□

6.3. Combinatorial Models for $NC^{(s)}(w)$

We review here for the classical types A, B, C, D the combinatorial models for the elements of $NC^{(s)}(W)$, that is, the s-element multichains $w_1 \leq \cdots \leq w_s$ in $NC(W)$. We also review how to read off the W-orbit $[X]$ of the subspace $X = V^{w_1}$, and the $\mathbb{Z}/sh\mathbb{Z}$-action on $NC^{(s)}(W)$.

6.3.1. Type *A*. One can identify $NC^{(s)}(A_{n-1})$ with the set of *s-divisible noncrossing set partitions* of $\{1, 2, \ldots, sn\}$, that is, those whose block sizes are all divisible by s; see [1, Chapter 3].

Under this identification, if $w_1 \leq \cdots \leq w_s$ corresponds to an s-divisible noncrossing partition having block sizes $s\lambda = (s\lambda_1, \ldots, s\lambda_\ell)$ for some partition λ of n, then the fixed space V^{w_1} will lie in the W-orbit $[X]$ where W_X is the parabolic subgroup $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_\ell}$ inside $\mathfrak{S}_n$.

Here, $h = n$, and Armstrong's $\mathbb{Z}/sh\mathbb{Z}$-action on $NC^{(s)}(A_{n-1})$ corresponds, under this identification with s-divisible partitions, to the $\mathbb{Z}/sn\mathbb{Z}$-action that cycles the label set $\{1, 2, \ldots, sn\}$ within the blocks via

$$\cdots \to 1 \to 2 \to \cdots \to sn-1 \to sn \to 1 \to \cdots.$$

6.3.2. Types *B*, *C*. One can identify $NC^{(s)}(B_n) = NC^{(s)}(C_n)$ with the subset of s-divisible noncrossing partitions of the label set $\{1, 2, \ldots, 2sn\}$ that are invariant under the involution ι swapping lablels $i \leftrightarrow i + sn \bmod 2sn$; see [1, Sect. 4.5]. For this reason, we relabel $\{1, 2, \ldots, 2sn\}$ as

$$\{+1, +2, \ldots, +sn, -1, -2, \ldots, -sn\} = \{\pm 1, \ldots, \pm sn\} \tag{6.5}$$

so that ι swaps $+i \to -i$ for each i. Note that, since every block B of the s-divisible noncrossing partition must have $\iota(B)$ as another block, the noncrossing condition implies that there can be at most one block B_0 with the property $\iota(B_0) = B_0$; we call such a block B_0, if it exists, a *zero block*, and call the pairs $\{B, \iota(B)\}$ with $\iota(B) \neq B$ the *nonzero blocks*.

Under the identification, if $w_1 \leq \cdots \leq w_s$ corresponds to an s-divisible noncrossing partition with nonzero blocks $\{B_1, \iota(B_1)\}, \ldots, \{B_\ell, \iota(B_\ell)\}$ of sizes $s\nu = (s\nu_1, \ldots, s\nu_\ell)$, then the partition $\nu = (\nu_1, \ldots, \nu_\ell)$ will have $|\nu| \leq n$, and the fixed space V^{w_1} will lie in the W-orbit $[X]$ where W_X is the parabolic subgroup $B_{n-|\nu|} \times \mathfrak{S}_{\nu_1} \times \cdots \times \mathfrak{S}_{\nu_\ell}$ inside B_n.

Here, $h = 2n$, and the $\mathbb{Z}/sh$-action corresponds to a $\mathbb{Z}/2sn\mathbb{Z}$-action cycling the labels $\{\pm 1, \ldots, \pm sn\}$ via

$$\cdots \to +1 \to +2 \to \cdots \to +sn \to -1 \to -2 \to \cdots \to -sn \to +1 \to \cdots.$$

6.3.3. Type D. One can identify $NC^{(s)}(D_n)$ with certain set partitions of the same label set of size $2sn$ as in (6.5), but this time arranged on the inner and outer boundary of an *annulus*, where the $2s(n-1)$ labels

$$+1, +2, \dots, +s(n-1), -1, -2, \dots, -s(n-1)$$

appear in this order *clockwise* on the outer boundary, and the remaining $2n$ labels

$$+(s(n-1)+1), +s((n-1)+2), \dots, +(sn-1), +sn,$$
$$-(s(n-1)+1), -s((n-1)+2), \dots, -(sn-1), -sn,$$

appear in this order *counterclockwise* on the inner boundary. Given a block in any such set partition, we say that the block is *entirely inner* (resp. *entirely outer*) if it only contains labels from the inner (resp. outer) boundary; we say that the block is *traversing* if it is neither entirely inner nor entirely outer. Then $NC^{(s)}(D_n)$ is identified with those set partitions that satisfy these conditions:

NCD1 *noncrossing-ness*: the vertices within a block can all be connected by simple closed curves staying within the annulus, in such a way that curves corresponding to distinct blocks do not cross.

NCD2 *ι-stability*: if B is a block, then $\iota(B)$ is also a block.

NCD3 *zero-block closure*: if there exists a zero-block $B_0 = \iota(B_0)$, then it is unique, and contains all the labels on the inner boundary.

NCD4 *strong s-divisibility*: when reading elements of a block in any clockwise order coming from the planar embedding, their sequence of absolute values pass through consecutive residue classes in $\mathbb{Z}/s\mathbb{Z}$.

NCD5 *determinacy*: if there are no traversing blocks, then the outer blocks completely determine the inner blocks in a certain fashion, whose details are not important to us here; see [24, Sect. 7] or [37, Sect. 7].

The conditions NCD1-4 were given by Krattenthaler–Müller [26, Sect. 7]; while condition NCD5 was inadvertently omitted, and recorded by Kim [24, Sect. 7].

Similarly to types B_n, C_n, under the identification, if $w_1 \leq \cdots \leq w_s$ corresponds to a partition with nonzero blocks $\{B_1, \iota(B_1)\}, \dots, \{B_\ell, \iota(B_\ell)\}$ of sizes $s\nu = (s\nu_1, \dots, s\nu_\ell)$, then the partition $\nu = (\nu_1, \dots, \nu_\ell)$ will either have $|\nu| = n$ if there is no zero block or $|\nu| \leq n-2$ if there is a zero block. Then the fixed space V^{w_1} will lie in the W-orbit $[X]$ where W_X is the parabolic subgroup $D_{n-|\nu|} \times \mathfrak{S}_{\nu_1} \times \cdots \times \mathfrak{S}_{\nu_\ell}$ inside D_n.

Here, $h = 2(n-1)$, and Rhoades [37, Sect. 7] observed that the $\mathbb{Z}/sh$-action corresponds to the $\mathbb{Z}/2s(n-1)$-action simultaneously

- cycling the outer labels (*clockwise*) via

$$\cdots \to -s(n-1) \to +1 \to +2 \to \cdots \to +s(n-1)$$
$$\to -1 \to -2 \to \cdots \to -s(n-1) \to +1 \to \cdots$$

- cycling the inner labels (*counterclockwise*) via

$$\cdots \to -sn \to +(s(n-1)+1) \to +(s(n-1)+2) \to \cdots \to +sn$$
$$\to -(s(n-1)+1) \to -(s(n-1)+2) \to \cdots$$
$$\to -sn \to +(s(n-1)+1) \to \cdots .$$

6.4. Putting it Together

We now assemble the proof of Theorem 1.7 in each type A, B, C, D. Thus we assume that $m = sh + 1$ for a positive integer s, and that d is a divisor of $m - 1 = sh$.

In each case, the strategy will be to first show that if $w_1 \leq \cdots \leq w_s$ is an element in $NC^{(s)}(W)$ having *d-fold symmetry*, that is, fixed by an element $c^{\frac{sh}{d}}$ having order d in the cyclic group $C = \langle c \rangle \cong \mathbb{Z}/sh\mathbb{Z}$, then the parabolic subgroup W_X fixing $X = V^{w_1}$ corresponds to a principal-in-a-Levi nilpotent orbit $\mathcal{O}_\lambda$ that falls into one of the cases from Proposition 6.4. where $\mathrm{Krew}(\Phi, \mathcal{O}_\lambda, m, q = \omega_d)$ is nonvanishing. Then we will count the number of such d-fold symmetric elements, generally relying on known formulas or bijections to objects with known formulas, and see that they agree with the evaluations in Proposition 6.4.

6.4.1. Type *A*. As in Sect. 6.3.1, we have identified the elements of $NC^{(s)}(A_{n-1})$ with the s-divisible noncrossing partitions of $\{1, 2, \ldots, sn\}$, that is, those noncrossing partitions having block sizes $s\lambda$ for some λ with $|\lambda| = n$.

When $d = 1$, one needs to count those which are fixed by the identity element in $\mathbb{Z}/sn\mathbb{Z}$, which are all such elements. In this case, the formula in Proposition 6.4(i) agrees with Kreweras's original count for such partitions.

If $d \geq 2$ and the s-divisible partition has the d-fold symmetry, then most of blocks will be in orbits of size d, with at most one block which is itself d-fold symmetric– two such blocks would cross each other.

Thus, either λ has $\mu_j(\lambda) \equiv 0 \bmod d$ for all j, or there exists one j_0 having $\mu_j(\lambda) \equiv 1 \bmod d$, matching the description for when $\mathrm{Krew}(\Phi, \mathcal{O}_\lambda, m, q = \omega_d)$ is nonvanishing from Proposition 6.4(i).

The proof that in this situation there are exactly

$$\binom{\frac{sn}{d}}{\lfloor \frac{\mu(\lambda)}{d} \rfloor} \tag{6.6}$$

elements with the d-fold symmetry, as in Proposition 6.4(i), was sketched in [8, Theorem 6.2], but we repeat it here for completeness. Such an element is completely determined by restricting each of its blocks to its intersection with the subset $\{1, 2, \ldots, \frac{2sn}{d}\}$; relabel these numbers by $\{1, 2, \ldots, \frac{sn}{d}, -1, -2, \ldots, -\frac{sn}{d}\}$. If there is a (unique) d-fold symmetric j_0-block, then call its restriction the "zero block". It is easily seen that this gives a bijection to the type $B_{\frac{sn}{d}}$ noncrossing partitions considered in [35] having $\hat{\mu}_j(\lambda)$ nonzero blocks of size sj for each j. The formula (6.6) then agrees with the count for such type B noncrossing partitions given by Athanasiadis [3, Theorem 2.3].

6.4.2. Types *B, C*. As in Sect. 6.3.2, we are identifying the elements of $NC^{(s)}(B_n)$ or $NC^{(s)}(C_n)$ with the s-divisible ι-stable noncrossing partitions of $\{\pm 1, \pm 2, \ldots, \pm sn\}$. Such a noncrossing partition has nonzero blocks $\{B_1, \iota(B_1)\}, \ldots, \{B_\ell, \iota(B_\ell)\}$ of sizes $s\nu$ where $\nu = (\nu_1, \ldots, \nu_\ell)$ with $|\nu| \leq n$. The principal-in-a-Levi nilpotent orbit $\mathcal{O}_\lambda$ corresponding to the same parabolic subgroup W_X has $\lambda = (\nu, \nu, N - 2|\nu|)$ where $N = 2n + 1$ in type B_n and $N = 2n$ in type C_n.

If the noncrossing partition additionally has the d-fold symmetry, then the nonzero blocks will all lie in orbits of size d. That is, only the zero block (if present) can have fewer than d distinct images under the d-fold symmetry, and will in fact, be itself d-fold symmetric. As there will be $2\mu_j(\nu) = 2\hat{\mu}_j(\lambda)$ nonzero blocks of size sj for each j, this means that $2\hat{\mu}_j(\lambda) \equiv 0 \bmod d$ for each j, matching the description for when $\mathrm{Krew}(\Phi, \mathcal{O}_\lambda, m, q = \omega_d)$ is nonvanishing from Proposition 6.4(ii).

Similarly to type A, there will be exactly

$$\binom{\frac{2sn}{d^+}}{\frac{2\mu(\lambda)}{d^+}} = \binom{\frac{sn}{d^-}}{\frac{\mu(\lambda)}{d^-}} \tag{6.7}$$

elements with the d-fold symmetry, matching Proposition 6.4(ii): restricting each block to its intersection with the subset $\{\pm 1, \pm 2, \ldots, \pm \frac{sn}{d}\}$ gives a bijection to the type $B_{\frac{sn}{d}}$ noncrossing partitions having $\hat{\mu}_j(\lambda)$ nonzero blocks of size sj for each j, and (6.7) again agrees with Athanasiadis' count [3, Theorem 2.3].

6.4.3. Type *D*. As in Sect. 6.3.3, we are identifying the elements of $NC^{(s)}(D_n)$ with certain "annular" noncrossing partitions. It is convenient to consider separately the two cases where a zero block is present or absent; these will correspond to the two cases in Proposition 6.4(iii) where $\hat{L}(\lambda) = 1$ and $\hat{L}(\lambda) = 0$, respectively.

The case with a zero block present. Due to the *zero-block closure condition NCD3*, removing the $2s$ elements on the inner boundary of the annulus from the zero block gives a bijection between the subset of elements of $NC^{(s)}(D_n)$ having a zero block, and the whole set $NC^{(s)}(B_{n-1})$. Furthermore, this bijection is equivariant with respect to the $\mathbb{Z}/s(n-1)\mathbb{Z}$-action on these sets.

Just as in the type B/C case, let us assume that the nonzero blocks $\{B_1, \iota(B_1)\}, \ldots, \{B_\ell, \iota(B_\ell)\}$ have sizes $s\nu$ where $\nu = (\nu_1, \ldots, \nu_\ell)$ with $|\nu| \le n-2$. Then the principal-in-a-Levi nilpotent orbit $\mathcal{O}_\lambda$ corresponding to the same parabolic subgroup W_X has $\lambda = (\nu, \nu, 1, j_0)$ where $j_0 = 2n-1-2|\nu|$. Again as in the type B/C case, the d-fold symmetry implies that the nonzero blocks all lie in orbits of size d. Hence, there will be $2\mu_j(\nu) = 2\hat{\mu}_j(\lambda)$ nonzero blocks of size sj for each j, so $2\hat{\mu}_j(\lambda) \equiv 0 \bmod d$ for each j. Also the number of such d-fold symmetric elements should be the same as the formula in Proposition 6.4(ii), replacing n by $n-1$. This exactly matches the conditions and the formula in the $\hat{L}(\lambda) = 1$ case of Proposition 6.4(iii).

The case with no zero block. Again assume that the nonzero blocks $\{B_1, \iota(B_1)\}$, $\ldots, \{B_\ell, \iota(B_\ell)\}$ have sizes $s\nu$ where $\nu = (\nu_1, \ldots, \nu_\ell)$ with $|\nu| = n$. Then the principal-in-a-Levi nilpotent orbit $\mathcal{O}_\lambda$ corresponding to the same parabolic subgroup W_X has $\lambda = (\nu, \nu)$.

We are claiming that this case will match with the conditions and formulas appearing in the $\hat{L}(\lambda) = 0$ cases of Proposition 6.4(iii). Thus one should expect the analysis to break into further subcases based on whether $d = 1, 2$ or at least 3.

The subcase with no zero block and $d = 1$. Here, one wishes to count *all* of the elements of $NC^{(s)}(D_n)$ whose annular noncrossing partition has no zero block and nonzero block sizes $s\nu$. This is given by a formula of Krattenthaler and Müller [26, Corollary 16], [25, Theorem 1.2]. Using the formulation in [25, Theorem 1.2], and the notational correspondences $\ell = 1$, $b = \mu$, $k = s$, so that $s_1 = n - b, s_2 = b$, one obtains a formula equivalent to the $\hat{L}(\lambda) = 0$ case of Proposition 6.4(iii) with $d = 1$.

The subcase with no zero block and $d = 2$. Here one wishes to count the elements of $NC^{(s)}(D_n)$ whose annular noncrossing partition has no zero blocks and nonzero block sizes $s\nu$, with the additional property that they are fixed by the element $c^{s(n-1)}$ of order 2 inside the cyclic group $C \cong \mathbb{Z}/2s(n-1)\mathbb{Z}$. Note that this element has different effects on the outer and inner boundary labels: on the outer boundary it rotates 180 degrees, acting as the map $\iota\colon i \leftrightarrow -i$, while on the inner boundary it iterates the 180-degree rotation $n-1$ times, acting as ι^{n-1}.

Thus, whenever the annular noncrossing partition has two (nonzero) inner blocks of size s, it is always fixed by $c^{s(n-1)}$: all blocks will be either entirely inner or outer, and hence the *ι-stability condition* $NCD2$ implies the stability of the blocks under any power of ι.

If the annular noncrossing partition does not have two inner blocks of size s, then all of the labels on its inner boundaries lie in the traversing blocks. We claim that it is then fixed by $c^{s(n-1)}$ if and only if n is even: on the outer labels one is acting by ι, and on the inner labels one is acting by ι^{n-1}, so one needs $\iota^{n-1} = \iota$ to be fixed, that is, n even.

Thus, for n even, all of these elements are fixed by $c^{s(n-1)}$, and therefore should have the same formula as in Proposition 6.4(iii) with $d = 1$. Indeed this agrees with the formula in Proposition 6.4(iii) when $\hat{L}(\lambda) = 0$, $d = 2$ and n even.

Meanwhile for n odd, the elements fixed by $c^{s(n-1)}$ are those with two (nonzero) inner blocks of size s. Removing these inner blocks gives an easy bijection to the elements of $NC^{(s)}(B_{n-1})$ with all nonzero blocks, of sizes $s\lambda - (s,s)$, where $s\lambda - (s,s)$ is obtained from $s\lambda$ by removing two copies of the part s (corresponding to these inner blocks). Therefore, this should have the same formula as Proposition 6.4(ii) but replacing λ with $\lambda - (1,1)$, namely

$$\binom{s(n-1)}{\hat{\mu}_1(\lambda) - 1, \hat{\mu}_{\geq 2}(\lambda)}.$$

Happily, one has an easily checked identity

$$\binom{s(n-1)}{\hat{\mu}_1(\lambda) - 1, \hat{\mu}_{\geq 2}(\lambda)} = \binom{s(n-1)}{\hat{\mu}_{\geq 2}(\lambda)} \binom{s(n-1)+1-|\hat{\mu}_{\geq 2}(\lambda)|}{\hat{\mu}_1(\lambda)} - \binom{s(n-1)}{\hat{\mu}(\lambda)}, \tag{6.8}$$

whose right side agrees with the value given in Proposition 6.4(iii) when $\hat{L}(\lambda) = 0$, $d = 2$ and n is odd.

The subcase with no zero block and $d \geq 3$. Note that there is a dichotomy in the conditions and formulas in Proposition 6.4(iii) when $\hat{L}(\lambda) = 0$ and $d \geq 3$. This will correspond to the following dichotomy in the annular noncrossing partitions that model $NC^{(s)}(D_n)$ and have no zero block.

Proposition 6.8. *In annular noncrossing partitions modelling $NC^{(s)}(D_n)$ with no zero block, either*

(A) *every label on the inner boundary lies in a traversing block, or*
(B) *there are no traversing blocks, only entirely outer blocks, and two entirely inner blocks each of size s.*

Proof. Assume case (B) fails, that is, some inner boundary label j lies in a traversing block B. Then $-j$ lies in $\iota(B)$, which will be a *different* traversing block, since we are assuming that there is no zero block. But then the strong s-divisibility condition $NCD4$ now prevents any of the inner boundary labels from lying in an entirely inner block: $NCD4$ implies that such a traversing block would contain elements having absolute values from every residue class in $\mathbb{Z}/s\mathbb{Z}$, which is impossible since j and $-j$ are the only inner boundary labels whose absolute values achieve their residue class. □

An immediate corollary is that if an element of $NC^{(s)}(D_n)$ corresponds to an annular noncrossing partition with no zero blocks and has d-fold symmetry, then either it is in case (A) of Proposition 6.8 and its blocks all come in orbits of size d (as they are all entirely outer or traversing), or it is in case (B), so that its entirely outer blocks come in orbits of size d, leaving only the two entirely inner blocks (each of size s). This means that its block sizes $s\lambda$ will satisfy

$$\mu_j(\lambda)(= 2\hat{\mu}_j(\lambda)) \equiv 0 \bmod d^+ \text{ for each } j \geq 2,$$
$$\mu_1(\lambda)(= 2\hat{\mu}_1(\lambda)) \equiv \begin{cases} 0 \bmod d^+ & \text{in case (A),} \\ 2 \bmod d^+ & \text{in case (B).} \end{cases}$$

Note that this matches the dichotomy of conditions in Proposition 6.4(iii) when $\hat{L}(\lambda) = 0$. It remains to check that the formulas there match the number of d-fold symmetric elements for $d \geq 3$ in each case.

In case (B), we claim that the two entirely inner blocks of size s are always stable under the element $c^{\frac{2s(n-1)}{d}}$ of order d inside the cyclic group $C = \langle c \rangle$. This is because one has

$$2n = \sum_j j\mu_j(\lambda) \equiv 2 \bmod d^+$$

so that d^+ divides $2(n-1)$, and hence s divides $\frac{2s(n-1)}{d}$. Thus, these two inner blocks are always d-fold symmetric, and they are completely determined by the entirely outer blocks (according to the *determinacy condition* $NCD5$). Therefore, the number of the d-fold symmetric elements in case (B) is the same as the number of d-fold symmetric elements of type B_{n-1} having block sizes $s\lambda - (s, s)$, that is, the formula from Proposition 6.4(ii), but replacing λ with $\lambda - (1, 1)$:

$$\binom{\frac{2s(n-1)}{d^+}}{\frac{\mu_1(\lambda)-2}{d^+}, \frac{\mu_{\geq 2}(\lambda)}{d^+}}.$$

This matches the desired formula in Proposition 6.4(iii), with $\hat{L}(\lambda) = 0, d \geq 3$ and $\hat{\mu}_1(\lambda) \equiv 0 \bmod d^+$.

In case (A), we need a further structural observation.

Lemma 6.9. *For $d \geq 3$, in case (A), the annular noncrossing partition is fixed by $c^{\frac{2s(n-1)}{d}}$ if and only if it has both d-fold rotational symmetry and d divides n.*

Proof. Since in case (A) d divides both $2n = \sum_j \mu_j(\lambda)$ and $2s(n-1)$, one concludes that d divides $2s$. Thus, d divides the number of labels on both the inner boundary and the outer boundary.

Given any block B of a d-fold symmetric annular noncrossing partition in case (A), decompose it uniquely as $B = B_i \sqcup B_o$ with inner boundary labels B_i and outer boundary labels B_o. Then under the action of $g := c^{\frac{2s(n-1)}{d}}$ one must have d disjoint images $B_o, g(B_o), \ldots, g^{d-1}(B_o)$ reading *clockwise*, and also d disjoint images $B_i, g(B_i), \ldots, g^{d-1}(B_i)$ reading *counterclockwise*. However, since each of the sets $g^j(B) = g^j(B_i) \sqcup g^j(B_o)$ is another block, the *noncrossing condition NCD1* (and $d \geq 3$) implies that the counterclockwise ordering $B_i, g(B_i), \ldots, g^{d-1}(B_i)$ must actually also be *clockwise*. In other words, the partition has d-fold rotational symmetry.

Furthermore, since the inner boundary has $2s$ elements and rotating it $\frac{2s(n-1)}{d}$ steps counterclockwise is the same as rotating it $\frac{2s}{d}$ steps clockwise, one concludes that

$$\frac{-2s(n-1)}{d} \equiv \frac{+2s}{d} \bmod 2s, \qquad \text{that is, } \frac{2sn}{d} \equiv 0 \bmod 2s, \qquad \text{or } d \text{ divides } n.$$

Conversely, when d divides n and the partition has d-fold rotational symmetry, one can reverse the above arguments to see that it is fixed by $c^{\frac{2s(n-1)}{d}}$. ∎

We can now complete the comparison of the number of d-fold symmetric elements in case (A) of Proposition 6.8 with the formula in Proposition 6.4(iii) at $\hat{L}(\lambda) = 0$ and $d \geq 3$

$$\left(1 + (-1)^{\frac{2n}{d}}\right) \binom{\frac{2s(n-1)}{d^+}}{\frac{\mu(\lambda)}{d^+}} = \begin{cases} 2 \binom{\frac{2s(n-1)}{d^+}}{\frac{\mu(\lambda)}{d^+}} & \text{if } d \text{ divides } n, \\ 0 & \text{if } d \text{ does not divide } n. \end{cases} \tag{6.9}$$

As noted at the start of the proof of Lemma 6.9, the assumptions of case (A) imply that d divides $2n$.

If d does not divide n, Lemma 6.9 implies that there are no d-fold symmetric elements, matching the value 0 on the right in (6.9).

So assume that d does divide n. By Lemma 6.9 we must count the annular noncrossing partitions modelling elements in $NC^{(s)}(D_n)$ which lie in case (A) of Proposition 6.8, having block sizes $s\lambda$, and which are additionally d-fold rotationally symmetric. The rotational symmetry means that such a partition is completely determined by restricting its blocks to the $\frac{2s(n-1)}{d^-}$ outer labels $\{\pm 1, \pm 2, \ldots, \pm\frac{s(n-1)}{d^-}\}$; the $\frac{2s}{d^-}$ inner labels that accompany these blocks are determined by the *strong s-divisibility condition NCD4*. Since d divides $m-1 = 2s(n-1)$, it also divides $2s$. Thus, setting $\tilde{s} := \frac{s}{d^-}$, the well-defined reduction map $\mathbb{Z}/s\mathbb{Z} \to \mathbb{Z}/\tilde{s}\mathbb{Z}$ shows that these blocks will also satisfy the *strong $\tilde{s}$-divisibility condition NCD4* and determine a unique element of $NC^{\tilde{s}}(D_n)$. Its blocks still have sizes of the form sj, of course, and the number of blocks of size sj will be $\frac{\mu_{sj}(s\lambda)}{d^-} = \frac{\mu_j(\lambda)}{d^-}$. This means that this element of $NC^{\tilde{s}}(D_n)$ has block sizes $\tilde{s}\tilde{\lambda}$ where $\tilde{\lambda}$ only has parts of the form $d^- \cdot j$. Note that $d \geq 3$ forces $d^- \geq 2$, so that $\mu_1(\tilde{\lambda}) = 0$ and $\mu_{\geq 2}(\tilde{\lambda}) = \mu(\tilde{\lambda})$. Also, $\mu_{d^-\cdot j}(\tilde{\lambda}) = \frac{\mu_{sj}(s\lambda)}{d^-} = \frac{\mu_j(\lambda)}{d^-}$. Hence, the number of such elements is counted by the formula of Krattenthaler and Müller already mentioned, that is, the $\hat{L}(\lambda) = 0$ case of Proposition 6.4(iii) with $d = 1$, replacing s with $\tilde{s}$:

$$\begin{aligned}
&\binom{\tilde{s}(n-1)}{\hat{\mu}_{\geq 2}(\lambda)}\binom{\tilde{s}(n-1)+1-|\hat{\mu}_{\geq 2}(\lambda)|}{\hat{\mu}_1(\lambda)} + \binom{\tilde{s}(n-1)}{\hat{\mu}(\lambda)} \\
&\quad = \binom{\tilde{s}(n-1)}{\frac{\hat{\mu}(\lambda)}{d^-}}\binom{\tilde{s}(n-1)+1-|\hat{\mu}_{\geq 2}(\lambda)|}{0} + \binom{\tilde{s}(n-1)}{\frac{\hat{\mu}(\lambda)}{d^-}} = 2\binom{\tilde{s}(n-1)}{\frac{\hat{\mu}(\lambda)}{d^-}} \\
&\quad = 2\binom{\frac{2s(n-1)}{d^+}}{\frac{\mu(\lambda)}{d^+}},
\end{aligned}$$

which matches (6.9) when d divides n.

The completes the proof of Theorem 1.7.

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