COUNTIN' LIKE THE WIND

C. CHAN, I. GESSEL, J. PROPP, V. REINER, L. ROSE, AND B. SAGAN

Dedicated to Richard P. Stanley on the occasion of his sixtieth birthday

Abstract. This work honors one of the greats of enumeration.

1. Countin' like the wind

How many ways can one direct a graph
oh, totally acyclic'lly?
Yes 'n' say that the graph can contain no self-loop
to avoid triviality.
If there are χ of t ways to color its nodes
with at most t colors properly,
then the answer my friend is χ of minus one
though we must take that absolutely.
[1, Corollary 1.3]

How many faces at most can there be
when triangulating a sphere
if we fix the dimension and number of nodes
not assuming convexity?
Yes 'n' also let's say that one need not assume
PL nor shellability.
The maximum's achieved, as Motzkin would agree,
on a cyclic polytope's boundary.
[2, Note added in proof, p. 135]

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How many ways can one order one to ten
so the number of descents is five,
while the number of inversions is thirty-two,
what a very strange thing to contrive!
Yes 'n' doesn't this just make your poor noggin sting
even cause you to break out in hives?
Well the answer's nearly free thinking q-binomially:
it's twenty-nine thousand and five.
[3, Corollary 3.6 with $k = r = 1$]

How many folks dare to count magic squares
with fourteen as their row and column sum
with five rows and five columns and (just to make it hard)
let's tack on a symmetry assumption?
Yes 'n' what's the result if you think to consult
a particular generating function?
One hundred seventy four million, two hundred forty eight thousand,
four hundred and fifty one.
[4, §5, p. 525]

How many subsets of an odd-sized set
of numbers can have the same sum?
Erdős and Moser puzzled long over this,
and it wasn't because they were dumb.
Yes 'n' what can we bound this number by
from the insight of Bernt Lindstrom?
The answer's the largest rank size in the product
of two type B minuscule posets, by gum!
[5, Corollary 5.3 with $k = 1$ and n odd]

How many ways can a permutation
bubble-sort to the identity,
if the sorting is done as fast as can be
and the permutation's vexillary?
Yes 'n' let's say that ϕ is the permutation's code
rearrangèd decreasing weakly.
The answer my friend is the number of tableaux
that are standard, Young, and have shape ϕ.
[6, Corollary 4.2]

How many ways can one tile with unit rhombs
a centrally-symmetric hexagon
if the tiling enjoys the symmetry
of antipodal rotation?
Yes 'n' let us assume that the hexagon's sides
have lengths all of which are even.
Then just square the tiling count for a half-sized hexagon
known to Percy Alexander MacMahon.
[7, equation (3a)]

How many sequences of integers
give a graph's vertex-degrees,
if we say that the graph contains no self-loop,
nor an edge appearing multiply?
Yes 'n' being concrete, let's assume that the graph
has exactly five vertices.
Just count the lattice points in a certain zonotope
and it's five hundred and thirty-three.
[8, Corollary 3.6]

How many ord'rings of 1 through n
distributed quasi-symmetric'lly
(this means by standardizing randomly chosen words
whose letters have fixed probabilities)
consist of increasing runs whose lengths ξ of i
form a composition of n we'll call ξ?
The answer's the ribbon Schur function indexèd by ξ,
evaluated at the probabilities.
[9, Theorem 3.6]

How many theses can one man inspire,
before he decides to retire?
Yes 'n' how many theorems can one man prove,
before he loses his groove?
Yes 'n' just how much fun can he pack in a pun,
and still make up another one?
The answer grows uncannily, though huge most certainly,
when the man's name is Richard P. Stanley.

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