

On the Partial Sum of the Laplacian Eigenvalues of Abstract Simplicial Complexes

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Abstract

We present progress made in showing the generalized Grone-Merris conjecture for the second partial sum for 3-dimensional simplicial complexes. We reduce the conjecture to a special subset of split simplicial complexes.

We also generalize Brouwer's conjecture to a case of simplicial complexes, which gives an upper bound for the partial sums of eigenvalues using only the number of $k - 1$ dimensional faces. In particular, we show that one possible generalization is satisfied for all shifted simplicial complexes.

1 Introduction

Let G be a finite, simple, undirected graph on n vertices, $\{v_1, v_2, \dots, v_n\}$. We denote the edge set of G by $e(G)$, which we will simply refer to as e when there is no ambiguity. The *degree* of any of the vertices is the number of edges incident to them.

Denote the degree of a vertex v_i by $deg(v_i)$. We define the *degree sequence* by $d(G) := \{d_1, d_2, \dots, d_n\}$, where the d_i 's are in non-increasing order. We define the *conjugate degree sequence* $d^T(G) := \{d_1^T, d_2^T, \dots\}$, which for each d_i^T gives the number of elements in $d(G)$ that are at least i .

Definition 1. The **oriented incidence matrix** of a directed graph G is the matrix that has a column corresponding to each edge and a row for each vertex of the graph. Given an edge e from vertex i to j , where $i < j$, the entries of the matrix are,

$$M_{e,v} := \begin{cases} 1 & \text{if } v = i \\ -1 & \text{if } v = j \\ 0 & \text{otherwise} \end{cases}$$

We can evaluate the incidence matrix of an undirected graph G by imposing an arbitrary orientation on the edges.

Definition 2. The **Laplacian matrix** of G is defined as,

$$L'(G) := M(G)^T M(G),$$

where $M(G)^T$ is the matrix transpose of $M(G)$.

In some cases, it might be more convenient to look at a second matrix,

$$L''(G) := M(G)M(G)^T,$$

which has the same eigenvalues as $L(G)$.

Several properties of the Laplacian matrix are known including the fact that it is positive-semidefinite, and therefore has non-negative, real numbers as eigenvalues. We denote the Laplacian spectrum of G by,

$$\lambda(G) = \{\lambda_1, \lambda_2, \dots, \lambda_n\},$$

where the sequence is in non-increasing order. There are as many 0 eigenvalues as there are connected components of G . In particular, $\lambda_n = 0$.

In spectral graph theory, there is an interest in how large these λ_i 's can get. Grone and Merris conjectured that the Laplacian spectrum of G is majorized by the conjugate degree sequence of G , i.e.

$$\sum_{i=1}^t \lambda_i \leq \sum_{i=1}^t d_i^T$$

This conjecture was recently proved in [Bai], by reducing the theorem to a special class of graphs called split graphs.

Definition 3. A **split graph** is a graph whose vertices can be partitioned into two sets C and Q , where any two vertices of C are connected by an edge (i.e. they form a complete graph) and no two vertices of Q are connected by an edge (i.e. they are independent).

There might be any number of edges between a vertex in C and a vertex in Q . We call C and Q the *clique* and *co-clique* of G , respectively. We denote their cardinality by c and q .

For instance, the following graph on vertex set $\{1, 2, 3, 4, 5, 6\}$, where $\{1, 2, 3\} \in C$ and $\{4, 5, 6\} \in Q$, is split.

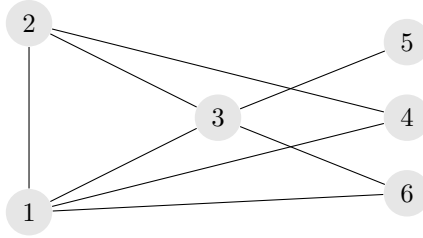


Figure 1: Example of a Split Graph

A variation of the Grone-Merris theorem that is still open for all graphs is,

Conjecture 4 (Brouwer's Conjecture). Let G be a graph with e edges and Laplacian eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq \lambda_n = 0$. Then, $\forall t \in \mathbb{N}$,

$$\sum_{i=1}^t \lambda_i \leq e + \binom{t+1}{2}$$

It is not hard to see that this conjecture holds for $t = 1$ and $n - 1$. It has also been shown for the second partial sum, and for special classes of graphs including trees, split graphs, regular graphs, co-graphs, and all graphs with at most 10 vertices.¹

Brouwer's conjecture is intimately connected to the Grone-Merris theorem. Despite the fact that it uses less information than the Grone-Merris theorem, it is shown in [M] that the t -th inequality of Brouwer's conjecture is sharper than that of the Grone-Merris inequality if and only if the graph is non-split.

These conjectures can be generalized to the case of abstract simplicial complexes, which is the topic of interest for this report.

Definition 5. An (abstract) **simplicial complex**, S , on a vertex set $\{1, 2, \dots, n\} = [n]$ is a collection of k -subsets of $[n]$, which are called the *faces* or *simplices*, that are closed under inclusion. i.e if F and F' are subsets of $[n]$, where $F \in S$ and $F' \subset F$, then $F' \in S$.

Given a face, F , we denote its cardinality by $|F|$, and its dimension by $|F| - 1$. The dimension of a simplicial complex is the maximum dimension of a face in it.

A face $F \in S$ is called a *facet* if $\dim(F) = \dim(S)$; and S is called *pure* if every face in S is contained in a facet of S . Throughout this report, we assume that all simplicial complexes are pure, undirected, and finite.

Definition 6. The f -vector of S is the sequence

$$f(S) := (f_{-1}(S), f_0(S), f_1(S), \dots)$$

where $f_i(S) := |S_i|$.

Given a simplicial complex, S , for all $i \in \mathbb{N}, i \geq -1$, we have *chain groups* $C_i(S)$ and maps between these chain groups.

$$0 \longrightarrow 0 \longrightarrow \dots C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

These C_i 's are vector spaces with the basis being the i^{th} dimensional faces of the simplicial complex. The ∂ 's, which we call the *boundary maps*, are defined \mathbb{R} -linearly by extending the following map on the basis elements.

$$\partial_i[v_0, \dots, v_i] = \sum_{j=0}^i (-1)^j [v_0, \dots, \hat{v}_j, \dots, v_i], \quad (1)$$

where $v_0 < v_1 < \dots < v_i$.

Here, $[v_0, \dots, v_i]$ are oriented chains, and so switching any of the v_j 's, where $1 \leq j \leq i$, switches the sign. The $i - 2$ -dimensional faces are each multiplied by a sign that depends on the vertex of $[v_0, \dots, v_i]$ that is removed. In general, for any simplicial complex S and a face $\{a_0, \dots, a_n\} \in S$ (where $a_0 \leq a_1 \leq \dots \leq a_n$), we define the *sign* of $\{a_0, \dots, a_n\} \setminus \{a_j\}$ as $(-1)^j$.

We can use (1) to write down these boundary maps matrices. For instance, the boundary map ∂_2 would have rows indexed by the 2-dimensional faces and the columns by the 1-dimensional faces.

¹These proofs can be found in [H] and [M]. We will include the discussion for co-graphs in Section 3.

Definition 7. Using the above boundary maps, we can define the following matrices:

$$L'(S) := \partial_{k-1} \partial_{k-1}^T$$

$$L''(S) := \partial_{k-1}^T \partial_{k-1}$$

Note that these definitions agree with those given for graphs. Just as in the graph case, $\partial_{k-1}^T \partial_{k-1}$ and $\partial_{k-1} \partial_{k-1}^T$ have the same non-zero eigenvalues counting multiplicity. Therefore, we may use either $L'(G)$ or $L''(G)$ when studying the Laplacian spectrum of simplicial complexes.

Before tackling the generalized form of the Grone-Merris theorem and Brouwer's conjecture, we note two special classes of simplicial complexes, which the reader might recognize as generalizations of those given in the graph section.

Definition 8. For a given k , **split simplicial complexes** are simplicial complexes whose vertices can be divided into a *clique* and a *co-clique*.

Just as in the case of graphs, none of the vertices in the co-clique form a k subset of $[n]$ (i.e. they are independent). The clique is a complete simplicial complex. We can have k -dimensional faces connecting vertices in the clique to those in the co-clique.²

An example is the simplicial complex $S = \{123, 124, 134, 234, 235, 237, 246\}$, for $k = 3$. Here, $\{1, 2, 3, 4\} \in C$ and $\{5, 6, 7\} \in Q$.

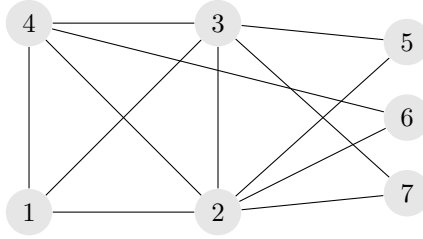


Figure 2: Example of a Split Simplicial Complex

Definition 9. A **shifted simplicial complex**, S , is a simplicial complex in which if $F \in S$, $v \subset F$ and $v' < v$, then $(F - v) \sqcup v' \in S$.

We are interested in finding a lower bound for the partial sum of Laplacian eigenvalues of simplicial complexes. As such, we will tackle a special case of the generalized Grone-Merris theorem (as conjectured by [DR]).

Conjecture 10. For a simplicial complex, S , with Laplacian spectrum $\lambda(S) = \{\lambda_1, \lambda_2, \dots\}$, the following majorization result holds:

$$\sum_{i=1}^t \lambda_i \leq \sum_{i=1}^t d_i^T$$

This conjecture has been shown for $k = 2$ (i.e. graphs in [Bai]), shifted simplicial complexes, and for $t = 1$ (see [DR]).

²We will discuss restrictions on this in Section 2.

In Section 2, we present progress made to show the conjecture holds for the second partial sum for $k = 3$. Just as in [Bai], we will reduce the conjecture to the case of split simplicial complexes. For a given c and q , we impose a partial ordering on all possible split simplicial complexes, and show that the conjecture can further be reduced to a subset of these split simplicial complexes.

In Section 3, we present four different possible generalizations of Brouwer's conjecture, and discuss the progress made to show that these bounds hold for different classes of simplicial complexes. In particular, we will show that one of these is satisfied for all shifted simplicial complexes.

2 Generalized Grone-Merris Conjecture

Let S be such a simplicial complex. Recall that we assume S to be finite, simple undirected and pure. In this section, we present attempts to prove the Grone-Merris conjecture for the second partial sum for 3-dimensional simplicial complexes. i.e.

$$\lambda_1(S) + \lambda_2(S) \leq d_1^t(S) + d_2^t(S)$$

Given any collection of 3 vertices $\{v_1, v_2, v_3\}$ of S , each of degree at least 2, we can add $\{v_1, v_2, v_3\}$ and its subsets to S to obtain a simplicial complex with the same value for $d_1^t + d_2^t$ and with eigenvalues at least as large as the original simplicial complex.

Using this observation, it follows that we can assume S is a split simplicial complex, *i.e.*, a simplicial complex whose vertices can be partitioned into two sets, the *clique* and *co-clique*, such that

- For any three vertices v_1, v_2, v_3 in the clique of S , the set $\{v_1, v_2, v_3\}$ is in S .
- All vertices in the co-clique have degree 1.

since forming this split simplicial complex can only increasing the left hand side of (2).

Note that we could disregard vertices not part of any 3-family, since they do not contribute to $\partial_2^t \partial_2$, and hence to the spectrum $\lambda(S)$. For example, in the case of a graph, we could disregard isolated vertices since they would simply add a row and column of all 0's.

It is easy to check (2) for $c = 1, 2$, since we are considering the case for $k = 3$. Thus, we assume that c is at least 3.

Define the clique subcomplex S_c of S as the subset of S consisting of all sets in S that do not contain any vertices in the coclique of S . Observe that S_c is indeed a subcomplex of S and is a complete simplicial complex.

Given the restrictions so far, faces in $\{v_1, v_2, v_3\} \in S \setminus S_c$ can have any of the following three forms:

- All three vertices v_1, v_2, v_3 are in the co-clique,
- Two vertices are in the co-clique and one is in the clique,
- Two vertices are in the clique and one is in the co-clique.

Type (a): Suppose S contains a face of type (a). Then, S is the union of S' and the power set of $\{v_1, v_2, v_3\}$, where S' consists of all sets in S that do not contain any of v_1, v_2, v_3 . The Laplacian $\partial_2^t \partial_2$ of S is a block diagonal matrix with blocks L and M , where L is the corresponding Laplacian for S' and M is the 1×1 matrix with entry 3. The spectrum of S is obtained by appending a 3 to the spectrum of S' ; also, adding the power set of $\{v_1, v_2, v_3\}$ to S increases $d_1^t(S) + d_2^t(S)$ by 3. Thus (2) holds for S if it holds for S' , *i.e.*, we can suppose S contains no 3-family $\{v_1, v_2, v_3\}$ with v_1, v_2, v_3 in the coclique.

Type (b): Now suppose S contains a face of type (b). We illustrate one possibility below:

In this case, if S' consists of all sets in S that do not contain any of v_1, v_2 , the Laplacian $\partial_2^t \partial_2$ of S is a block diagonal matrix with blocks L and M , where L is the corresponding Laplacian for S' and M is the 1×1 matrix with entry 3. The spectrum of S is obtained by appending a

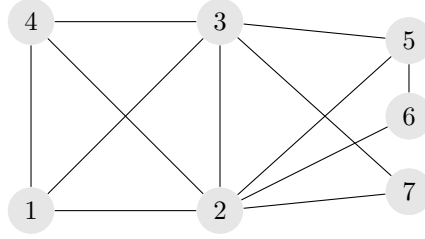


Figure 3: Split Simplicial Complex of Type (b)

3 to the spectrum of S' . Since v_3 is in the clique, S' contains at least one 3-family containing v_3 , and so its largest eigenvalue is at least 3. It follows that $\lambda_1(S) + \lambda_2(S) = \lambda_1(S') + \lambda_2(S')$ unless $\lambda_2(S) = 3$.

Suppose (2) holds for S' . If $\lambda_1(S) + \lambda_2(S) = \lambda_1(S') + \lambda_2(S')$, then (2) clearly holds for S' . Otherwise, $\lambda_2(S) = 3$, and (2) holds for S because

- $\lambda_1(S) \leq d_1^t(S)$ (see [DR]), and
- $\lambda_2(S) = 3 \leq c = d_2^t(S)$.

Thus we can suppose S contains no 3-family $\{v_1, v_2, v_3\}$ with v_1, v_2 in the coclique and v_3 in the clique, and we have narrowed down the definition of a split simplicial complex to that of type (c). (Note that the example given in the introduction is a type (c) split simplicial complex.)

For simplicity, we assume that the vertices of the clique are labeled, $\{1, 2, \dots, c\}$, and those of the co-clique are labelled, $\{c + 1, c + 2, \dots, n\}$.

Definition 11. The **(simplicial) multigraph** $M(S)$ of S is constructed as follows:

- The vertices of $M(S)$ are the vertices of S .
- For each pair of vertices $v, w \in M(S)$, the number of edges in $M(S)$ joining v, w equals the number of vertices u in the coclique of S for which $\{u, v, w\}$ is in $M(S)$.

For example, the multigraph corresponding to Figure 2 is,

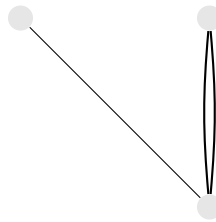


Figure 4: Multigraph of the Split Simplicial Complex where $c = 4$ and $q = 3$

This multigraph encodes all the information we need to form the simplicial complex. Thus, from here onward, we will denote the split simplicial complexes being considered by their multigraphs.

After fixing the clique and co-clique size, we can put a partial ordering of all the possible simplicial eigenvalues using the first and second partial sum of their Laplacian eigenvalues, which we can show using these multigraphs.³ For example, if we fix $c = 4$ and $q = 4$, we get nine different split simplicial complexes ordered as,

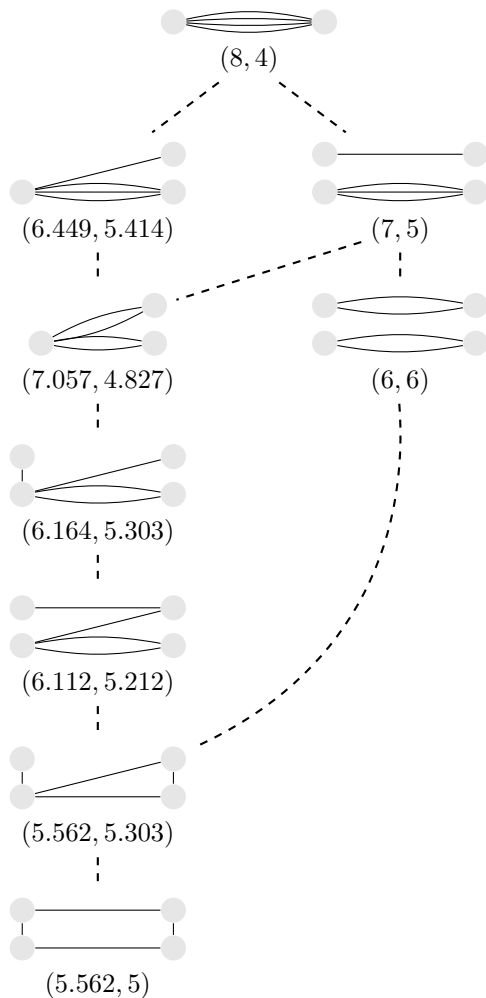


Figure 5: Poset of Split Simplicial Complexes for $c = 4$ and $q = 4$

The simplicial incidence matrix ∂_S of S has the form

$$\begin{pmatrix} \partial & X \\ 0 & I \\ 0 & -I \end{pmatrix}$$

³See Appendix A for more examples.

where ∂ denotes the simplicial incidence matrix of S_c .

Hence the Laplacian of S is

$$\partial_S^t \partial_S = \begin{pmatrix} \partial^t \partial & \partial^t X \\ X^t \partial & X^t X + 2I \end{pmatrix} \quad (2)$$

The rows and columns of $\partial_S^t \partial_S$ are indexed by the 3-faces of S , with the 3-faces of S_c indexed first. (The block $\partial^t \partial$ is the Laplacian of the subcomplex S_c ; both its rows and its columns are indexed by the 3-faces of S_c .)

Next we will show that all other eigenvectors are contained in a collection of subspaces V_P corresponding to the connected components P of the multigraph $M(S)$.

Notation 12. Denote by $\{p_1, q_1\}, \dots, \{p_d, q_d\}$ the pairs of vertices joined by one or more edges in the connected component P_i . For any face $f \in S$ and i between 1 and d , let $k_i(f)$ equal the sign of $\{p_i, q_i\}$ in f if f contains the vertices p_i, q_i , and zero otherwise.

Definition 13. *Defining the subspaces V_P .* Let v be an arbitrary vector with the same dimension as the columns (and rows) of $\partial_S^t \partial_S$. We index the entries of v by the faces of S (in the same order as we indexed the rows and columns of $\partial_S^t \partial_S$). Say that v is in the subspace V_P if, for some real numbers $x_1, \dots, x_d, y_1, \dots, y_d$, the vector v has f -entry $\sum_{i=1}^d k_i(f)x_i$ for $f \in S_c$ and f -entry $\sum_{i=1}^d k_i(f)y_i$ for $f \in S \setminus S_c$. (Note that $k_i(f)$ equals either 0 or 1 for $f \in S \setminus S_c$, since the vertices of the clique are labeled with the lowest indices.)

To state the next result, we need to introduce more notation:

Notation 14. For each $1 \leq i, j \leq n$, with i, j distinct, let m_i be the number of vertices in the coclique that share a face with the vertices p_i, q_i , and let m_{ij} be defined as follows:

- If $\{p_i, q_i\}$ and $\{p_j, q_j\}$ are disjoint, then $m_{ij} = 0$.
- If $\{p_i, q_i\}$ and $\{p_j, q_j\}$ intersect, then m_{ij} equals the product of the signs of $\{p_i, q_i\}$ and $\{p_j, q_j\}$ in the (unique) face

Observe that $m_{ij} = m_{ji}$.

We have the following result:

Theorem 15. *There exists a 1-1 correspondence between eigenvectors of $\partial_S^t \partial_S$ contained in V_P and eigenvectors of the matrix*

$$L_P = \begin{pmatrix} cI & P \\ (c-2)I + Q & P + 2I \end{pmatrix},$$

where I is the $d \times d$ identity matrix, P is the diagonal matrix with i -th diagonal entry m_i , and Q is the matrix with i -th diagonal entry m_i and ij -entry m_{ij} . Corresponding eigenvectors have the same eigenvalues.

Proof. Let $f = \{\alpha, \beta, \gamma\} \in S_c$, with $\alpha \leq \beta \leq \gamma$. Let $g \in S$. There are three cases to consider.

- $\mathbf{g \in S_c}$. The f -row of $\partial^t \partial$ (and therefore the f -row of the Laplacian $\partial_S^t \partial_S$ of S) has g -entry the dot product of the column vectors of ∂ corresponding to the faces f and g . The dot product is 3 if $g = f$ and zero if g, f share at most one vertex; if g, f shares exactly two vertices, the dot product is the product of the signs in g, f of the common edge.

- $\mathbf{g} = \{\mathbf{p}_i, \mathbf{q}_i, \mathbf{c}\} \in \mathbf{S} \setminus \mathbf{S}_c$ for some $1 \leq i \leq d$. The g -column of $\partial^t X$ is just the row of ∂ corresponding to $\{p_i, q_i\}$; hence the f -row of the Laplacian $\partial_S^t \partial_S$ has g -entry $k_i(f)$.
- $\mathbf{g} \in \mathbf{S} \setminus \mathbf{S}_c$ but does not contain the pair $\{\mathbf{p}_i, \mathbf{q}_i\}$ for any i . The f -row g -column entry of the Laplacian $\partial_S^t \partial_S$ will not affect future calculations, since vectors in V_P have g -entry zero.

Thus, for a vector $v \in V_P$, its image $\partial_S \partial_S^t v$ under the Laplacian map has f -entry

$$\begin{aligned}
& 3 \sum_i c_i(f) x_i + \sum_{g=\{\alpha, \beta, \delta\}, \delta \neq \gamma \in S_c} (\text{sign of } \{\alpha, \beta\} \text{ in } g) \sum_i k_i(g) x_i \\
& - \sum_{g=\{\alpha, \gamma, \delta\}, \delta \neq \beta \in S_c} (\text{sign of } \{\alpha, \gamma\} \text{ in } g) \sum_i k_i(g) x_i \\
& + \sum_{g=\{\beta, \gamma, \delta\}, \delta \neq \alpha \in S_c} (\text{sign of } \{\beta, \gamma\} \text{ in } g) \sum_i k_i(g) x_i \\
& + \sum_{i=1}^d m_i k_i(f) y_i,
\end{aligned}$$

where m_i is the number of vertices in the coclique that share a face with the vertices p_i, q_i . Let $1 \leq i \leq d$ be arbitrary.

- If f contains neither p_i nor q_i , then $k_i(g) = 0$ for all faces g sharing at least two vertices with f ; so x_i must have coefficient zero.
- Suppose that f contains p_i but not q_i . If $p_i = \beta$, then x_i has coefficient

$$\begin{aligned}
& (\text{sign of } \{\alpha, \beta\} \text{ in } \{\alpha, \beta, q_i\}) k_i(\{\alpha, \beta, q_i\}) + (\text{sign of } \{\beta, \gamma\} \text{ in } \{\beta, \gamma, q_i\}) k_i(\{\beta, \gamma, q_i\}) \\
& = (\text{sign of } \{\alpha, \beta\} \text{ in } \{\alpha, \beta, q_i\}) \cdot (\text{sign of } \{\beta, q_i\} \text{ in } \{\alpha, \beta, q_i\}) \\
& \quad + (\text{sign of } \{\beta, \gamma\} \text{ in } \{\beta, \gamma, q_i\}) \cdot (\text{sign of } \{\beta, q_i\} \text{ in } \{\beta, \gamma, q_i\}) \\
& = -(\text{sign of } \{\alpha, q_i\} \text{ in } \{\alpha, \beta, q_i\}) - (\text{sign of } \{\gamma, q_i\} \text{ in } \{\beta, \gamma, q_i\})
\end{aligned}$$

If $\beta < q_i$ then $\{\alpha, q_i\}$ has sign -1 in $\{\alpha, \beta, q_i\}$ and $\{\gamma, q_i\}$ has sign 1 in $\{\beta, \gamma, q_i\}$; if $\beta > q_i$ then $\{\alpha, q_i\}$ has sign 1 in $\{\alpha, \beta, q_i\}$ and $\{\gamma, q_i\}$ has sign -1 in $\{\beta, \gamma, q_i\}$. Hence x_i has coefficient zero. Similar reasoning shows x_i has coefficient zero when p_i equals either α or γ .

- Finally, suppose that f contains both p_i and q_i . Then x_i has coefficient

$$3k_i(f) + \sum_{g=\{p_i, q_i, \delta\}, \delta \in S_c \setminus f} (\text{sign of } \{p_i, q_i\} \text{ in } f) (\text{sign of } \{p_i, q_i\} \text{ in } g) k_i(g) x_i$$

or $ck_i(f)$.

We deduce that $\partial_S \partial_S^t v$ has f -entry

$$\sum_{i=1}^d ck_i(f) x_i + \sum_{i=1}^d m_i k_i(f) y_i \tag{3}$$

Now suppose $f = \{\alpha, \beta, \gamma\}$ with γ in the coclique. Then, for $g \in S_c$, the g -entry of the f -row of $\partial_S \partial_S^t$ is the sign of $\{\alpha, \beta\}$ in g . For $g \in S \setminus S_c$, the g -entry of the f -row of $\partial_S \partial_S^t$ is 3 if $g = f$, 1 if $g \neq f$ but $\{\alpha, \beta\} \in g$, and 0 otherwise. Thus, for a vector $v \in V_P$, its image $\partial_S \partial_S^t v$ under the Laplacian map has f -entry

$$\sum_{g \in S_c} (\text{sign of } \{\alpha, \beta\} \text{ in } g) \left(\sum_i k_i(g) x_i \right) + (m_i + 2) y_i \quad (4)$$

If $\{\alpha, \beta\}$ is not in P , but in a different component of the multigraph $M(S)$, then $k_i(g)$ and the sign of $\{\alpha, \beta\}$ in g cannot both be nonzero. If $\{\alpha, \beta\}$ is in P , then the sign of $\{\alpha, \beta\}$ in g is just $\sum_j k_j(f) k_j(g)$. Therefore (4) reduces to

$$\sum_{g \in S_c} \sum_j k_j(f) k_j(g) \left(\sum_i k_i(g) x_i \right) + (m_i + 2) y_i \quad (5)$$

Fix $1 \leq j \leq d$. Let g_i be the element of S_c containing both $\{p_i, q_i\}$ and $\{p_j, q_j\}$ for each $1 \leq i \leq d$ for which such a g_i exists. (If no such g_i exists for some i , let g_i be an arbitrary face; it does not matter.) Observe that the g_i are not necessarily pairwise distinct. The expression (5) is just

$$\sum_j (c - 2) k_j(f) x_j + \sum_{i \neq j} \sum_j k_j(f) k_j(g_i) k_i(g_i) x_i + (m_i + 2) y_i$$

or

$$\sum_i \sum_j k_j(f) m_{ij} x_i + (m_i + 2) y_i, \quad (6)$$

where m_{ij} is defined as follows:

- If $\{p_i, q_i\}$ and $\{p_j, q_j\}$ are disjoint, then $m_{ij} = 0$.
- If $\{p_i, q_i\}$ and $\{p_j, q_j\}$ intersect, then m_{ij} equals the product of the signs of $\{p_i, q_i\}$ and $\{p_j, q_j\}$ in the (unique) face

Suppose that the vector $v' = (x_1 \ x_2 \ \cdots \ x_d \ y_1 \ y_2 \ \cdots \ y_d)$ is an eigenvector of the matrix ... with eigenvalue λ . Then

$$\lambda x_i = c x_i + m_i y_i$$

and

$$\lambda y_i = (c - 2) x_i + \sum_{j \neq i} m_{ij} x_j + (m_i + 2) y_i,$$

and the expressions (3) and (6) become $\sum_i \lambda k_i(f) x_i$ and $\sum_i \lambda k_i(f) y_i$, respectively. We conclude that, if v' is an eigenvector of the matrix L_P , then v is an eigenvector of the matrix L_P . Since the matrix L_P is just the restriction of the Laplacian $\partial_S \partial_S^t$ to the subspace V_P the matrix L_P has a basis of eigenvectors. These eigenvectors correspond to $2d$ eigenvectors of $\partial_S \partial_S^t$ in V_P . Since V_P has dimension $2d$, we see that *every* eigenvector of $\partial_S \partial_S^t$ in V_P corresponds to an eigenvector of the matrix L_P with the same eigenvalue. \square

3 Generalized Brouwer's Conjecture

First, we discuss the proofs of Brouwer's conjecture for some special classes of graphs, which will come in handy in the following sub-sections. We will then detail different possible generalizations of Brouwer's conjecture to the case of simplicial complexes.

3.1 Brouwer's Conjecture for Graphs

First, we note two ways to obtain new graphs from old ones in such a way that the Laplacian eigenvalues of the new graph can be easily extracted from the old ones.

Given two graphs, G_1 and G_2 , we can obtain their **disjoint union**, $G_1 \sqcup G_2$, by simply taking the union of their vertices and edges. The Laplacian spectrum of this new graph is simply the direct sum of the Laplacian spectra of G_1 and G_2 .

The **complement** of G , G^c , is the graph that has two vertices being adjacent if and only if they were not adjacent in G . If the Laplacian spectrum of G in non-increasing order is, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, then that of G^c is $0 \leq n - \lambda_{n-1} \leq \dots \leq n - \lambda_1$ in non-decreasing order, which we relabel as $\lambda_1^c \leq \dots \leq \lambda_{n-1}^c \leq \lambda_n^c = 0$.

3.1.1 Co-graphs

Definition 16. A **cograph** is a graph that does not contain P_4 , the path graph on four vertices, as a subgraph.

There is an inductive way of defining co-graphs, using the following rules.

- a single vertex is a cograph
- if G is a cograph, then so is G^c
- if G_1 and G_2 are cographs, then so is $G_1 \sqcup G_2$

We note the following lemmata, whose proofs can be found in [M], or can easily be reproduced by noting how the Laplacian spectrum changes when considering the complement graph and the disjoint union of two graphs.

Lemma 17. *A graph G satisfies Brouwer's conjecture if and only if G^c also satisfies Brouwer's conjecture.*

Lemma 18. *If two graphs, G_1 and G_2 satisfy Brouwer's conjecture, so does $G_1 \sqcup G_2$.*

Lemma 19. *Cographs satisfy Brouwer's conjecture.*

Proof. This easily follows from the fact that the single vertex satisfies Brouwer's conjecture, Lemma 17 and Lemma 18. □

Definition 20. A **threshold graph** is a special type of co-graph that can be defined inductively as follows.

- Isolated points are threshold graphs.

- A graph on n vertices is threshold if some subgraph of $n - 1$ vertices is threshold and the n -th remaining vertex is either an isolated vertex or a cone vertex (*i.e.*, a vertex connected to all other vertices by an edge).

Threshold graphs are a special type of co-graphs. Two special cases of threshold graphs are the *complete graph* and the *star graph*. The former is obtained by adding a cone vertex at each stage and the latter by adding $n - 1$ empty vertices and a final cone vertex. Threshold graphs clearly satisfy Brouwer's conjecture as they are co-graphs. We can, in fact, deduce a stronger result for threshold graphs.

Lemma 21. *Given a threshold graph, G , Brouwer's conjecture admits an equality for $t = c$, where c is the number of cone vertices.*

Proof. We may first assume that the last vertex added is a cone vertex, because if it was an isolated vertex, then it simply appends a 0 at the end of the spectrum, which does not affect our result.

We prove the above result inductively. The base case is the star graph, which has Laplacian eigenvalue n with multiplicity one and eigenvalue 1 with multiplicity $n - 2$. For $t = 1$, we get $n = (n - 1) + \binom{2}{2} = n$.

For the inductive step, we note the following equality from [DR].

$$\sum_{i=1}^t \lambda_i = \sum_{i=1}^t d_i^T$$

Here, d^T —the conjugate degree sequence—denotes for each d_i the number of vertices that have degree at least i .

Assume that Brouwer's conjecture holds for some threshold graph with n vertices, of which c are cone vertices.

$$\sum_{i=1}^c d_i^T = e + \binom{c+1}{2}$$

We consider what happens when we add a cone vertex. We want to show that,

$$\begin{aligned} \sum_{i=1}^{c+1} d_i^T &= e + n + \binom{c+2}{2} \\ \sum_{i=1}^c d_i^T + n + c + 1 &= e + \binom{c+1}{2} + n + c + 1, \end{aligned}$$

The last step follows from the fact that, adding a cone vertex to a graph that has n vertices increases the number of edges by n and increases the degree of the n original vertices by 1 each. The cone vertex will have degree n . \square

Note that, in addition to getting an equality for the first partial sum for star graphs, we also get an equality for $n - 1$ for complete graphs.

3.1.2 Trees

We recall that a **tree** is an undirected graph in which any two vertices are connected by exactly one simple path. In [H], it is shown that an even stronger bound holds for the case of trees.

Lemma 22. *Let T be a tree with n vertices, then*

$$\sum_{i=1}^t \lambda_i \leq e + 2t - 1$$

where $1 \leq t \leq n$.

Proof. We will show this by induction on the number of vertices. If T is a star graph, then we have observed that the t^{th} partial sum of the eigenvalues is $n + t - 1$, and we're done.

If T is not a tree, then there is an edge whose removal leaves a forest F consisting of two trees, T_1 and T_2 , both of which have at least one edge. Now, suppose that t_i of the t largest eigenvalues of F come from the Laplacian spectrum of T_i for $i = 1, 2$. We note that $t_1 + t_2 = t$. We now have one of two possibilities.

Case 1: One of the t_i 's, say t_2 , is 0. Then,

$$\begin{aligned} \sum_{i=1}^t \lambda_i(T) &= \sum_{i=1}^t \lambda_i(F \sqcup K_2), \text{ where } K_2 \text{ is an edge} \\ &\leq \sum_{i=1}^t \lambda_i(T_1) + \sum_{i=1}^t \lambda_i(K_2) \\ &\leq (e(T_1) + 2t_1 - 1) + 2 \\ &\leq n + 2t - 2 \\ &= e(T) + 2t - 1 \end{aligned}$$

Case 2: Otherwise, neither of the t_i 's is 0. Then, we have that

$$\begin{aligned} \sum_{i=1}^t \lambda_i(T) &= \sum_{i=1}^t \lambda_i(T_1 \sqcup T_2 \sqcup K_2) \\ &\leq \sum_{i=1}^t \lambda_i(T_1) + \sum_{i=1}^t \lambda_i(T_2) + \sum_{i=1}^t \lambda_i(K_2) \\ &= e(T_1) + 2t_1 - 1 + e(T_2) + 2t_2 - 1 + 2 \\ &= e(T) + 2t - 1 \end{aligned}$$

as desired. □

3.2 Brouwer's Conjecture for Simplicial Complexes

Denote the complete simplicial complex, which can also be regarded as the complete k -family on $[n]$ by $\binom{[n]}{s}$. The complete graph has $\binom{n}{k}$ faces of dimension $\binom{n-1}{k-1}$. The star simplicial complex on $[n]$ is the k -family, $\{\{1, 2, \dots, k-1, k\}, \{1, 2, \dots, k-1, k+1\}, \dots, \{1, 2, \dots, k-1, n\}\}$. Therefore, the star graph has $k(n - k + 1)$ faces of dimension k .

The lexicographic ordering of the faces in simplicial complexes allows us to see a special property of shifted simplicial complexes called shelling.

Definition 23. Shelling is a way to build a simplicial complex one facet at a time, so that each facet added includes a new minimal face to the simplicial complex. The lexicographic order of the faces induces a canonical shelling order.

One can check for shiftedness of a simplicial complex, S , by taking the complete simplicial complex on the same vertices and looking at the partially ordered set of its faces in lexicographic order, P . Take the sub-poset induced by S , P_S , and look at its maximal elements, m_i . If $\forall p_i \in P$, $p_i \leq m_i$, we have that $p_i \in P_S$, then S is shifted.

Just as in the case of graphs, there are operations that allow us to construct new simplicial complexes from old ones such that the Laplacian eigenvalues change in a predictable manner. We note the following equalities from [DR]:

Given two simplicial complexes, S_1 and S_2 , the Laplacian spectra of $S_1 \sqcup S_2$ is simply the direct sum of the Laplacian spectra of S_1 and S_2 .

The complement of S , which we will denote by S^c , is

$$S^c := \binom{[n]}{k} \setminus S = \{F \subseteq [n] : |F| = k, F \notin S\}$$

and the star of S , which we will denote by S^* , is

$$S^* := \{[n] \setminus S : F \in S\}$$

The complement of S is a k -family while S^* is an $(n - k)$ -family. Moreover, both the star and the complementation operation are involutive ($(S^c)^c = (S^*)^* = S$) and they also commute with one another ($S^{c*} = S^{*c}$).

These final two operations affect the Laplacian spectra in the following way. If S has Laplacian spectrum,

$$\lambda = (\lambda_1, \lambda_2, \dots)$$

then the Laplacian spectrum of S^* is

$$\lambda_i^* = n - \lambda_{|S|+1-i}$$

and the Laplacian spectrum of S^c is

$$\lambda_i^c = n - \lambda_{\binom{n-1}{k-1}+1-i}$$

We can finally present the different possible generalizations of Brouwer's conjecture. For each of the possible forms, we note the progress that has been made to show that the bounds holds for special classes of simplicial complexes, and also under the different operations on simplicial complexes. No counter examples have been found from the SAGE computations for any of these bounds.

3.2.1 Possible Generalization I

$$\sum_{i=1}^{\binom{t+k-2}{k-1}} \lambda_i \leq f_{k-1} + (k-1) \binom{t+k-1}{k}$$

When considering the graph case, we let $k = 2$, and the generalized form reduces to Brouwer's conjecture that we saw earlier in this report. This generalized bound preserves some of the properties that we've already seen.

Lemma 24. *The star simplicial complex on $[n]$ for all $k \geq 2$ satisfies generalized Brouwer's conjecture I, and with equality for $t = 1$.*

Proof. The star simplicial complex, S , has Laplacian spectra of the following form:

$$\lambda(S) = \{n\}^1, \{k\}^{n-k}$$

We prove the statement inductively. The base case clearly gives an equality. Now, assume the generalized bound holds for some partial sum $s \geq 1$. i.e

$$n + sk = (n - k + 1) + (k - 1) \binom{s + k - 1}{k}$$

We want to show that

$$\begin{aligned} n + (s + 1)k &\leq (n - k + 1) + (k - 1) \binom{s + k}{k} \\ n + sk + k &\leq (n - k + 1) + (k - 1) \binom{s + k - 1}{k - 1} + (k - 1) \binom{s + k - 1}{k} \\ k &\leq (k - 1) \binom{s + k - 1}{k - 1} \end{aligned}$$

which is clearly true. □

Lemma 25. *The complete simplicial complex satisfies generalized Brouwer's conjecture I, and with equality for $t = n - k + 1$*

Proof. Note that the complete simplicial complex, S , has Laplacian spectra of the form

$$\lambda(S) = n \binom{n-1}{k-1}$$

For $t = n - k + 1$, generalized Brouwer's conjecture states that

$$\begin{aligned} n \binom{n-1}{k-1} &= \binom{n}{k} + (k-1) \binom{n}{k} \\ &= \binom{n}{k} + (k) \binom{n}{k} - \binom{n}{k} \\ &= n \binom{n-1}{k-1} \end{aligned}$$

Note that, when considering the graph case ($k = 2$), the lemma reduces to the analogue shown in Section 2. \square

Of all the possible generalizations for Brouwer's conjecture considered, this gives the tightest bound. However, it is also one of the most difficult to work with to show the desired properties that we noted in the graph case.

3.2.2 Possible Generalization II

A close imitation the previous bound, another generalization of Brouwer's conjecture that we consider is the following:

$$\sum_{i=1}^t \lambda_i \leq f_{k-1} + (k-1) \binom{t+k-1}{k}$$

This bound clearly gives an equality for $t = 1$, for the star simplicial complex. However, since the left hand side for this form grows more slowly than that in the previous section, while the bound is still satisfied, we do not get an equality $t = n - k + 1$ for complete simplicial complexes.

We also present partial progress made to show that this bound holds under complementation and star operation.

Conjecture 26. A simplicial complex, S , satisfies generalized Brouwer's Conjecture II if and only if S^c satisfies generalized Brouwer's conjecture II.

Proof. Here is partial progress made in an attempt to prove this conjecture.

We assume that generalized Brouwer's conjecture is satisfied for some S . We want to show that it also holds for S^c .

$$\begin{aligned} \sum_{i=1}^s \lambda_i^c &\leq f_{k-1}^c + (k-1) \binom{s+k-1}{k} \\ sn - \sum_{i=1}^s \lambda_{\binom{n-1}{k-1}+1-i} &\leq \binom{n}{k} - f_{k-1} + (k-1) \binom{s+k-1}{k} \\ sn - kf_{k-1} + \sum_{i=1}^{\binom{n-1}{k-1}-s} \lambda_i &\leq \binom{n}{k} - f_{k-1} + (k-1) \binom{s+k-1}{k} \\ \sum_{i=1}^{\binom{n-1}{k-1}-s} \lambda_i &\leq \binom{n}{k} + (k-1)f_{k-1} + (k-1) \binom{s+k-1}{k} - sn \end{aligned}$$

\square

Conjecture 27. A simplicial complex, S , satisfies generalized Brouwer's Conjecture II if and only if S^* satisfies generalized Brouwer's conjecture II.

Proof. Here is partial progress made in an attempt to prove this conjecture.

We assume that generalized Brouwer's conjecture is satisfied for some S , and we want to show that it holds for S^* .

$$\begin{aligned}
\sum_{i=1}^s \lambda_i^* &\leq f_{k-1}^* + (n-k-1) \binom{s+n-k-1}{n-k} \\
sn - \sum_{i=1}^s \lambda_{|k|+1-i} &\leq f_{k-1} + (n-k-1) \binom{s+n-k-1}{n-k} \\
sn - kf_{k-1} + \sum_{i=1}^{|k|-s} \lambda_i &\leq f_{k-1} + (n-k-1) \binom{s+n-k-1}{n-k} \\
\sum_{i=1}^{|k|-s} \lambda_i &\leq (k+1)f_{k-1} + (n-k-1) \binom{s+n-k-1}{n-k}
\end{aligned}$$

□

3.2.3 Possible Generalization III

Another bound which grows less quickly than the right hand side of either of the above bounds is:

$$\sum_{i=1}^t \lambda_i \leq f_{k-1} + \binom{t+k-1}{2}$$

Lemma 28. *This generalized Brouwer's conjecture III holds for the star simplicial complex.*

Proof. We recall that the star simplicial complex has one eigenvalue n with multiplicity 1, and another eigenvalue k with multiplicity $n-k$.

For $t = 1$, we have that

$$\begin{aligned}
n &\leq (n-k+1) + \binom{k}{2} \\
k-1 &\leq \binom{k}{2}
\end{aligned}$$

This last line holds for all $k \geq 2$. For latter inequalities, we see that both sides grow by k for all the non-zero eigenvalues. □

Lemma 29. *This generalized Brouwer's conjecture III holds for the complete simplicial complex.*

Proof. We recall that the star simplicial complex has one eigenvalue n with multiplicity $\binom{n-1}{k-1}$.

We will show this by induction. For $t = 1$, we have $n \leq \binom{n}{k} + 1$. For later inequalities, we note that the right hand side grows by

$$(s+1)n \leq \binom{n}{k} + \binom{s+k}{k}$$

For $t = \binom{n-1}{k-1}$, we have that

$$\begin{aligned} n \binom{n-1}{k-1} &\leq \binom{n}{k} + \binom{k}{2} \\ k-1 &\leq \binom{k}{2} \end{aligned}$$

This last line holds for all $k \geq 2$. For latter inequalities, we see that both sides grow by k for all the non-zero eigenvalues. \square

While a simpler form than the previous generalized bounds, this bound is not any easier to work with when trying to show that it holds under complementation and star operation.

3.2.4 Possible Generalization IV

$$\sum_{i=1}^t \lambda_i \leq (k-1)f_{k-1} + \binom{t+k-1}{k}$$

For this bound does not given an equality for the complete simplicial complex for $t = \binom{n-1}{k-1}$ (take the the complete simplicial complex on 4 vertices for $k=3$, for example) or the star simplicial complex for $t = 1$ (the star simplicial complex on 4 vertices for $k=3$ is a counter example). However, the bound holds for all shifted simplicial complexes.

Theorem 30. *Shifted simplicial complexes satisfy Brouwer's conjecture for simplicial complexes.*

Proof. First, recall that shifted simplicial complexes are shellable. The lexicographic order on the faces gives us a canonical shelling. Moreover, for shifted simplicial complexes, we have that,

$$\sum_{i=1}^t \lambda_i = \sum_{i=1}^t d_i^T$$

where d_i^T is the conjugate degree transpose.

Now, assume that Brouwer's conjecture for simplicial complexes is not satisfied for some t and k . We choose a minimal counterexample, S , for which this bound is not satisfied. That is, while S does not satisfy Brouwer's conjecture, if you remove any of its maximal faces, F , then the new simplicial complex, $S \setminus F$ does. (Note that F must be maximal, because if it was not, then our new simplicial complex would no longer be shifted.) We have two cases:

Case 1: S has $\geq t+k$ vertices

There exists a maximal face $F = \{v_1, v_2, \dots, v_k\}$ where $v_1 \geq v_2 \geq \dots \geq v_k$, such that v_k is at least $t+k$. To see this, choose any face such that $v_k \geq t+k$. If it is maximal, then we are done. If it is not, then we can increase the indices by going up the partially ordered set until we reach a maximal element. Relabel this face $F = \{v_1, v_2, \dots, v_k\}$. We will still have that $v_k \geq t+k$ since going up the partially ordered set can only increase the indices of the vertices.

Since S is shifted, it follows that it contains $F' = \{v_1, v_2, \dots, v_{k-1}, v_k'\}$ for all $v_k' < v_k$, such that the cardinality of F' is still k . Thus, each of $\{v_1, v_2, \dots, v_{k-1}\}$ has degree at least $t+1$.

We delete F from our simplicial complex, to get $S \setminus F$. This reduces the degrees of $\{v_1, v_2, \dots, v_{k-1}\}$ by one each. Thus, when we delete F the left hand side of the bound goes down by less than $k < 1$, while the right hand side goes down by exactly $k - 1$. Thus, if $S \setminus F$ had satisfied the bound, then S should as well, and we reach a contradiction.

Case 2: S has $< t + k$ vertices

$$\begin{aligned} f_{k-1} &\leq \binom{t+k-1}{k} \\ (k-1)f_{k-1} + f_{k-1} &\leq \binom{t+k-1}{k} + (k-1)f_{k-1} \\ kf_{k-1} &\leq \binom{t+k-1}{k} + (k-1)f_{k-1} \\ \sum \lambda &\leq \binom{t+k-1}{k} + (k-1)f_{k-1} \end{aligned}$$

□

In [H], it is shown that a sharper bound than that given by Brouwer's conjecture holds for the case of trees. We note a similar result here for the case of simplicial trees. The notion of simplicial trees that we are using in this report is that given in [F].

Definition 31. A facet, F , of a simplicial complex is called a **leaf** if either F is the only facet of the simplicial complex, T , or for some facet $G \in T \setminus (F)$ we have that $F \cap T \setminus (F) \subseteq G$.

Definition 32. A connected simplicial complex, T , is a **tree** if every nonempty subcollection of T , that is a subcomplex of T , whose facets are also facets of T has a leaf.

The following is a generalization of the graph case, which gives a stronger bound than the generalized Brouwer's conjecture for simplicial complexes.

Theorem 33. *Let T be a simplicial tree on n vertices. Then we have that,*

$$\sum_{i=1}^t \lambda_i \leq (k-1)f_{k-1} + kt - k + 1$$

for all $1 \leq t \leq n$.

Proof. We first deal with the isolated case of a simplicial star, where the equality holds by noting that t is at least 1 and $n \leq k$.

$$\begin{aligned} n + (t-1)(k-1) &\leq (k-1)(n-k+1) + kt - k + 1 \\ n(2-k) &\leq k(2-k) \\ n &\geq k \end{aligned}$$

If T is not a simplicial star, then there exists a $k-1$ dimensional face whose removal results in a forest F with two components T_1 and T_2 that are not connected by such a face. Say that t of the largest eigenvalues of F come from t_1 and t_2 , i.e. $t = t_1 + t_2$. We have one of two cases:

Case 1: One of the t_i 's, say t_2 , is 0, i.e $t_1 = t$.

$$\begin{aligned}
\sum_{i=1}^t \lambda_i(T) &= \sum_{i=1}^t \lambda_i(F \sqcup K_k), \text{ where } K_k \text{ is one } k \text{ face} \\
&\leq \sum_{i=1}^t \lambda_i(T_1) + \sum_{i=1}^t \lambda_i(K_k) \\
&\leq (k-1)f_{k-1}(T_1) + kt_1 - k + 1 + k \\
&\leq (k-1)f_{k-1}(T) + kt - 1
\end{aligned}$$

as desired.

Case 2: Otherwise, neither of the t_i 's is 0. Then, we have that

$$\begin{aligned}
\sum_{i=1}^t \lambda_i(T) &= \sum_{i=1}^t \lambda_i(T_1 \sqcup T_2 \sqcup K_k) \\
&\leq \sum_{i=1}^t \lambda_i(T_1) + \sum_{i=1}^t \lambda_i(T_2) + \sum_{i=1}^t \lambda_i(K_k) \\
&\leq (k-1)f_{k-1}(T_1) + (k-1)f_{k-1}(T_2) + kt_1 + kt_2 - k + 2 \\
&\leq (k-1)f_{k-1}(T) + kt - k + 2
\end{aligned}$$

which also holds.

□

A Partially Ordered Sets of Multigraphs

In this section, we present some examples of partially ordered sets of multigraphs that we've worked out. Recall that an example was given in the Grone-Merris section for clique size 4, and coclique size 4.

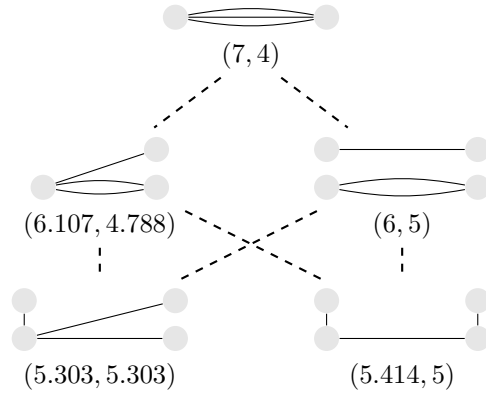


Figure 6: Poset of Split Simplicial Complexes for $c = 4$ and $q = 3$

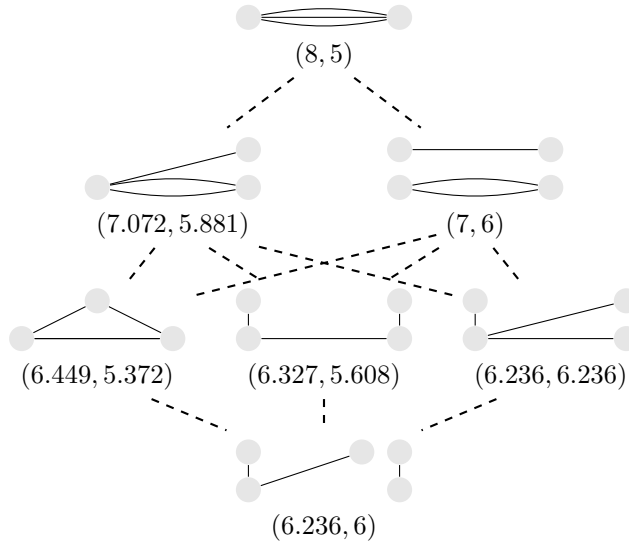


Figure 7: Poset of Split Simplicial Complexes for $c = 5$ and $q = 3$

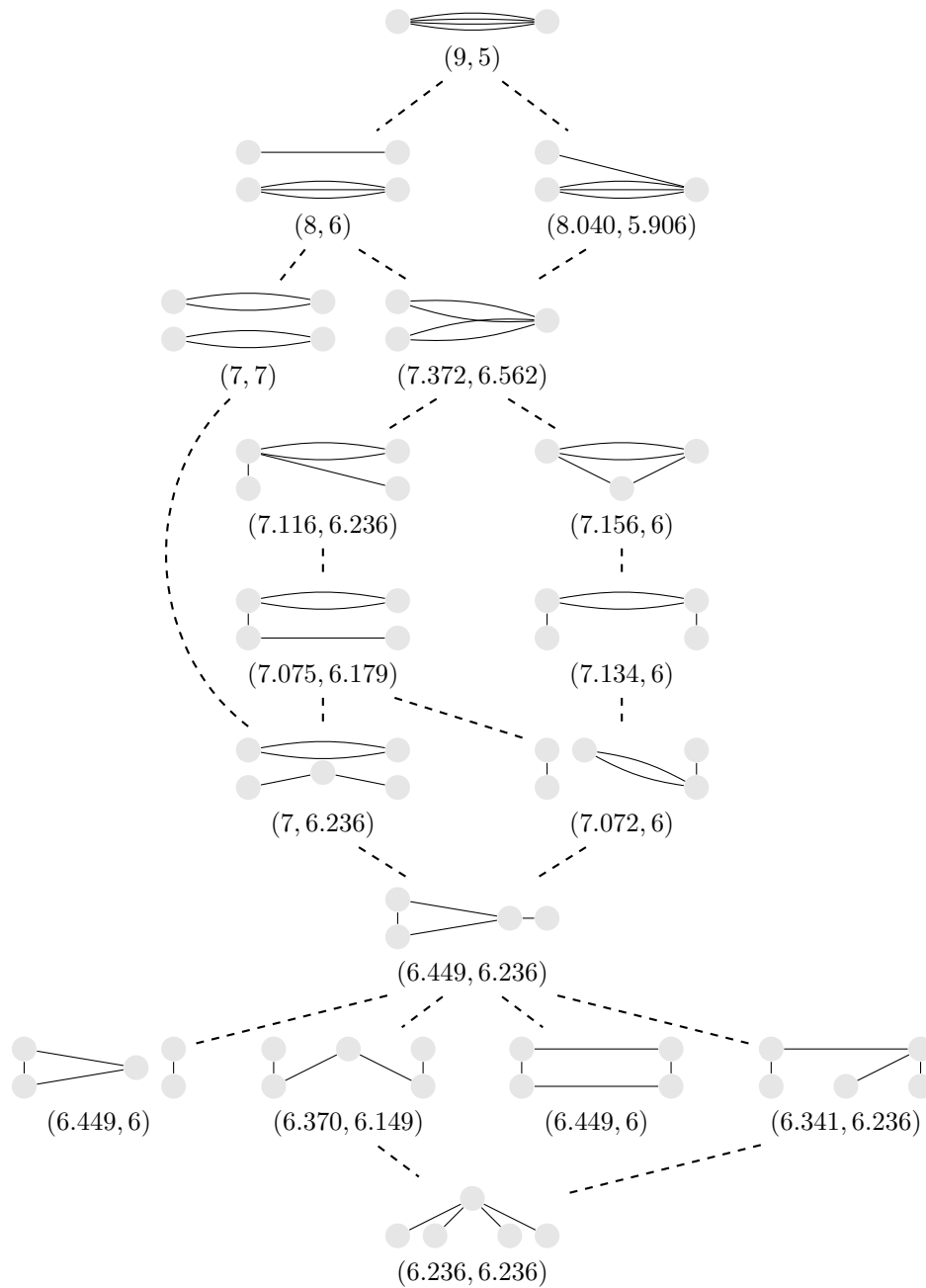


Figure 8: Poset of Split Simplicial Complexes for $c = 5$ and $q = 4$

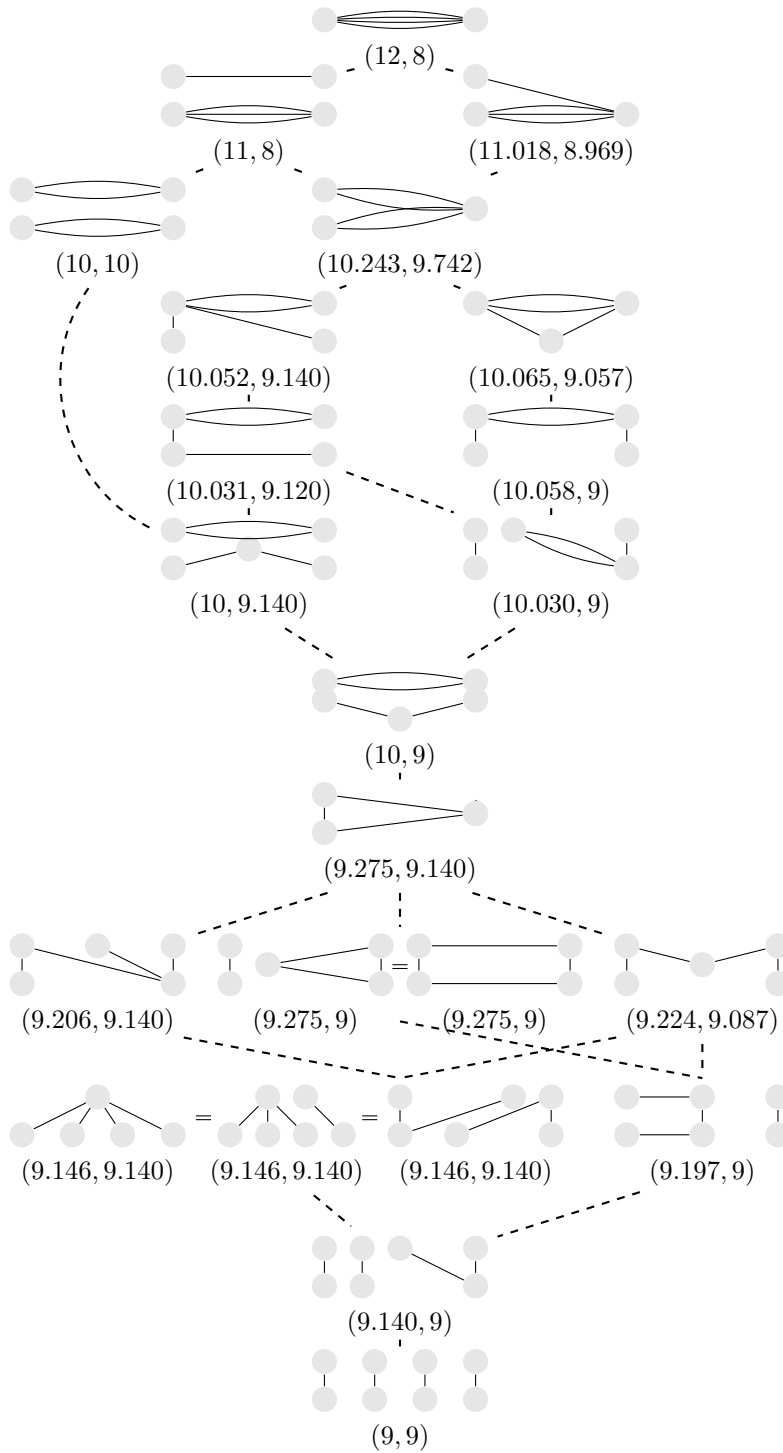


Figure 9: Poset of Split Simplicial Complexes for $c = 8$ and $q = 4$

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