Recovering Conductances of Resistor Networks in a Punctured Disk

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Abstract

It has been proven by Curtis, Ingerman, and Morrow that for circular planar resistor networks, there exists a necessary and sufficient condition for recovering the conductance of each edge in the network uniquely from the response matrix. We generalize their results to certain types of resistor networks on a punctured disk. First, we define certain circular planar graphs that are electrically equivalent to standard graphs. We then turn them into networks on a punctured disk by adding a boundary vertex in the middle and prove such networks are recoverable. We then generalize this result to a much broader family of networks, thus obtaining a sufficient condition for recoverability. A necessary condition for recoverability is also introduced. We also prove several results about medial graphs of resistor networks on a punctured disk, define the notion of z-sequences for such graphs, and introduce new local moves.

1 Introduction

Curtis, Ingerman, and Morrow [1] studied a class of electrical networks known as circular planar resistor networks. They proved many useful results related to connectivity and $Y-\Delta$ equivalence for such networks. They also defined a class of networks called "critical" and prove that recoverability and criticality are equivalent. R. Kenyon [3] set up the background for many related concepts in his paper. The author defined the medial graph and described local reductions that maintain electrical equivalence, while also defining the general inverse problem for electrical networks. Curtis, Mooers, and Morrow [2] were also interested in studying circular networks and computing conductances from voltages and currents measured at the boundary. In their paper, they presented an algorithm for recovering conductances using that approach and gave a characterization of various boundary measurements that arise in the study of circular networks. Other types of electrical networks in addition to circular planar resistor networks have also been studied. In [4], Lam and Pylyavskyy studied the inverse problem for electrical networks on a cylinder. They gave a conjectural solution for general cylindrical electrical networks, and showed it held for a special class of these networks known as "purely cylyndrical" networks.

We introduce a new class of electrical networks closely related to circular planar resistor networks. We look at networks that can be embedded on a punctured disk, meaning we allow for one boundary vertex to be placed inside the disk such that it cannot be moved to the boundary of the disk without breaking planarity. We study medial graphs of such networks and provide a sufficient condition for their recoverability.

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2 Background and Definitions

Defnition 1. A *resistor network* is a finite graph (V, E) with a specified set $B \subseteq V$ of *boundary vertices* and a real nonnegative conductance c_e, for each $e \in E$. The remaining vertices, $I = V - B$, are called *internal vertices.*

Defnition 2. A *circular planar resistor network* (cprn) is a resistor network that can be embedded in a disk so that it is planar and all the boundary vertices are on the boundary of the disk. A *resistor network in a punctured disk* (rnpd) is a resistor network that can be embedded in a disk so that it is planar and all boundary vertices but one are on the boundary of the disk.

Remark 1. The interior boundary vertex of an rnpd must be surrounded by a polygon. Otherwise, we are able to redraw the network so that the interior boundary vertex is on the boundary of the disk.

Defnition 3. A potential function assignment to the boundary vertices of Γ induces a net current at boundary vertices. The linear map from the potential function to the net current is represented by the *response matrix* of Γ, $\Lambda(\Gamma)$.

Defnition 4. Call graphs Γ_1 and Γ_2 *electrically equivalent*, if, for every assignment of conductances for Γ_1, there exists an assignment of conductances to Γ_2 such that the resulting electrical networks have the same response matrix, and vice versa.

Defnition 5. Call an edge a *boundary edge* if the vertices it connects are both boundary vertices.

Defnition 6. Call an edge e a *boundary spike* if it is connected to a boundary vertex v, and e is the only edge connected to v.

Defnition 7. Call an graph *reducible* if it is electrically equivalent to a graph with fewer edges, and *irreducible* otherwise.

Defnition 8. A *connection* of a resistor network $\Gamma = (V, E)$ is a tuple (P, Q) s.t. $P = \{p_1, \cdots, p_k\}$ and $Q = \{q_1, \cdots, q_k\}$ are subsets of V and there exists k disjoint paths that do not pass through boundary vertices connecting p_i to q_i, $\forall i$. For a cprn, we require that $p_1, \cdots, p_k, q_k, \cdots, q_1$ occur in clockwise order around the disk's boundary. For an rnpd, we have the same requirement, with the exception that the interior boundary vertex can appear in either Q or P, at any index.

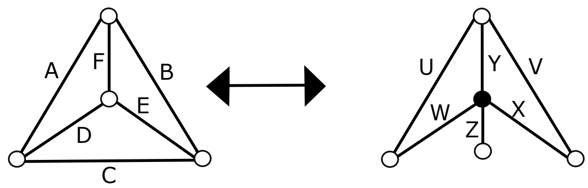
Defnition 9. [1] Let Γ be a cprn with n boundary vertices on a disk D and let $t_1, ..., t_{2n}$ be the endpoints of strands on D, moving consecutively clockwise. For each $i \in \{1, 2, \cdots, 2n\}$ let z_i be the number associated with the geodesic which intersects D at t_i. The sequence $z_1, ..., z_{2n}$ is called the *z-sequence* of Γ.

Defnition 10. We'll call a graph *standard*, denoted by Σ, if the z-sequence associated to it is $1, 2..., n, 1, 2, ..., n$.

3 Local Moves

A *local move* is a transformation that can be applied to a resistor network without changing the response matrix. There are five possible local moves that can be used on cprns: pendant removal, parallel edges, loop removal, series transformation, and Y-Delta transformations. We are able to use all of these moves in the rnpd case as well, but have a few new moves:

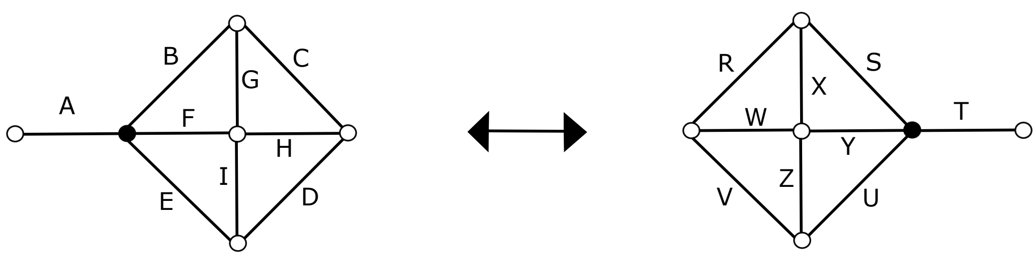
- Antenna Jumping
- Antenna Absorption



- Generalized Antenna Absorption

$$A = U + \frac{YW}{Z + X + W + V}$$
$$B = V + \frac{YX}{Z + X + W + V}$$
$$C = \frac{WX}{Z + X + W + V}$$
$$D = \frac{WZ}{Z + X + W + V}$$
$$E = \frac{XZ}{Z + X + W + V}$$
$$F = \frac{YZ}{Z + X + W + V}$$

- Antenna Jumping: Square Case



$$A = \frac{RSU + RSV + RTV + RUV + RVY + SUV + SUW}{SU}$$
$$B = \frac{RSU + RSV + RTV + RUV + RVY + SUV + SUW}{v(S + T + U + Y)}$$
$$C = \frac{ST}{S + T + U + Y}$$
$$D = \frac{TU}{S + T + U + Y}$$
$$E = \frac{RSU + RSV + RTV + RUV + RVY + SUV + SUW}{R(S + T + U + Y)}$$
$$F = \frac{W(RSU + RSV + RTV + RUV + RVY + SUV + SUW)}{RV(S + T + U + Y)}$$
$$G = \frac{-SUW + SVX + SVY + TVX + UVX + VXY}{V(S + T + U + Y)}$$
$$H = \frac{TY}{S + T + U + Y}$$
$$I = \frac{RSZ + RTZ + RUY + RUZ + RYZ - SUW}{R(S + T + U + Y)}$$

4 Medial Graphs and Z-sequences

Defnition 11. For an rnpd Γ, define the *medial graph* $M(\Gamma)$ of Γ to be as in [1], treating the interior boundary vertex as internal.

In particular, suppose Γ has n boundary vertices (not including the interior boundary vertex), indexed v_1 through v_n in clockwise order. For each edge e, let m_e be its midpoint. Place $2n$ points $t_1, \cdots t_n$ around the boundary such that $t_1 < v_1 < t_2 < t_3 < v_3 < \cdots < t_{2n-1} < v_n < t_{2n}$ in clockwise order.

Then the vertices of the $M(\Gamma)$ consist of $t_1, \cdots, t_{2n}$ and m_e for all edges e of Γ.

For each face F of Γ, consider the clockwise boundary of edges of F $e_{F,0}, e_{F,2}, \cdots e_{F,k_F-1}$, where spikes into F are considered to be on the boundary twice.

Then $M(G)$'s edges are $e_{F,i}$ to $e_{F,i+1}$ (with indices modulo K) for faces F of G, with multiplicity.

Remark 2. For an edge e of Γ, m_e has degree 4 in $M(\Gamma$. Each vertex t_i has degree 1 in $M(\Gamma)$. Proof is left as an exercise.

For a graph Γ, the *medial strands* of $M(\Gamma)$ equivalence classes of edges of $M(\Gamma)$ induced by the following relations $\equiv$:

Let e be any edge of Γ. let m_e be its corresponding vertex in $M(\Gamma)$. Let the four medial edges adjacent to m_e be $v_{e,1}, v_{e,2}, v_{e,3}$, and $v_{e,4}$ in clockwise order. Then $v_{e,1} \equiv v_{e,3}$ and $v_{e,2} \equiv v_{e,4}$.

Defnition 12. Define a medial circle to be a medial strand that does not have endpoints.

Defnition 13. Define a *medial lens* to be the region enclosed by two medial strands which intersect each other twice.

Defnition 14. Define a *medial loop* to be the region enclosed by any self-intersecting medial strand.

Theorem 1. *In an irreducible graph,*

1. *Any medial lens contains the interior boundary vertex.*
2. *Any medial loop contains the boundary vertex.*
3. *Any medial circle contains the boundary vertex.*

Proof. In progress. ☐

Theorem 2. *Medial circles do not exist.*

Proof. In progress. ☐

Theorem 3. *In an irreducible graph, there exists at most one medial loop.*

Theorem 4. *Any two medial strands intersect at most twice.*

Proof. Assume for contradiction we have two medial strands S_1 and S_2 that intersect (at least) three times. Without loss of generality, assume S_1 contains no self-intersection. As we move along S_2 let our three intersections be v_1, v_2, and v_3. Then S_1 and S_2 form two lenses - one with endpoints v_1 and v_2, and one with endpoints v_2 and v_3. Since these lie on opposite sides of S_1, they have disjoint interiors. Thus at least one medial lens does not contain the interior boundary vertex, a contradiction. ☐

Defnition 15. For an rnpd Γ, define the z-sequence of Γ $z(\Gamma)$ similarly as in [1] - except, for each strand S associated with s in the z-sequence, we perform the following modification:
If S does not contain a self-intersection, it divides the graph into two regions. Let A be the one not containing the interior boundary vertex. If S does contain a self intersection, it divides the graph into three regions. Let A be the region which does only touches the medial loop at a single vertex. Then label one s in the z-sequence s'. such that A contains the clockwise arc from s to s'.

Theorem 5. *Two minimal rnpds have the same z-sequence if and only if they are related by $Y-\Delta$ moves.*

Proof. Note motions don't change the relative order of strand endpoints, or whether strands go around the interior boundary vertex clockwise or counterclockwise, so the z-sequence is preserved by motions and thus $Y-\Delta$ moves.
For the other direction, we use strong induction on the number of strands. The base case - one strand - is clear.
Suppose we have we have a graphs G and H with k strands and the same z-sequence. Take a strand S_G in G which does not intersect itself. It divides the graph into two regions. Call the region which does not contain the boundary vertex A_G, and the other B_G. Without loss of generality, we may assume every strand in region A_G intersects S_G - if some strand S'_G does not - we could have picked that strand as S_G, decreasing the size of A_G.

Now, if two strands intersect in A_G, A_G has an almost-boundary triangle T using part of S_G as a side. Then we have T contains a boundary triangle, so we may use a motion to reduce the number of strand intersections in A_G.

Thus, G is $Y-\Delta$ equivalent to a graph G' where no strands intersect on the side of S not containing the interior boundary vertex. Furthermore the other side of S in G' is an rnpd, whose with z-sequence may be computed from from the z-sequence of G by removing strand S. By similarly writing H $Y-\Delta$ equivalent to H'(we pick the same strand S), we have one side of S in H' and G' is the same, while the other sides of H' and G' are $Y-\Delta$ equivalent by induction.

From G equivalent to G' equivalent to H' equivalent to H, we have G and H are $Y-\Delta$ equivalent as desired.

☐

5 4-Periodic cprns and Spider Graphs

In this section, we will define a canonical family of rnpds and prove that all the graphs in this family are recoverable.

5.1 4-Periodic cprns

We begin by defining a family of cprns that are electrically equivalent to standard graphs. We'll refer to them as 4-periodic graphs, denoted as Π_n for each n. The construction is as follows: For each $n \in \mathbb{N}, n \geq 3$, we start with an n-gon. Then we extend our polygon to have exactly $\lfloor \frac{n+1}{4} \rfloor$ layers, where a layer is another n-gon connected to the original n-gon by n edges between respective vertices (see Fig. 2). For $n \equiv 0 \mod 4$ we will add $\frac{n}{2}$ boundary spikes where all outermost vertices are boundary. For $n \equiv 1 \mod 4$, we will add all n boundary spikes. For $n \equiv 2 \mod 4$, we first add all the boundary spikes and then connect $\frac{n}{2}$ consecutive boundary vertices. Finally, the $n \equiv 3 \mod 4$ needs no modifications, we just need to note that all the outermost vertices are assumed to be boundary.

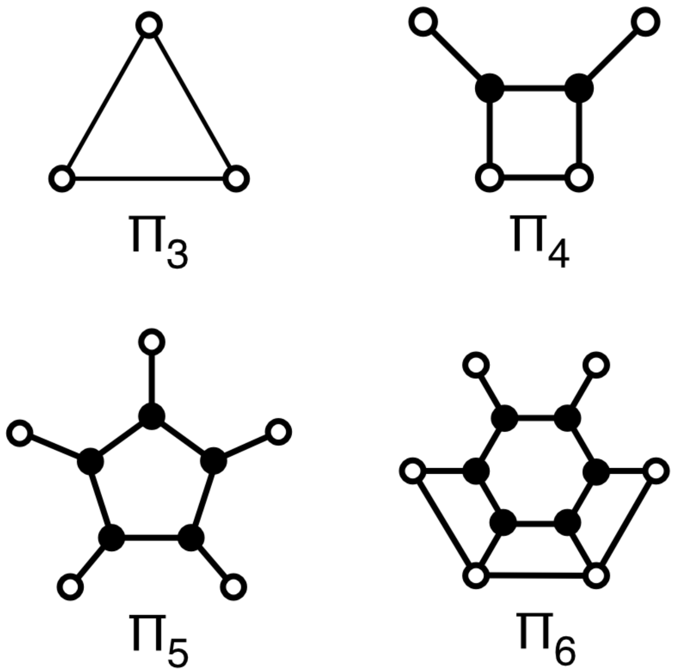


Figure 1: 4-periodic cprns

We will show that such graphs are equivalent to standard graphs.

Lemma 1. Let Π_n be a 4-periodic graph and let ℓ be the number of layers in this graph. Let S be a medial strand in Π_n that starts at s_1 and ends at s_2. Then:

1. If $n \equiv 0 \mod 4$, then the number of strand endpoints between s_1 and s_2 is $4\ell - 1$.
2. If $n \equiv 1 \mod 4$, then the number of strand endpoints between s_1 and s_2 is 4ℓ.
3. If $n \equiv 2 \mod 4$, then the number of strand endpoints between s_1 and s_2 is $4\ell - 3$.
4. If $n \equiv 3 \mod 4$, then the number of strand endpoints between s_1 and s_2 is $4\ell - 2$.

Proof. First observe that the four base cases ($n = 3, 4, 5, 6$) are true by inspection. We'll prove (2) and (1) by induction, and the other two cases are similar. In fact, we will prove a stronger statement first: if S starts by crossing a spike and ends by crossing a spike as well, then the number of strand endpoints between s_1 and s_2 is 4ℓ. Suppose the claim is true for a graph with $\ell - 1$ layers, so the number of strands between the beginning and the end of every strand is $4\ell - 4$ in such a graph. When we add a layer to that graph, we can observe that adding a layer forces us to have two extra sides of the polygon in-between the beginning and the end of each strand, which corresponds to exactly 4 extra strand endpoints. Thus, the number of strands between s_1 and s_2 is $4\ell - 4 + 4 = 4\ell$, as desired. We can now observe that in any Π_n where $n \equiv 1 \mod 4$, any strand will always start by crossing a spike and end by crossing a spike. Thus, (2) holds. (A similar argument holds for strands starting and ending at a boundary edge.)

Now suppose that $n \equiv 0 \mod 4$. Then we claim that each strand must start crossing a spike and end crossing an edge, or vice versa. To prove this, suppose, toward a contradiction, that the strand S starts by crossing a spike and ends by crossing a spike (the other case is similar). Then, as proved in the previous case, there are exactly 4ℓ strands between s_1 and s_2. This means that there are at least $\frac{4\ell-2}{2} + 2$ spikes in the graph ($4\ell - 2$ internal strands and 2 spikes that S crosses). Since $n \equiv 0 \mod 4$, we get the following relation:

$$\frac{4\ell - 2}{2} + 2 = 2\ell + 1 = 2\left\lfloor \frac{n+1}{4} \right\rfloor + 1 = 2 \cdot \frac{n}{4} + 1 = \frac{n+2}{2} > \frac{n}{2}$$

This is a contradiction, since the number of boundary spikes in such graphs is exactly $\frac{n}{2}$. Now we will prove that in any such graph, any strand that starts by crossing a spike and ends by crossing an edge has exactly $2\ell - 1$ strand endpoints between its endpoints. Assume that the lemma holds for a graph with $\ell - 1$ layer. Again, adding a layer adds exactly two sides of the polygon between s_1 and s_2, and consequently 4 more strand endpoints, and this concludes the proof.

□

Corollary 1. For all $n \in \mathbb{N}, n \geq 3$, Π_n is electrically equivalent to Σ_n.

Proof. Again, we will only prove the result for the case when $n \equiv 1 \mod 4$, as the other cases are similar. We first note that the number of medial strands in Π_n is n. Also, the number of endpoints is $2n$, as every boundary edge or boundary spike corresponds to exactly two endpoints. Take any strand $S = (s_1, s_2)$. By Lemma 1, we know that the number of strands between s_1 and s_2 is 4ℓ on one side. Since $n \equiv 1 \mod 4$, we know that $4\ell = 4\lfloor \frac{n+1}{4} \rfloor = 4 \cdot \frac{n-1}{4} = n - 1$, so there are $n - 1$ strand endpoints in-between s_1 and s_2. Since there are $2n-2$ endpoints other than s_1 and s_2, it follows that the number of endpoints between s_1 and s_2 on the other side is also $n - 1$. Since S was an arbitrary strand, the graph is standard. □

5.2 Spider Graphs

Defnition 16. We now introduce a new family of graphs called spider graphs. A *spider graph*, Ξ_n, is defined for each $n \geq 3 \in \mathbb{N}$. Ξ_n is constructed by placing a boundary vertex inside the center face of Π_n, and connecting it to each vertex on the boundary of that face.

We now work towards proving recoverability of the spider graph family.

Lemma 2. Let Γ be a critical cprn. Then, Γ has a boundary edge or a boundary spike.

Proof. This is Lemma 11.3 from [1]. □

Lemma 3. Let Γ be a critical cprn. Then, Γ remains critical after deleting a boundary edge or contracting a boundary spike.

Proof. This is a result of Lemmas 11.1 and 11.2 in [1]. □

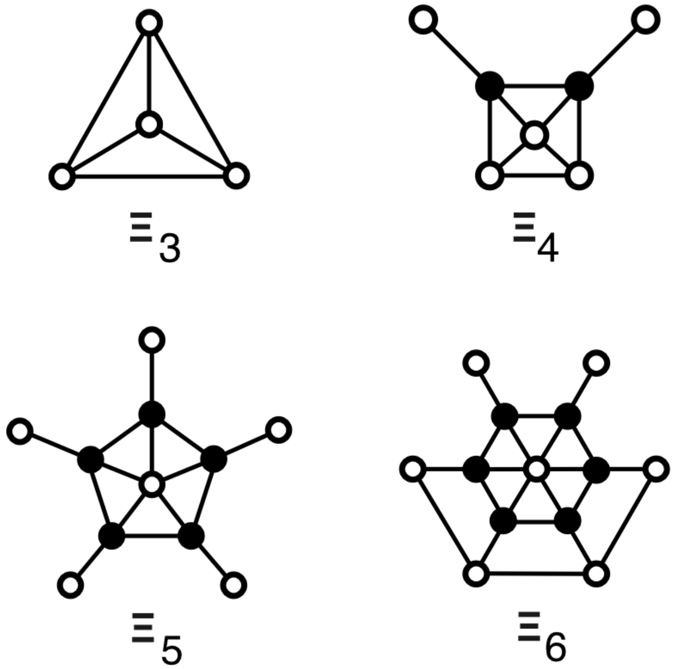


Figure 2: Spider Graph

Lemma 4. Removing an edge of a critical cprn breaks a (P, Q) connection, for some P and Q.

Proof. This is a result of [1]. □

Remark 3. Let Γ be a spider graph with interior boundary vertex b, and let $(P, Q) = (\{p_1, \cdots, p_k\}, \{q_1, \cdots, q_k\})$ be a pair of boundary nodes, with P and Q not interweaving. For any (P, Q) not containing b, the following statement is true. If (P, Q) is connected by k disjoint paths through internal vertices (connecting p_i to $q_{\pi(i)}$ for some $\pi \in S_k$), then $(-1)^k \det \Lambda(P; Q) > 0$. Otherwise, $\det \Lambda(P; Q)$ is zero.

The above remark is proved in Theorem 4.2 of [1] for the cprn case. Since the disjoint paths cannot pass through boundary vertices, when we restrict P and Q to not contain b, we are not able to use the center face of the spider graph in our (P, Q) possibilities, and thus have the same set of possibilities as the 4-periodic graphs (cprns) would. Other than properties true for all planar graphs, the proof of Theorem 4.2 relied only on the signs of the permutations for all (P, Q) possibilities. Since these are the same as for the 4-periodic graph, the same proof holds.

Lemma 5. Let Γ be an rnpd with interior boundary vertex b, and let e be a boundary edge or a boundary spike. If deleting or contracting $e = pq$ breaks some connection (P, Q) s.t. $b \notin P \cup Q$, then we can derive the conductance of e from $\Lambda(\Gamma)$.

Proof. Corollaries 4.3 and 4.4 of [1] prove this for the cprn case. However, this remains true if we restrict P and Q to not include the interior boundary vertex. □

Defnition 17. A resistor network is said to be *recoverable* if given its graph and its response matrix, we are able to uniquely recover the conductance of every edge.

Theorem 6. *Spider graphs are recoverable.*

Proof. Let $\Xi_n = (V, E)$ and the center face of Ξ_n containing the interior boundary vertex be f. Let $G = (V', E')$ be the interior graph of f, i.e. the star graph. Assume we are given $\Lambda(\Xi_n)$. First, note that $(V - V', E - E') = \Pi_n$, which is known to be critical. So, by Lemma 2, we always have a boundary edge or boundary spike with which to begin the following process. First, pick a boundary edge or boundary spike, e, to delete or contract. Notice that deleting e must break a connection (P, Q) in Π_n by Lemma 4. Since paths in connections cannot pass through other boundary vertices, adding back in G does not repair the broken connection, and (P, Q) is broken in Ξ_n. Thus, by Lemma 5, we know the conductance of e. Knowing this allows us to derive the response matrix of the graph after contracting or deleting e, which we will call Ξ'_n (see section 8 of [1]). Now, by Lemma 3, our $\Xi'_n - G$ is still critical. Then, we can continue this process, deleting and contracting boundary edges and spikes one by one, until we are left with G. In other words, we will eventually know the conductance of every edge in Ξ_n except for those in G, but will know the response matrix of G.
However, at that point, G will be a cprn with only boundary vertices, and hence its response matrix is exactly its Kirchoff's matrix. Therefore, Ξ_n is recoverable. □

5.3 Z-sequences of Spider Graphs

Theorem 7. *Let Ξ_n be a spider graph. Consider the z-sequence for Ξ_n. Let $S = (s^+, s^-)$ be an arbitrary strand. Then if n is odd, then there are exactly $n-3$ strands between s^+ and s^-, where we always measure the distance moving clockwise. If n is even, then for all but two strands, there are exactly $n-3$ strands between s^+ and s^-. One will have $n-2$ strands between its endpoints, and the other will have $n-4$.*

Proof. First assume n is odd. We first note that for odd n, each strand of Ξ_n will start by crossing a spike (or a boundary edge) and end by by crossing a spike (or a boundary edge) as well. We know that the number of strand endpoints between s^+ and s^- in Π_n is exactly $n-1$. Note that having a boundary vertex and edges in the middle forces us to have 2 more strand endpoints between s^- and s^+. So the number of medial strand endpoints between s^- and s^+ is $n+1$. Since we have $2n-2$ strand endpoints in total excluding s^+ and s^-, this means that the number of strands between s^+ and s^- is $n-3$.

Let n be even. Recall that in every Π_n where n is even, we have that every strand must start by crossing a spike and end by crossing an edge. Then, when we inset the star graph in the middle, as mentioned before, we get two more strand endpoints between s^- and s^+. This implies that if we start at s^- in Ξ_n, then s^+ gets shifted by exactly one spike or one edge. Hence, all but two strands will still have one endpoint on a boundary edge, and one on a boundary spike. These are identical to the odd case. One of the strands will start at a spike and end at a spike, and another one will start at an edge and end at an edge, due to the shift. Assume n is 2 mod 4. Then, one of the remaining strands, S_1 has s_1^+ and s_1^- on boundary edges, and the other, S_2, has s_2^+ and s_2^- on edges adjacent to the rightmost and leftmost vertices on the boundary edge side of the graph. The strand passing from a boundary edge to a boundary edge will have $n-2$ strands between s_1^+ and s_1^-. The final strand will have $n-4$ strands between s_2^+ and s_2^-. Now, assume n is 0 mod 4. Then, S_1 has s_1^+ and s_1^- on edges adjacent to the rightmost and leftmost vertices on the boundary edge side of the graph, and the other, S_2, has s_2^+ and s_2^- on boundary edges. The strand passing from a boundary edge to a boundary edge will have $n-4$ strands between s_2^+ and s_2^-. The final strand will have $n-4$ strands between s_1^+ and s_1^-. □

Remark 4. By Theorem 5, any irreducible graph whose z-sequence is the same as that of a spider graph is recoverable.

6 Recoverability Conditions

We are able to generalize the result for spider graphs to provide a necessary condition and a sufficient condition for recoverable rnpd.

Defnition 18. A *star graph* is a central boundary vertex with only boundary spikes or pendants attached. *Inserting a star graph into a polygon* means inserting a boundary vertex into the polygon, and connecting it to any number of vertices on the polygon.

Theorem 8. *Assume* Γ *is a cprn. Let* Γ' *be the rnpd resulting from inserting a star graph into any face of* Γ*. Then* Γ *is critical only if* Γ' *is recoverable.*

Proof. The proof of Theorem 6 only relies on the following assumptions:

- Removing the interior boundary vertex and its incident edges results in a critical cprn.
- After repeatedly removing and contracting boundary edges and boundary spikes of the outer critical cprn, we are left with a cprn with only boundary vertices.

Both of these assumptions will be satisfied if Γ' is recoverable and Γ' is formed by inserting a star graph into a face of Γ'. Therefore, we can follow the same process to show Γ' is recoverable. □

Before stating the necessary condition, we first need to define the following algorithm for obtaining a cprn from a certain type of rnpd. Suppose we are given an rnpd with the internal boundary vertex B that is only adjacent to some of the vertices forming the face around it.

Algorithm 1.

1. Turn all the neighbors of B into boundary vertices.
2. Remove B from the graph (with all the edges incident to it).
3. If the resulting graph is a cprn, stop.
4. Otherwise, apply the above steps to each boundary vertex (in any order) that can't be placed on the disk until it results in a cprn.

Recall that in any rnpd, the interior boundary vertex has to be inside a polygon (one of the faces of a cprn).

Theorem 9. *Assume* Γ' *is an rnpd where the internal boundary vertex is the center of a star graph in the interior of some face* f *of the cprn* Γ*. Apply Algorithm 1 iteratively starting at the internal boundary vertex, and call the resulting cprn* Γ''*. Then* Γ'' *is critical if* Γ *is recoverable.*

Proof. Assume Γ'' is not critical. Then the number of edges can be reduced. Note that Γ'' is structurally isomorphic to a subgraph of Γ, except it may have more boundary vertices. Having more boundary vertices may only decrease the number of possible moves. This implies that the number of edges in Γ could be reduced to begin with, and thus Γ is not recoverable.

□

7 ABC-Conjectures

a. An rnpd is recoverable if and only if it is reducible.

b. There exists a necessary and sufficient condition for recoverability of rnpds.

c. The moves we present are exhaustive.

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