COINCIDENCES AMONG SKEW GROTHENDIECK POLYNOMIALS

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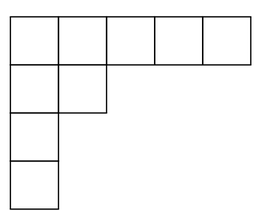
Abstract. The question of when two skew shapes are Schur-equivalent has been well-studied. We investigate the same question in the case of Grothendieck polynomials. We prove a necessary condition for two skew shapes to give rise to the same dual stable Grothendieck polynomial. We also provide a necessary and sufficient condition in the case where the two skew shapes are ribbons.

1. Introduction

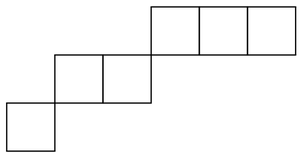
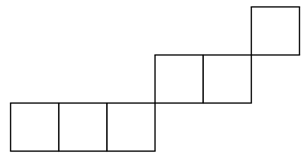
It is well known that the Schur functions s_λ form a linear basis for the algebra of symmetric functions. However, for general skew shapes λ/μ, this is not the case. In fact two different skew shapes can give rise to the same Schur function. Such skew shapes are called *Schur equivalent*. It is natural to ask when these coincidences occur. Coincidences among skew Schur functions have been well-studied. In [1] the authors provide necessary and sufficient conditions for two ribbons α and β to be Schur equivalent. In [5] the authors establish some necessary conditions and some sufficient conditions for two skew shapes to be Schur equivalent. Less is known about coincidences among stable and dual stable Grothendieck polynomials of skew shapes. In this paper we provide a necessary condititíon for two skew shapes to give rise to the same dual stable Grothendieck polynomial. We also provide a necessary and sufficient condition in the case where the two skew shapes are ribbons. Lastly, we document our approaches towards the skew stable Grothendieck polynomials and highlight areas for further research.

2. Background

A *partition* λ of a positive integer n is a weakly decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ whose sum is n. We write $\lambda \vdash n$, read as "λ is a partition of n." The integer λ_i is called the i-th part of λ. We call n the *size* of λ, denoted by $|\lambda| = n$. Throughout this document λ will refer to a partition. We may visualize a partition λ using a *Young diagram*. The Young diagram of a partition λ is a collection of left-justified boxes where the i-th row from the top has λ_i boxes. For example, the Young diagram of $\lambda = (5, 2, 1, 1)$ is shown below.



A *skew shape* λ/μ is a pair of partitions $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\mu = (\mu_1, \ldots, \mu_k)$ such that $k \leq m$ and $\mu_i \leq \lambda_i$ for all i. We form the Young diagram of a skew shape λ/μ by superimposing the Young diagrams of λ and μ and removing the boxes which are contained in both. A skew shape where μ is empty is called a straight shape. Given a skew shape λ/μ, we define its antipodal rotation $(\lambda/\mu)^*$ as the skew shape obtained by rotating the Young diagram of λ/μ by 180 degrees. For example, the Young diagrams of the skew shapes $(6,3,1)/(3,1)$ and $((6,3,1)/(3,1))^*$ are respectively

A *semistandard Young tableau* of shape λ/μ is a filling of the boxes of the Young diagram of λ/μ with positive integers such that the entries weakly increase from left to right across rows and strictly increase from top to bottom down columns. For example,

1	1	4	7
2	6		
9			

is a semistandard Young tableau of shape $\lambda = (4, 2, 1)$.

A *set-valued tableau* of shape λ/μ is a filling of the boxes of the Young diagram of λ/μ with nonempty sets of positive integers such that the entries weakly increase from left to right across rows and strictly increase from top to bottom down columns. For two sets of positive integers A and B, we say that $A \leq B$ if $\max A \leq \min B$ and $A < B$ if $\max A < \min B$. For a set-valued tableau T, we define $|T|$, the *size* of T, to be the sum of the sizes of the sets appearing as entries in T. For example,

$1,2$	$2,3$	6	9
3	5		
6	$6,7$		

is a set-valued tableau of shape $\lambda = (4, 2, 2)$ and size 11.

A *reverse plane partition* of shape λ/μ is a filling of the boxes of the Young diagram of λ/μ with positive integers such that the entries weakly increase both from left to right across rows and top to bottom down columns. For example,

1	1	2	7
1	1	5	9
1	6	9	

is a reverse plane partition of shape $\lambda = (4, 4, 3)$.

To each of the above fillings of a Young diagram we may associate a monomial. Given a semistandard Young tabelau or set-valued tableau T, we associate a monomial x^T given by

$$x^T = \prod_{i \in \mathbb{N}} x_i^{m_i},$$

where m_i is the number of times the integer i appears as an entry in T. Below is an example of a semistandard Young tableau with associated monomial $x_1^2 x_2 x_4 x_6 x_7 x_9$.

1	1	4	7
2	6		
9			

Below is an example of a set-valued tableau with associated monomial $x_1 x_2^2 x_3^3 x_5 x_6^3 x_7 x_9$.

$1,2$	$2,3$	6	9
3	5		
6	$6,7$		

Given a reverse plane partition T, the associated monomial x^T is given by

$$x^T = \prod_{i \in \mathbb{N}} x_i^{m_i},$$

where m_i is the number of columns of T which contain the integer i as an entry. Below is an example of a reverse plane partition with associated monomial $x_1^{@} x_2 x_5 x_7 x_9^2$.

1	1	2	7
1	1	5	9
1	6	9	

We can now define the Schur functions and the stable and dual stable Grothendieck polynomials.

We define the *Schur function* $s_{\lambda/\mu}$ by

$$s_{\lambda/\mu} = \sum_T x^T,$$

where the sum is across all semistandard Young tableau of shape λ/μ. We define the *stable Grothendieck polynomial* $G_{\lambda/\mu}$ by

$$G_{\lambda/\mu} = \sum_T (-1)^{|T|-|\lambda|} x^T,$$

where the sum is across all set-valued tableau of shape λ/μ. We define the *dual stable Grothendieck polynomial* $g_{\lambda/\mu}$ by

$$g_{\lambda/\mu} = \sum_T x^T,$$

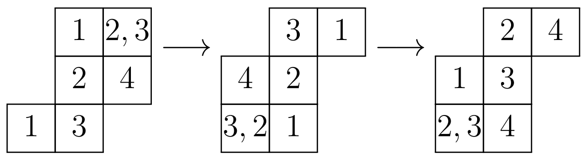
where the sum is across all reverse plane partitions of shape λ/μ.

We say that two skew shapes D_1 and D_2 are *G-equivalent* or *g-equivalent* if $G_{D_1} = G_{D_2}$ or $g_{D_1} = g_{D_2}$, respectively. Since semistandard Young tableaux are set-valued tableaux where each set has size 1, we see $G_{\lambda/\mu}$ is a sum of the Schur function $s_{\lambda/\mu}$ and terms of higher degree. Similarly, semistandard Young tableaux are also the reverse plane partitions where the columns strictly increase. Hence, $g_{\lambda/\mu}$ is a sum of the Schur function $s_{\lambda/\mu}$ and terms of lower degree. It follows that for two skew shapes to be G-equivalent or g-equivalent, they must also be Schur equivalent. Furthermore, it is easy to check that two skew shapes that are equivalent in any of the three aforementioned senses must have the same number of rows and columns. We will implicitly use this fact throughout.

Note that all three notions of skew equivalence are preserved under *. For completeness, we give a bijective proof for G-equivalence. The same bijection proves the proposition for Schur equivalence and g-equivalence as well.

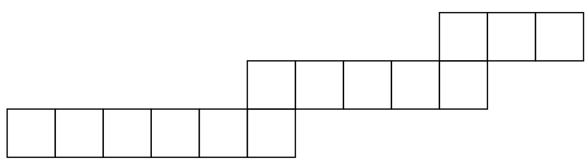
Proposition 2.1. *For any skew shape λ/μ, $G_{\lambda/\mu} = G_{(\lambda/\mu)^*}$*

Proof. Let $x_I = x_{i_1}^{p_1} x_{i_2}^{p_2} \ldots x_{i_k}^{p_k}$ be a monomial with $i_1 < i_2 < \ldots < i_k$. It suffices to show that the x_I-coefficient of each of the two polynomials is equal. To do so, we construct a bijection between set-valued tableaux of shape λ/μ with weight monomial x_I and set-valued tableau of shape $(\lambda/\mu)^*$ with weight monomial $x_{I'} = x_{i_k+1-i_1}^{p_1} x_{i_k+1-i_2}^{p_2} \ldots x_1^{p_k}$. This bijection, which is in fact an involution, maps a tableau T to the tableau T' given by rotating T and then replacing every entry j with $i_k + 1 - j$. An example is given below where $i_k = 5$:

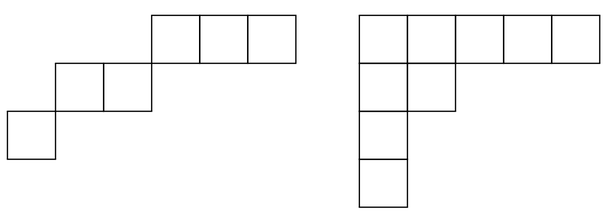


Thus, the $x_{I'}$ coefficient of $G_{(\lambda/\mu)^*}$ is equal to the x_I coefficient of $G_{\lambda/\mu}$. By symmetry, the $x_{I'}$ coefficient of $G_{(\lambda/\mu)^*}$ is equal to the x_I coefficient of $G_{(\lambda/\mu)^*}$. So, the x_I coefficients of $G_{\lambda/\mu}$ and $G_{(\lambda/\mu)^*}$ are equal as desired. $\square$

We will be interested in a special class of skew shapes known as *ribbons*. A skew shape α is called a ribbon if it is connected and contains no 2×2 rectangle. Below is an example of a ribbon.



The skew shapes below are not ribbons. The left shape is not connected while the right shape contains a $2x2$ rectangle.



A composition of a positive integer n is an m-tuple of positive integers that sum to n. For example, $(2, 7, 4, 9)$ is a composition of 22. It is easy to see that ribbons of size n are in bijection with compositions of n by letting the sizes of the rows of a ribbon correspond to a summand in a composition of n. Note that one can also construct the same bijection using instead the sizes of the columns of α. We will thus often write a ribbon α as $(\alpha_1, \ldots, \alpha_k)$, where α_i is the size of the i-th row of α. We will also write a ribbon β as $[\beta_1 \ldots, \beta_\ell]$, where β_i is the size of the i-th column of β. We note that the antipodal rotation α^* of α is the ribbon $(\alpha_k, \ldots, \alpha_1)$. We refer to α^* as the *reverse ribbon* of α.

We now define several binary operations on the set of ribbons. Here we let $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_m)$ be ribbons. We define the concatenation operation

$$\alpha \cdot \beta = (\alpha_1, \ldots, \alpha_k, \beta_1 . \ldots, \beta_m)$$

and the near concatenation operation

$$\alpha \odot \beta = (\alpha_1, \ldots, \alpha_{k-1}, \alpha_k + \beta_1, \beta_2, \ldots, \beta_m).$$

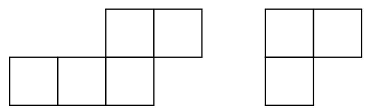
We let

$$\alpha^{\odot n} = \underbrace{\alpha \odot \cdots \odot \alpha}_{n}.$$

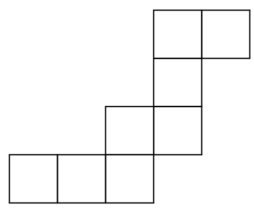
We can combine the two concatenation operations to yield a third operation $\circ$, defined by

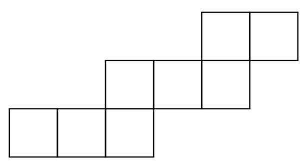
$$\alpha \circ \beta = \beta^{\odot \alpha_1} \cdots \beta^{\odot \alpha_k}.$$

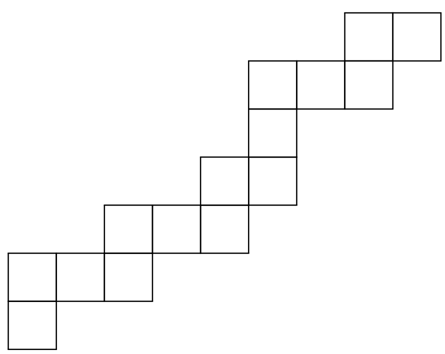
Visually, the operation $\circ$ replaces each square of α with a copy of β. For example, let $\alpha = (3, 2)$ and $\beta = (1, 2)$. Below are the Young diagrams of α and β, respectively.



Below are the Young diagrams of $\alpha \cdot \beta$, $\alpha \odot \beta$, and $\alpha \circ \beta$, respectively.







If a ribbon α can be written in the form $\alpha = \beta_1 \circ \cdots \circ \beta_\ell$, we call this a *factorization* of α. A factorization $\alpha = \beta \circ \gamma$ is called *trivial* if any of the following conditions hold:

(1) One of β or γ has size 1.
(2) Both β and γ contain a single row.
(3) Both β and γ contain a single column.

A factorization $\alpha = \beta_1 \circ \cdots \circ \beta_\ell$ is called irreducible if none of the factorizations $\beta_i \circ \beta_{i+1}$ are trivial and each β_i has no nontrivial factorization. In [1] the authors prove that every ribbon α has a unique irreducible factorization. They then prove the following theorem:

Theorem 2.2. *Two ribbons α and β satisfy $s_\alpha = s_\beta$ if and only if α and β have irreducible factorizations*

$$\alpha = \alpha_1 \circ \cdots \circ \alpha_k \quad \text{and} \quad \beta = \beta_1 \circ \cdots \circ \beta_k,$$

where each β_i is equal to either α_i or α_i^*.

In the next section, we use the above theorem to prove a necessary and sufficient condition for two ribbons to be g equivalent. We also provide a necessary condition for two skew shapes to be g equivalent.

3. Coincidences in Dual Stable Grothendieck Polynomial

3.1. **Ribbons.** The main result of this section is that for two ribbons α and β, $g_\alpha = g_\beta$ if and only if $\alpha = \beta$ or $\alpha = \beta^*$. We will obtain restrictions on α and β by writing the dual stable Grothendieck polynomials in terms of ribbon Schur functions and comparing the coefficients in the resulting expansions.

Proposition 3.1. *For a ribbon $\alpha = [\alpha_1, \ldots, \alpha_n]$, the polynomial g_α can be decomposed into a sum of ribbon Schur functions as*

$$g_\alpha = \sum_{\gamma \le \alpha} \left(\prod_{i=1}^{n} \binom{\alpha_i - 1}{\alpha_i - \gamma_i} \right) s_\gamma$$

where the sum is over all ribbons $[\gamma_1, \ldots, \gamma_n]$ with the same number of columns as α such that $\gamma_i \le \alpha_i$ for each i.

Proof. We define a map from reverse plane partitions of α to the set of all semistandard Young tableaux of shape γ where $\gamma \le \alpha$. Given a reverse plane partition T, map T to a semistandard Young tableau of shape $\gamma = [\gamma_1, \ldots \gamma_n]$ where γ_i is the number of distinct entries in column i in T. Fill column i of γ with the distinct entries of column i in T in increasing order. This gives a semistandard Young tableau, since columns are clearly strictly increasing and rows will remain weakly increasing.

This map preserves the monomial which the reverse plane partition corresponds to. The map is also surjective, since any semistandard Young tableau of shape γ where $\gamma \le \alpha$ is mapped to by any reverse plane partition with the same entries in each column but with some entries copied. It remains to show each semistandard Young tableau is mapped to by exactly $\prod \binom{\alpha_i - 1}{\alpha_i - \gamma_i}$ reverse plane partitions. Fix some semistandard Young tableau of shape $\gamma \le \alpha$. We construct all possible reverse plane partitions of α mapping to this semistandard Young tableau column by column. Given column i of α, consider the $\alpha_i - 1$ pairs of adjacent squares in the column. Since there are γ_i distinct entries in the column and the entries are written in weakly increasing order, $\alpha_i - \gamma_i$ of these pairs must match. A size $(\alpha_i - \gamma_i)$ subset of the $\alpha_i - 1$ pairs of adjacent squares gives a unique filling, where the given subset is the set of adjacent squares that match. Thus the number of possible fillings for each column is $\binom{\alpha_i - 1}{\alpha_i - \gamma_i}$, giving the desired formula. $\square$

Lemma 3.2. *Let $\alpha = [\alpha_1, \ldots, \alpha_n]$ and $\beta = [\beta_1, \ldots, \beta_n]$ be ribbons such that $g_\alpha = g_\beta$. Then for all $i = 1, \ldots, n$ we have $\alpha_i + \alpha_{n-i+1} = \beta_i + \beta_{n-i+1}$.*

Proof. Use Proposition 3.1 to write g_α and g_β as a sum of ribbon Schur functions. Note that all terms of degree $n+1$ in both sums are of the form s_γ where γ is a ribbon $(i, n-i+1)$.

It is shown in Proposition 2.2 of [1] that the set of ribbon Schur functions $r_\lambda, \lambda \vdash n+1$ (i.e. the Schur functions α where $\alpha = (\alpha_1, \dots, \alpha_r)$ has size $n+1$ and is in decreasing order) forms a basis for Λ_{n+1}. Then since each ribbon $(i, n-i+1)$ is Schur equivalent to $(n-i+1, i)$, it follows that the set of Schur functions of such ribbons is linearly independent. Comparing coefficients in the respective sums gives the desired equality. □

Lemma 3.3. *Suppose α and β are ribbons such that $g_\alpha = g_\beta$, $\alpha \neq \beta$, and there exist ribbons σ, τ, and μ such that $\alpha = \sigma \circ \mu$ and $\beta = \tau \circ \mu$. Then $\mu = \mu^*$.*

Proof. Let $\mu = [\mu_1, \dots, \mu_t]$, $\alpha = [\alpha_1, \dots, \alpha_n]$, and $\beta = [\beta_1, \dots, \beta_n]$. By hypothesis, we have that $\alpha = \mu \Box_1 \cdots \Box_r \mu$ and $\beta = \mu \diamond_1 \cdots \diamond_s \mu$, where each $\Box_i$ and $\diamond_i$ is one of the operations $\cdot$ or $\odot$. Thus each α_i and β_i is equal to one of $\mu_1, \dots, \mu_t$ or $\mu_1 + \mu_t$. Since $\alpha \neq \beta$, let r be the minimal index such that $\alpha_r \neq \beta_r$. We see that $\{\alpha_r, \beta_r\} = \{\mu_t, \mu_1 + \mu_t\}$, since the first index where α and β disagree corresponds to the first index i where $\Box_i \neq \diamond_i$. By Lemma 3.2 it follows that if $\alpha_i = \beta_i$ then $\alpha_{n-i+1} = \beta_{n-i+1}$. Hence $n - r + 1$ is the maximal index where α and β disagree. Note that by the same argument we similarly have $\{\alpha_{n-r+1}, \beta_{n-r+1}\} = \{\mu_1, \mu_1 + \mu_t\}$.

We have $\alpha_r + \alpha_{n-r+1} = \beta_r + \beta_{n-r+1}$ by Lemma 3.2. Substituting the possible values of $\alpha_r \neq \beta_r$ and $\alpha_{n-r+1} \neq \beta_{n-r+1}$, we find that this equation is either

$$\mu_1 + \mu_t = 2(\mu_1 + \mu_t)$$

or

$$2\mu_1 + \mu_t = \mu_1 + 2\mu_t.$$

The first equation is a contradiction. Thus the second equation holds, implying that $\mu_1 = \mu_t$. We will show by induction that $\mu_i = \mu_{t-i+1}$, completing the proof. We have just shown the base case. For the general case, we have by Lemma 3.2

$$\alpha_{r+i} + \alpha_{n-r-i+1} = \beta_{r+i} + \beta_{n-r-i+1}.$$

Since α and β are interchangeable we may assume without loss of generality that $\alpha_r = \mu_t$. Then we have $\alpha_{n-r+1} = \mu_1 + \mu_t$,$\beta_r = \mu_1 + \mu_t$ and $\beta_{n-r+1} = \mu_1$. Therefore

$$\begin{aligned} \alpha_{r+i} &= \mu_i \\ \beta_{r+i} &= \mu_{i+1} \\ \alpha_{n-r-i+1} &= \mu_{t-i} \\ \beta_{n-r-i+1} &= \mu_{t-i+1} \end{aligned}$$

We thus have

$$\mu_i + \mu_{t-i} = \mu_{i+1} + \mu_{t-i+1}$$

By the inductive hypothesis $\mu_i = \mu_{t-i+1}$, so canceling gives $\mu_{i+1} = \mu_{t-i}$, finishing the proof. □

Theorem 3.4. *For ribbons α, β, we have $g_\alpha = g_\beta$ if and only if $\alpha = \beta$ or $\alpha = \beta^*$*

Proof. Suppose $g_\alpha = g_\beta$. Then $s_\alpha = s_\beta$. By Theorem 2.2 let

$$\begin{aligned} \alpha &= \alpha_k \circ \cdots \circ \alpha_1 \\ \beta &= \beta_k \circ \cdots \circ \beta_1 \end{aligned}$$

be the irreducible factorizations for α and β, respectively. We prove by induction on r that for $r = 1, \dots, k$ we have

$$\alpha_r \circ \cdots \circ \alpha_1 \in \{\beta_r \circ \cdots \circ \beta_1, (\beta_r \circ \cdots \circ \beta_1)^*\}$$

By Theorem 2.2 we have $\alpha_1 \in \{\beta_1, \beta_1^*\}$ so the base case is satisfied. Now suppose $r \geq 2$. By the inductive hypothesis we have

$$\alpha_{r-1} \circ \cdots \circ \alpha_1 \in \{\beta_{r-1} \circ \cdots \circ \beta_1, (\beta_{r-1} \circ \cdots \circ \beta_1)^*\}$$

If $\alpha = \beta$ we are done, so we may assume otherwise. Then by letting $\mu = \alpha_{r-1} \circ \cdots \circ \alpha_1$ and applying Lemma 3.3 to α and either β or β^*, we have

$$\beta_{r-1} \circ \cdots \circ \beta_1 = (\beta_{r-1} \circ \cdots \circ \beta_1)^*$$

Since we also have that $\alpha_r \in \{\beta_r, \beta_r^*\}$ we are done. □

3.2. **Necessary Condition: Bottlenecks.**

Definition 3.5. Given a skew shape λ/μ with n columns and m rows, let the *number of bottleneck edges in column i* for $i = 1, 2, \ldots, n$ be

$$b_i^{\lambda/\mu} = |\{1 \leq j \leq m-1 \mid \mu_j = i-1, \lambda_{j+1} = i\}|$$

This is the number of horizontal interior edges in column i touching both the left and right boundaries of the skew shape.

Bottleneck edges are related to the *row overlap compositions* defined in [5].

Definition 3.6. The *k-row overlap composition* $r^{(k)}$ of a skew diagram λ/μ with m rows is $(r_1^{(k)}, \ldots, r_{m-k+1}^{(k)})$, where $r_i^{(k)}$ is the number of columns containing squares in all the rows $i, i+1, \ldots, i+k-1$.

In particular, $r^{(2)} = (\lambda_2 - \mu_1, \lambda_3 - \mu_2, \ldots, \lambda_m - \mu_{m-1})$. Thus bottleneck edges correspond to 1s in the 2-row overlap composition. When the $2, 3, \ldots, m$ row overlap compositions are written they form a triangular array of nonnegative integers. A column having i bottleneck edges corresponds in the array to an equilateral triangle of 1s with side length i. In [5] it is proven that given two Schur equivalent skew shapes, for each k the k-row overlap composition of one shape is a permutation of the k-row overlap composition of the other.

Example 3.7. For a ribbon $[\alpha_1, \alpha_2, \ldots, \alpha_n]$, we have $b_i = \alpha_i - 1$ for $i = 1, 2, \ldots, n$. Hence Theorem 3.10 generalizes Lemma 3.2.

Example 3.8. Let $\lambda/\mu = (5, 5, 4, 2, 2, 2)/(4, 2, 1, 1, 1, 0)$. Then the number of bottleneck

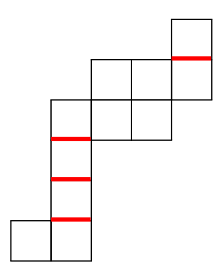


Figure 1. $(5, 5, 4, 2, 2, 2)/(4, 2, 1, 1, 1)$ has 3 bottleneck edges in column 2 and 1 bottleneck edge in column 5.

edges in each column is $(b_1, b_2, b_3, b_4, b_5) = (0, 3, 0, 0, 1)$. The row overlap compositions $r^{(2)}, \ldots, r^{(6)}$ are

$$\begin{array}{lccccccccc}
r^{(6)} & & & & & 0 & & & & \\
r^{(5)} & & & & 0 & & 0 & & & \\
r^{(4)} & & & 0 & & 0 & & 1 & & \\
r^{(3)} & & 0 & & 0 & & 1 & & 1 & \\
r^{(2)} & 1 & & 2 & & 1 & & 1 & & 1
\end{array}$$

Definition 3.9. Let a *1,2-RPP* be a reverse plane partition involving only 1's and 2's. A *mixed* column of a 1,2-RPP contains both 1's and 2's while for $i = 1, 2$, an *i-pure* column contains only i's.

Theorem 3.10. *Suppose $g_{\lambda/\mu} = g_{\gamma/\nu}$. Then*

$$b_i^{\lambda/\mu} + b_{n-i+1}^{\lambda/\mu} = b_i^{\gamma/\nu} + b_{n-i+1}^{\gamma/\nu}$$

for $i = 1, 2, \ldots, n$ where n is the number of columns in λ/μ (which is necessarily the same as the number of columns in γ/ν).

Proof. Note the 1,2-RPP's of a given shape are in bijection with lattice paths from the lower left vertex of the shape to the upper right vertex of the shape. The corresponding 1,2-RPP can be generated from such a lattice path by filling the squares below the path with 2's and the squares above the path with 1's. The corresponding lattice path can be recovered from a 1,2-RPP by drawing horizontal segments below the last 1 (if there are any) in a column and above the first 2 (if there are any) in a column. Vertical segments can the be drawn to connect these horizontal segments into a lattice path.

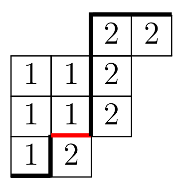


Figure 2. 1,2-RPPs correspond to lattice paths inside the skew shape. Note the red interior horizontal edge corresponds to the boundary between the 1s and 2s in the mixed column.

The coefficient of $x_1 x_2^n$ is the number of lattice paths with one mixed column and $n-1$ 2-pure columns. The lowest 1 in the mixed column must be in the first square in its row. For any row other than the last one, we can place a 1 in the first square in that row, necessarily place all 1's above it, and fill the rest of the shape with 2's to form such an RPP. We cannot do this with the last row since that would produce a pure column of all 1's. Thus the coefficient of $x_1 x_2^n$ is the number of rows in the shape, minus 1, so both λ/μ and γ/ν, have the same number of rows, m.

Let $k = \lceil \frac{n}{2} \rceil$ and $f_i = b_i + b_{n-i+1}$ for $i = 1, 2, \ldots, k-1$. If n is even, let $f_k = b_k + b_{n-k+1}$ and if n is odd, let $f_k = b_k$. Let $1 \leq r \leq k$. The coefficient of $x_1^r x_2^{n-r+1}$ for a skew shape is the number of 1,2-RPP's of the shape that have $r-1$ 1-pure columns, $n-r$ 2-pure columns, and 1 mixed column.

The lattice path corresponding to an 1,2-RPP with $r-1$ 1-pure columns, $n-r$ 2-pure columns, and one mixed column has one interior horizontal edge. If such a horizontal edge

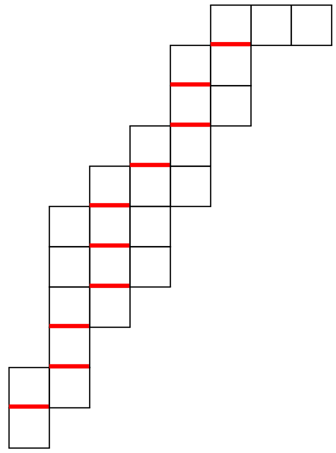


FIGURE 3. The first term in our sum, **S**, for the number of 1,2-RPP's with one mixed column, $r-1$ pure columns of 1's, and $n-r$ pure columns of 2's, consists of one 1,2-RPP for each of the $m-1$ red interior horizontal edges. If the red edge lies in columns $1,\dots,r-1$ then it touches the right boundary, and if it lies in columns $r+1,\dots,n$ then it touches the left boundary.

touches neither the right or left boundary of the shape, the lattice path must be vertical on either side of the edge until it hits the boundary, which it then follows, resulting in only one lattice path, which only gives the correct number of pure columns of all 1's if the chosen edge is in column r.

For an interior edge in column c touching only the right boundary of the shape, the path is forced to be vertical to the left of this edge and to then follow the lower boundary. On the right of this edge, it can move vertically from the lower boundary to the upper boundary after any column from c through n. Thus if c is $1,2,\dots r$, there is one lattice path resulting in the correct number of 2-pure columns but if $c>r$, there is not. Likewise if an interior edge in column c touches the left boundary, but not the right, the path is forced to be vertical on the right of this edge and to then follow the upper boundary. On the left of this edge, the path can move vertically from the upper boundary to the lower boundary before any column from c through 1. Thus if c is $r,r+1,\dots n$, there is one lattice path resulting in the correct number of columns of pure 1's but if $c<r$, there is not.

For an interior edge in column c touching both boundaries, there is flexibility on both sides: the number of 1-pure columns to the left can be anywhere from 0 to $c-1$ and the number of 2-pure columns on the right can be anywhere from 0 to $n-c$.

If $c \le r$, the number of 1-pure columns to the left of column c can range from 0 to $c-1 \le r-1$. The corresponding number of 1-pure columns to the right of column c ranges from $r-c$ to $r-1$. There are at least $n-r \ge r-1$ columns to the right of column c so all of these totals are possible, for a total of c 1,2-RPP's.

If $n-r+1 \ge c > r$, the number of 1-pure columns to the right of column c can range from 0 to $r-1$. The corresponding number of 1-pure columns to the left of column c ranges from 0 to $r-1$. So there are only r such 1,2-RPP's.

If $c > n-r+1$, the number of 1-pure columns to the right of column c ranges from 0 to $n-c$. The corresponding number of 1-pure columns to the left of column c can range from $r-n+c-1$ to $r-1$. So there are only $n-c+1$ such 1,2-RPP's.

So the total number of 1,2-RPP's with one mixed column, $r-1$ 1-pure columns, and $n-r$ 2-pure columns is given by the sum of the following 6 quantities:

(1) The number of interior edges in column r on neither boundary
(2) The number of interior edges in column $c \leq r$ on only right boundary
(3) The number of interior edges in column $c \geq r$ on only left boundary
(4) c times the number of interior edges in column $c \leq r$ touching both boundaries
(5) r times the number of interior edges in column $n-r+1 \geq c > r$ touching both boundaries
(6) $(n-c+1)$ times the number of interior edges in column $c > n-r+1$ touching both boundaries

Equivalently, this is the sum, **S**, of

(1) Number of interior edges in column r + number of interior edges in column $c < r$ touching right boundary + number of interior edges in column $c > r$ touching left boundary
(2) $(c-1)$ times the number of interior edges in column $c \leq r$ touching both boundaries
(3) $(r-1)$ times the number of interior edges in column $n-r+1 \geq c > r$ touching both boundaries
(4) $(n-c)$ times the number of interior edges in column $c > n-r+1$ touching both boundaries

For the rows below the entirety of column r, the top edges of the last square in each row are the interior edges in columns $c < r$ touching the right boundary. For the rows above the entirety of column r, the bottom edges of the first square in each row are the interior edges in columns $c > r$ touching the left boundary. The number of interior edges in column r is one less than the height of the column. Thus the first term in the sum is just $(m-1)$.

We conclude that when $r \leq k$, the coefficient of $x_1^r x_2^{n-r+1}$ is

$$\begin{aligned}(m-1) &+ (b_2 + 2b_3 + 3b_4 + \ldots (r-2)b_{r-1}) \\ &+ (r-1)(b_r + b_{r+1} + \ldots + b_{n-r+1}) \\ &+ ((r-2)b_{n-r+2} + (r-3)b_{n-r+3} + \ldots + 2b_{n-2} + b_{n-1}) \\ &= (m-1) + f_2 + 2f_3 + 3f_4 + \ldots + (r-1)f_r + (r-1)f_{r+1} + \ldots + (r-1)f_k\end{aligned}$$

for $1 \leq r \leq k$.

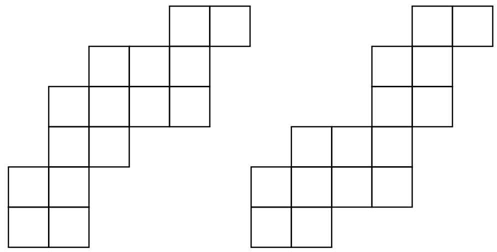
Now let the coefficient of $x_1^r x_2^{n-r+1}$ be t_r. Note that for $2 \leq r \leq k-1$, we have $2t_r - t_{r-1} - t_{r+1} = f_r$. Thus for $2 \leq r \leq k-1$, we have $f_r^{\lambda/\mu} = f_r^{\gamma/\nu}$. Since

$$\begin{aligned}&(m-1) + f_2^{\lambda/\mu} + 2f_3^{\lambda/\mu} + 3f_4^{\lambda/\mu} + \ldots + (k-1)f_k^{\lambda/\mu} \\ =\ &(m-1) + f_2^{\gamma/\nu} + 2f_3^{\gamma/\nu} + 3f_4^{\gamma/\nu} + \ldots + (k-1)f_k^{\gamma/\nu},\end{aligned}$$

it then follows that $f_k^{\lambda/\mu} = f_k^{\gamma/\nu}$.

By Corollary 8.11 in [5], it follows that $b_1 + \cdots + b_n = f_1 + \cdots + f_k$ is invariant, since the total number of bottleneck edges is the number of 1s in the 2-row overlap composition. Hence f_1 is invariant as well. □

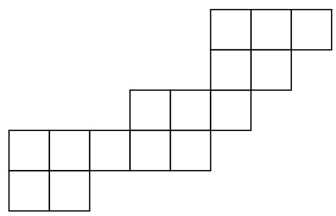
Example 3.11. It is noted in [5] that the shapes



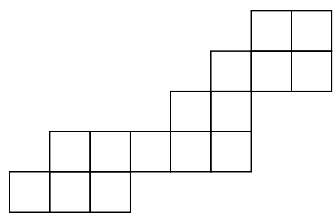
are Schur equivalent. But since $b_2 + b_5 = 2$ for the first shape and $b_2 + b_5 = 1$ for the second shape, it follows that the two shapes are not g equivalent.

Example 3.12. Having the same bottleneck edge sequence is not sufficient for two skew shapes to be g equivalent. Let α be the ribbon $(2, 1)$ and D be the partition $(3, 2)$. Then $\alpha \circ D$ and $\alpha \circ D^*$ are Schur equivalent by Theorem 7.6 in [5] and have the same b_i but are not g equivalent.

$\alpha \circ D =$



$\alpha \circ D^* =$



Since the bottleneck condition followed as a result of comparing terms of g with degree $n+1$ and 2 variables, it is natural to compute coefficients for terms with higher degree or more variables. The following result shows that terms with degree $n+1$ and more than 2 variables do not impose additional constraints for two skew shapes to be g equivalent.

Proposition 3.13. *Suppose two skew shapes λ/μ and γ/ν have the same number of rows and the polynomial $g_{\lambda/\mu}$ and $g_{\gamma/\nu}$ have same coefficient for every term of degree $n+1$ with two variables. Then in fact these polynomials have the same coefficient for any term of degree $n+1$.*

Proof. Fix positive integers $i_1, i_2, \ldots, i_k$ where k is some positive integer ≥ 2, and let $n = (\sum_{j=1}^{k} i_j) - 1$. We claim that there exists integers $c_2, \ldots, c_{n-1}$ such that for all skew shapes λ/μ with n columns we have that the coefficient of $x_1^{i_1} \cdots x_k^{i_k}$ in $g_{\lambda/\mu}$ is $(k-1)(m-1) + c_2 b_1 + \cdots + c_{n-1} b_{n-1}$, where m is the number of rows in λ/μ. We proceed by induction on k.

The base case $k = 2$ is given in the proof of Theorem 3.10, so henceforth let $k \geq 3$. We count the number of RPPs giving the monomial $x_1^{i_1} \cdots x_k^{i_k}$. Suppose first that every column containing a 1 is in fact 1-pure. Then the first i_1 columns must be filled with 1's. Note the remaining squares form a skew shape with $n - i_1$ columns, as depicted in Figure 4. We use henceforth use $(\lambda/\mu)_{i_1}$ to denote the skew shape given by removing the first i_1 columns of λ/μ. Note $(\lambda/\mu)_{i_1}$ must be filled with a reverse plane partition giving the monomial $x_2^{i_2} \cdots x_k^{i_k}$.

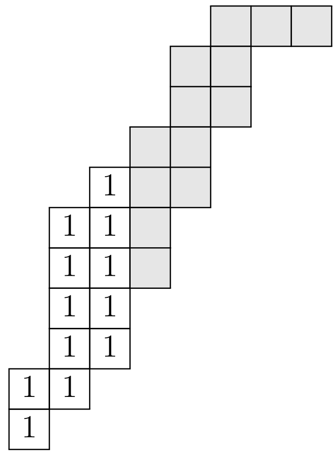


FIGURE 4. The remaining shape shaded in gray is a skew shape with $n - i_1$ columns, denoted $(\lambda/\mu)_{i_1}$.

Let m' be the number of rows in the shape attained by removing the first i_1 columns from λ/μ. Then by induction the number of ways to fill in this shape is $(k-2)(m'-1) + c'_{i_1+2}b_{i_1+2} + \cdots c'_{n-1}b_{n-1}$ for some integers $c'_{i_1+2}, \ldots, c'_{n-1}$.

The remaining case is when there is a mixed column containing a 1. As noted in Figure 3, there are $m-1$ possibilities for the unique interior horizontal edge. Consider first the interior horizontal edges in columns $1, \ldots, i_1$ touching the bottom boundary of λ/μ. This case is depicted in Figure 5. Note that there are $m - m'$ such edges, since in total there are $m-1$ edges touching the bottom boundary and exactly $m'-1$ of them lie in columns $i_1+1, \ldots, n$. Given any of these $m-m'$ edges, one possible lattice path starting from the top right is traveling along the top boundary until the boundary between column i_1 and column i_1+1, dropping to the bottom boundary and traveling to the edge, and then immediately dropping back down to the bottom boundary. This determines which squares are filled with 1s. The remaining shape is a disconnected skew shape where one component is a single column and the other component is $(\lambda/\mu)_{i_1}$. There are $(k-1)$ fillings using this lattice path, since the column below the edge may be filled with any of $2, \ldots, k$ and the remaining columns must fill $(\lambda/\mu)_{i_1}$ in increasing order. Note that unless the edge is a bottleneck edge, this is the unique lattice path using this edge.

The remaining $m'-1$ edges are those in column i_1 not touching the bottom boundary and the edges in column $i_i+1, \ldots, n$ touching the top boundary. For each of these edges, one possible lattice path is traveling along the top boundary until the edge, dropping down to the edge, and then dropping down to the bottom at the boundary between column i_1-1 and column i_1. This path determines which squares are filled with ones. The remaining squares form a (possibly disconnected) skew shape, which must be filled with no mixed columns. Note that the remaining skew shape is connected if and only if the horizontal edge was not a bottleneck edge. If the shape is connected, then filling the columns in increasing order is the only possible filling. Otherwise, this is one possible filling but there may be more.

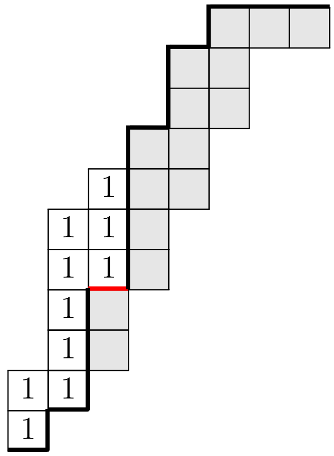


FIGURE 5. Case 1: the lattice path uses an edge in columns $1, \ldots, i_1$ touching the bottom boundary. Then the remaining shape is the union of $(\lambda/\mu)_{i_1}$ and a single column.

Thus far, this gives us $(k-2)(m'-1)+c'_{i_1+2}b_{i_1+2}+\cdots c'_{n-1}b_{n-1}+(k-1)(m-m')+(m'-1) = (k-1)(m-1)+c'_{i_1+2}b_{i_1+2}+\cdots c'_{n-1}b_{n-1}$ fillings. It remains to show each bottleneck edge in column i contributes a fixed number of additional fillings.

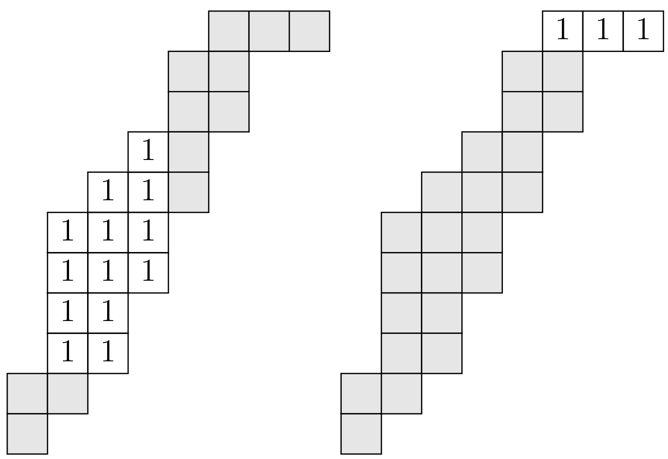


FIGURE 6. The remaining shape will have 1 or 2 components. The number of fillings is determined by $i_2, \ldots, i_k$ and the number of columns in the components.

As noted in the proof of Theorem 3.10, each bottleneck edge in column i has $\min(i, n-i+1, i_1)$ possible lattice paths using that edge. Each lattice path determines which squares will be filled with ones. Note the remaining squares will form a possibly disconnected skew shape with $n-i_1+1$ columns (depicted in Figure 6), which must then be filled with no mixed columns. There are a fixed number of ways to fill this shape, which depends only on $i_2, \ldots, i_k$ and the number of columns in the two components. The possible number of columns in each component is in turn determined by which column the bottleneck edge is in. This finishes the proof of the claim.

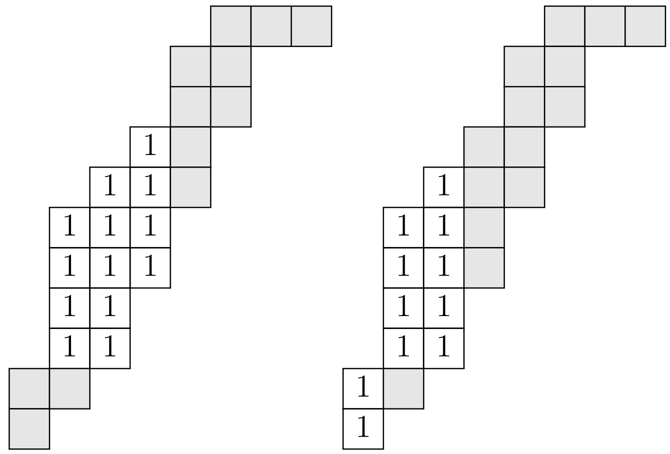


FIGURE 7. The possible numbers of columns in the components of the remaining shape are determined by which column the bottleneck edge is in. In this example, since the bottleneck edge is in column 2, the components have 1 and $n - i_1 + 1$ columns or 2 and $n - i_1 + 2$ columns.

Thus we have that the coefficient of $x_1^{i_1} \cdots x_k^{i_k}$ for any shape with $n := (\sum i_j) - 1$ columns is $(k-1)(m-1) + c_2 b_1 + \cdots + c_{n-1} b_{n-1}$ for some integers $c_2, \ldots, c_{n-1}$. Recall that every shape is equivalent to its 180 degree rotation, and note 180 degree rotation reverses the sequence $b_1, \ldots, b_n$. Since there are shapes with arbitrary sequences of $b_1, \ldots, b_n$ (for example, the ribbon $[b_1 + 1, \ldots, b_n + 1]$), it follows that $c_i = c_{n-i+1}$ for $i = 2, \ldots, n-1$. Recall also that the proof of Theorem 3.10 shows each sum $b_i + b_{n-i+1}$ for $i = 2, \ldots, n-1$ must be the same for any two shapes such that the terms in g of degree $n+1$ with two variables are the same. Since the number of rows m must be the same as well, it follows that the sum $(k-1)(m-1) + c_2 b_1 + \cdots + c_{n-1} b_{n-1}$ must also be the same. $\square$

Proposition 3.14. *The coefficient of $x_1^2 x_2^n$ in $g_{\lambda/\mu}$ is*

$$\binom{m}{2} - \sum_{i=1}^{n} \binom{b_i + 1}{2}$$

Proof. A 1,2-RPP giving the monomial $x_1^2 x_2^n$ must have no 1-pure columns, $n-2$ 2-pure columns, and 2 mixed columns. Hence the corresponding lattice paths have 2 interior horizontal edges. Consider the *heights* of the interior horizontal edges. By an interior horizontal edge at height i we mean the edge lies between row i and row $i+1$. Observe that given the height of the two interior horizontal edge, there is at most one lattice path using the heights; since there are no 1-pure columns, the lattice path is completely determined by the heights chosen.

There are $\binom{m}{2}$ ways to choose a pair of heights from $1, \ldots, m-1$ (with possible repetition). Since each pair of heights contributes either 1 or 0 lattice paths, the desired coefficient is thus $\binom{m}{2}$ minus the number of pairs not giving a lattice path. These are exactly the pairs of heights where the only interior horizontal edges at those heights lie in a single column. This is precisely the pairs of bottlenecks edges from the same column. For each column i there are $\binom{b_i+1}{2}$ ways to choose 2 of the bottlenecks in column i (with possible repetition), giving the desired formula. $\square$

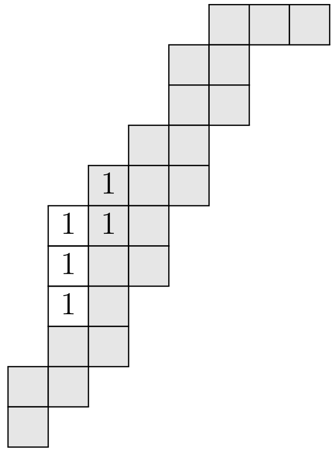


FIGURE 8. The heights determine the filling, since the column containing 1's more to the left must touch the left boundary of the shape, and the other column must touch the left boundary of the remaining shape.

Proposition 3.15. *The coefficient of* $x_1x_2x_3^n$ *in* $g_{\lambda/\mu}$ *is*

$$(m-1)^2 - \sum_{i=1}^{n} \binom{b_i+1}{2}$$

Proof. Since g is symmetric, we may equivalently compute the coefficient of $x_1x_2^nx_3$. As before, the lattice path must have two interior horizontal edges. Note the column containing 1's must touch the left boundary and the column containing 3's must touch the right boundary. Hence the squares containing 1s are determined by the height above which the 1s appear. Similarly, the squares containing 3s are determined by the height below which the 3s appear.

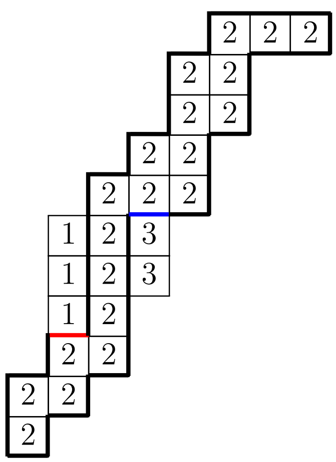


FIGURE 9. A pair of heights determines the filling.

There are $(m-1)^2$ ways to choose these two heights. The only pairs of heights not giving valid fillings come from bottleneck edges, which are the heights that touch both the left and right boundary. More precisely, the heights fail to give a valid filling exactly when they are both bottlenecks in the same column and the height for 1s is below or equal to the height for the 3s. For each column there are $\binom{b_i+1}{2}$ such pairs, giving the desired formula. □

By Corollary 8.11 in [5], the number of rows m and the sum $b_1 + \cdots + b_n$ are invariant when two shapes are g-equivalent. Hence we attain the following as a direct consequence of either of the above propositions.

Corollary 3.16. *Suppose $g_{\lambda/\mu} = g_{\gamma/\nu}$. Then*

$$\sum_{i=1}^{n} (b_i^{\lambda/\mu})^2 = \sum_{i=1}^{n} (b_i^{\gamma/\nu})^2$$

Equivalently, the sums of the areas of the equilateral triangles of 1s in the row overlap compositions $r^{(2)}, \ldots, r^{(m)}$ are the same.

For terms in g of higher degree $n+r$ for $r > 1$, the coefficient is affected not only by areas of width one touching both boundaries but also areas of the shape with width at most r. This leads us to define the following generalization of bottleneck edges.

Definition 3.17. For $i = 1, \ldots, \lambda_1 - w + 1$ the *number of width w bottlenecks in position i* is

$$b_i^{(w)} = |\{1 \le j \le m-1 \mid \mu_j = i-1, \lambda_{j+1} = i+w-1\}|$$

Example 3.18. Let $\lambda/\mu = (5,5,4,2,2,2)/(4,2,1,1,1,0)$. Then the number of bottleneck edges of each width is

$$\begin{array}{lccccccccc}
b^{(5)} & & & & & 0 & & & & \\
b^{(4)} & & & & 0 & & 0 & & & \\
b^{(3)} & & & 0 & & 0 & & 0 & & \\
b^{(2)} & & 0 & & 0 & & 1 & & 0 & \\
b^{(1)} & 0 & & 3 & & 0 & & 0 & & 1
\end{array}$$

Note $b^{(1)}$ is just the previously defined bottleneck edges.

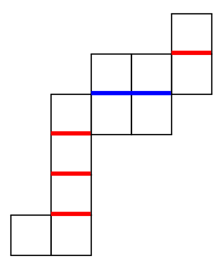


FIGURE 10. $(5,5,4,2,2,2)/(4,2,1,1,1,0)$ has 3 width 1 bottlenecks in column 2, 1 width 1 bottleneck in column 5, and 1 width 2 bottleneck in column 3.

Proposition 3.19. *The coefficient of $x_1^3 x_2^{n-1}$ in $g_{\lambda/\mu}$ is*

$$\left(\binom{m}{2} - \sum_{i=1}^{n} \binom{b_i^{(1)}+1}{2}\right) + \sum_{i=2}^{n-2} \binom{b_i^{(2)}+1}{2} + (m-2)\sum_{i=2}^{n-1} b_i^{(1)}$$
$$- \left(b_2^{(1)}(m - \mu_1' - 1) + b_{n-1}^{(1)}(\lambda_n' - 1) + \sum_{i=2}^{n-2} b_i^{(1)} b_{i+1}^{(1)}\right)$$

Proof. As is the case in Proposition 3.14, a lattice path identified with this monomial must have two interior horizontal edges. It can be checked that the pairs of heights not giving any lattice paths are the same as before, namely pairs of bottleneck edges in the same column. This contributes the term $\binom{m}{2} - \sum \binom{b_i^{(1)}+1}{2}$. It remains to account for pairs of heights giving more than one possible lattice path.

It can be checked that the only pairs of heights with more than one possible lattice path are those involving bottleneck edges and bottlenecks of width two. Consider the case where the two heights are both width two bottlenecks in the same column i where $i \in \{2, \ldots, n-2\}$. Then there are two possible fillings, since the last column containing 1s may be either to the left or to the right of the bottleneck of width 2. Since there are $\binom{b_i^{(2)}+1}{2}$ ways to choose a pair of width two bottlenecks from column i, this accounts for the second term in our formula.

Next, consider when one of the heights is a bottleneck in columns $2, \ldots, n-1$ (similar to the case in Theorem 3.10, bottlenecks in column 1 and n do not contribute additional paths). Then there are $(m-2)$ possibilities for the other height. Suppose the other interior horizontal edge is not a bottleneck edge. We show that there are 2 possible lattice paths. There are several cases to check. Let the heights of the bottleneck and the other interior horizontal edge by i and j respectively. Also, let the column in which the bottleneck belongs be c_i and the column containing the leftmost edge of height j be c_j.

Suppose first $i < j$. Then $c_i \leq c_j$ since λ/μ is a skew shape. If $c_j - c_i \geq 2$, then there are two lattice paths. Note the interior horizontal edge must be chosen so that it touches the left boundary. Then one lattice path is given by using column $c_i + 1$ as the last column containing 1, and the other is given by using column 1 as the last column containing 1. Now suppose $c_j = c_i$ or $c_j = c_i + 1$. Then, as depicted in Figure 11, there will be two lattice paths (except in a certain border case, which will be accounted for later).

Next is the case $i > j$. If $c_j > 1$ note there are two lattice paths using heights i and j. This is because the non-bottleneck interior horizontal edge must touch the left boundary, and then the last column containing 1s can be either column 1 or column $c_i + 1$. It remains to consider $c_i = 1$. In this case there are in fact still two lattice paths. One is given as before by using column $c_i + 1$ as the extra column. Unless $c_i = 2$, another lattice path is given by choosing the interior horizontal edge at height j in column 2, and dropping down to use column 1 as the last column.

Next we suppose both heights are bottlenecks (in different columns). Then unless they are in adjacent columns, the last column containing a 1 may be to the left, right or in between the two bottlenecks, giving 3 possible lattice paths. If the bottlenecks are in adjacent columns, then having a column in between is impossible, giving 2 lattice paths. This gives a total of $(m-2)\sum_{i=2}^{n-1} b_i^{(1)} - \sum_{i=2}^{n-2} b_i^{(1)} b_{i+1}^{(1)}$ extra paths.

As noted earlier, an exception is if the bottleneck is in column 2 and the other edge is in column 1. In this case, it is impossible for the last column to be to the left of the two edges, so there is only 1 lattice path. Similarly, a bottleneck in column $n-1$ and the edge in the last column is an exception. Thus we must subtract $b_2^{(1)}$ times the number of heights in the first column and $b_{n-1}^{(1)}$ times the number of heights in the last column. This gives the last two terms in the formula, finishing the proof. $\square$

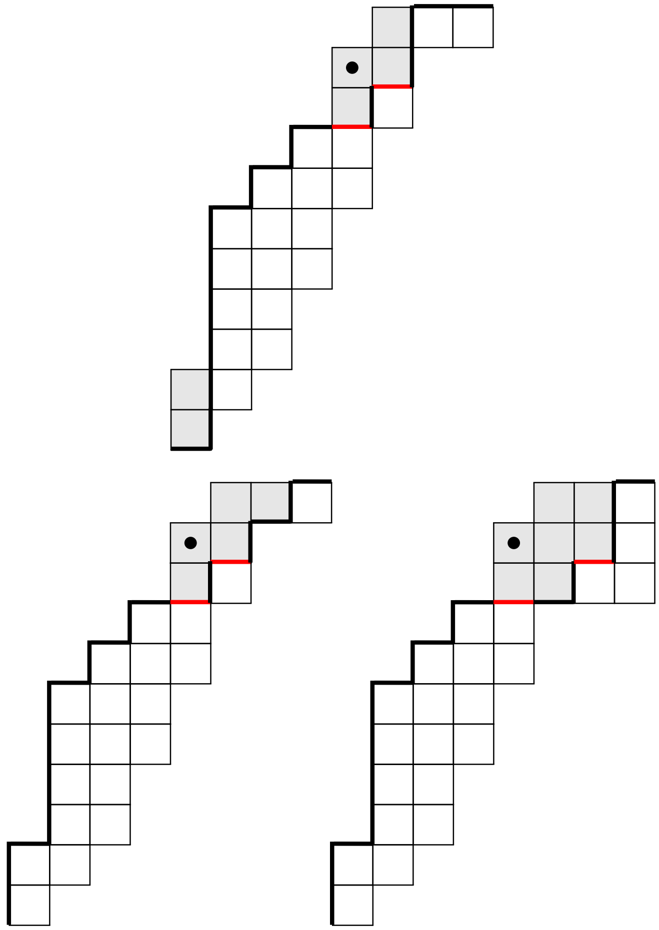


FIGURE 11. If $i < j$ and $c_i = c_j$ there are two lattice paths. One is given by dropping down between column 1 and 2. The other lattice path is given by dropping down between column $c_i + 2$ and $c_i + 3$, though whether the non-bottleneck interior horizontal edge is in column $c_i + 1$ or $c_i + 2$ depends on whether height j is a bottleneck of width 2. Note that if the square marked with a bullet and i and j are still the same, this example will now have $c_j = c_i + 1$. However, the permissible lattice paths will still be the exact same, giving 2 lattice paths in either case. An exception is when $c_i = n - 1$, in which there will only be the first lattice path.

Proposition 3.20. *The coefficient of $x_1^3 x_2^n$ in $g_{\lambda/\mu}$ is*

$$\binom{m+1}{3} - \sum_{i=1}^{n}\left((m-1)\binom{b_i^{(1)}+1}{2} - 2\binom{b_i^{(1)}}{3} - b_i^{(1)}(b_i^{(1)}-1)\right)$$
$$-\sum_{i=1}^{n-1}\left(\binom{b_i^{(2)}+2}{3} + (b_i^{(1)} + b_{i+1}^{(1)})\binom{b_i^{(2)}+1}{2} + b_i^{(1)}b_i^{(2)}b_{i+1}^{(1)}\right)$$

Proof. Note that a 1,2-RPP corresponding to the monomial $x_1^3 x_2^n$ must have no 1-pure columns, $n-3$ 2-pure columns, and 3 mixed columns. Thus the corresponding lattice path has exactly 3 interior horizontal edges. Next observe that given the 3 heights of the interior horizontal edges, there is at most one lattice path using these heights. Since there are no 1-pure columns, the left-most column containing a 1 must touch the left boundary, the next left-most column containing a 1 touches the left boundary of the remaining shape to be filled, and similarly the last column containing a 1 touches the left boundary of the remaining shape. Hence we can count the coefficient by starting with the $\binom{m+1}{3}$ ways to choose 3 heights (with possible repetition) from the $m-1$ heights and subtracting the combinations of heights that are not used by any lattice path.

One way a triple of heights fails to give a valid filling is if two of the interior horizontal edges are bottlenecks in the same column, and the last horizontal edge is at any of the $(m-1)$ heights. There are $\sum_{i=1}^{n}(m-1)\binom{b_i^{(1)}+1}{2}$ ways to choose such edges. However, this overcounts cases where all 3 edges are bottlenecks in the same column. Suppose all 3 are distinct heights. Then the given triple of heights has been counted 3 times, so we must subtract it twice, contributing a $-\sum_{i=1}^{n} 2\binom{b_i^{(1)}}{3}$ term. If 2 of the heights are the same and the last is distinct, then the given triple of heights has been counted twice. Subtracting gives a term $-\sum_{i=1}^{n} b_i^{(1)}(b_i^{(1)}-1)$. Finally, if all 3 heights are the same then the triple has beeen counted only once, so we need not subtract anything for this case. This gvies the first sum in the desired formula.

Suppose 3 heights i, j, k are chosen such that no two heights are bottlenecks in the same column and there are at least 3 columns containing edges at any of the 3 given heights. Then there will be a unique corresponding lattice path, given by choosing the leftmost edge at height i, the leftmost edge at height j after removing the squares above the first edge chosen, and then the leftmost edge at height k after removing the squares above the first two edges chosen.

Hence the only other way for a triple of heights to fail to give a valid filling is if there are only two columns containing edges at the 3 given heights. These are the terms involving the width two bottlenecks $b^{(2)}$. There are 3 possible cases. The first case is if all 3 edges are at heights that are width two bottlenecks in a given column. With repetition, this gives $\binom{b_i^{(2)}+2}{3}$ for each column. The second case is if 2 of the heights are at width two bottlenecks of a given column, and the last height is at one of the adjacent bottleneck edges. There are $\binom{b_i^{(2)}+1}{2}$ ways to choose the two width two bottlenecks and $(b_i^{(1)}+b_{i+1}^{(1)})$ ways to choose the remaining bottleneck edge, giving the second term in the sum. Finally, since we have already counted all the cases where two of the heights are bottlenecks in the same column, the remaining case is when one height is a bottleneck in column i, another height is a width two bottleneck in column i, and the last height is a bottleneck in column $i+1$. This gives the last term in the sum, giving the desired formula. $\square$

The proof of Proposition 3.19 may possibly be adapted to terms of degree $n+2$ with more x_1's, but the argument seems to become increasingly complicated. Similarly to the formulas for the terms of degree $n+1$ described in the proof of Theorem 3.10, the coefficients in the sum of terms $\binom{b_i^{(1)}+1}{2}$ will not all be 1. However, other terms in the formula become more complicated as well. For example, take $x_1^4x_2^{n-2}$, the next simplest monomial. Rather than terms involving just the first and last column height, the formula would also require terms involving the second and second to last column heights. Also, rather than just a sum over adjacent pairs b_ib_{i+1}, a sum over all products b_ib_j is necessary, as well as another sum over pairs $b_i b_{i+2}$. Given this drastic increase in complexity involving the interplay of several different terms, it is unclear that computing the coefficients of more complicated monomials will lead to nice conditions on the bottlenecks $b^{(w)}$.

4. Coincidences of Ribbon Stable Grothendieck Polynomials

The combinatorics of stable Grothendieck polynomials of ribbon seem to be more difficult than their dual stable Grothendieck and Schur counterparts.

For example, it is not immediate that the product of two ribbon Grothendieck polynomials can be expanded out as a sum of other ribbon Grothendieck polynomials, as is the case for dual stable Grothendieck polynomials. As one may recall, the proof method uses the fact that any two ribbon reverse plane partitions can be concatenated in some manner to form a new ribbon reverse plane partition. This does not extend to set-valued tableaux since the boxes along which we must concatenate may contain incomparable entries.

However, we still conjecture that coincidences among ribbon Grothendieck polynomials arise in exactly the same way as the dual case. Let α, β be ribbons.

Conjecture 4.1. *$G_\alpha = G_\beta$ if and only if β is equal to α or α^*.*

While the "if" direction is immediate, the "only if" direction has proven to be much more difficult. Here we document a few of the approaches taken.

4.1. Barely Set-Valued Tableaux.

For a ribbon α of size n, the $x_1 x_2 \dots x_{n+1}$-coefficient in G_α counts the number of *barely set-valued tableaux* on α, which are defined to be set-valued tableaux containing the numbers $1, \dots, n+1$.

An enumerative formula is given in [3] for the number of these tableaux on any skew shape. Given a skew shape σ, let f^σ be the number of standard Young tableaux on σ. Furthermore, let ${}^i\sigma$ and i^σ be the shapes obtained by adding a box to the right and left of row i in σ respectively. We define f^{σ^i} and $f^{{}^i\sigma}$ analogously as the number of standard Young tableaux on these shapes, although we set these quantities to 0 if the superscript is not a skew shape.

Lemma 4.2. [3] *The number of barely set-valued tableaux for a skew shape σ with n boxes and k rows is*

$$k(n+1)f^\sigma + \sum_{i=1}^{k}(k-i)f^{{}^i\sigma} - \sum_{i=1}^{k}(k-i+1)f^{\sigma^i}.$$

The authors attempted to evaluate this formula for ribbons by expanding the f^σ terms with Aitken's determinantal formula for counting skew standard Young tableaux but it proved to be impossible to obtain a closed form.

Furthermore, one should note that the number of barely set-valued tableaux on the ribbons $(1,2)\circ(2,1)$ and $(2,1)\circ(1,2)$ are the same, so this is not a strong enough invariant to prove our conjecture.

4.2. Unstable Grothendieck Polynomials.

Another attempt at unraveling the structure of Grothendieck polynomials is to examine their unstable versions as defined in [2].

These unstable polynomials are defined recursively for general permutations. Define the length $l(w)$ of a permutation w to be the smallest number k such that w can be expressed as a product of simple reflections $s_{i_1}s_{i_2}...s_{i_k}$ where s_i is the transposition $(i\ i+1)$.

Let $w_0 \in S_n$ be the longest permutation sending 1 to n, 2 to $n-1$, and so on.

Definition 4.3. The *(single-variable) Grothendieck polynomial* $\mathcal{G}_{w_0}(x)$ is the monomial

$$\prod_{1\le i\le n-1} x_i^{n-i}.$$

Now let w be any other permutation in S_n. Since w is not the longest permutation, there exists s_i such that $l(ws_i) = l(w) + 1$. Now we can define the Grothendieck polynomial for general w.

Definition 4.4. The *Grothendieck polynomial* $\mathcal{G}_w(x)$ is given by the recursive identity

$$\pi_i(\mathcal{G}_{ws_i}(x))$$

where π_i is the isobaric divided difference operator

$$\pi_i(f) = \frac{(1 - x_{i+1})f - (1 - x_i)f(x_1, x_2, \ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}.$$

It is known that the choice of s_i does not matter. We can obtain the stable Grothendieck polynomial by taking the limit

$$G_w = \lim_{m\to\infty} \mathcal{G}_{1^m\times w}$$

where $1^m \times w$ is the permutation in S_{m+n} fixing $1, 2, ..., m$ and sending $m + i$ to $m + w(i)$ for i between 1 and n, inclusive.

We now defined $\mathcal{G}_{\lambda/\mu)}$. The unstable Grothendieck polynomial $\mathcal{G}_{\lambda/\mu}$ is given by $\mathcal{G}_{w_{\lambda/\mu}}$ where $w_{\lambda/\mu}$ is a permutation determined by λ and μ.

Definition 4.5. Given a skew diagram λ/μ, define the permutation $w_{\lambda/\mu} \in S_{|\lambda/\mu|}$ as follows.

Starting from the lower-left box, fill in each square in λ/μ with the number of its corresponding diagonal, increasing as we travel up or right in the diagram. An example for $\lambda = (4, 3, 3, 3), \mu = (1)$ is given below:

	5	6	7
3	4	5	
2	3	4	
1	2	3	

Then, set $w_{\lambda/\mu}$ to be the permutation with reflection word given by reading the diagram from right to left, bottom to top. For example, we have

$$w_{(4,3,2)/(1)} = s_2s_1s_4s_3s_2s_6s_5s_4.$$

In a ribbon $\alpha = (\alpha_1, \cdots, \alpha_k)$, no two squares share the same diagonal. Therefore, we have

$$w_\alpha = s_{\alpha_1}s_{\alpha_1-1} \ldots s_1s_{\alpha_2+\alpha_1}s_{\alpha_2+\alpha_1-1} \ldots s_{\alpha_1+1} \ldots s_{|\alpha|}s_{|\alpha|-1} \ldots s_{(\sum_{i=1}^{k-1}\alpha_i)+1}$$

It is straightforward to calculate a sequence of reflections that send $1^m \times w_\alpha$ to the longest permutation.

Proposition 4.6. *Define B_i from $i = 1$ to k as the composition of the reflection word*

$$s_{|\alpha|-\alpha_i+m}s_{|\alpha|-\alpha_i+m-1} \cdots s_{1+\sum_{j=i+1}^k \alpha_j}$$

with the words $s_{|\alpha|}...s_p$ from $p = 2 + \sum_{j=i+1}^k \alpha_j$ to $\sum_{j=i}^k \alpha_j$.

Define M to be the composition of reflections

$$s_{|a|+m}s_{|a|+m-1}\cdots s_{|a|+1}s_{|a|+m}\cdots s_{|a|+2}\cdots s_{|a|+m}s_{|a|+m-1}s_{|a|+m}.$$

Then $w_\alpha B_1 B_2 \dots B_k M = w_0$.

Proof. It suffices to plug in the definitions of w_α, B_i, and M to verify that their composition is the longest word w_0. $\square$

From the above, we have a purely algebraic expression for the Grothendieck polynomial of a ribbon. Although the unstable polynomials do not have nice expressions and are not symmetric in general, they may be useful for comparing coefficients.

4.3. **Littlewood-Richardson Rule.** The stable Grothendieck polynomials on straight shapes form a basis for the ring of symmetric power series. Therefore, we can compare skew stable Grothendieck polynomials by comparing their expansions as a sum of straight stable Grothendieck polynomials.

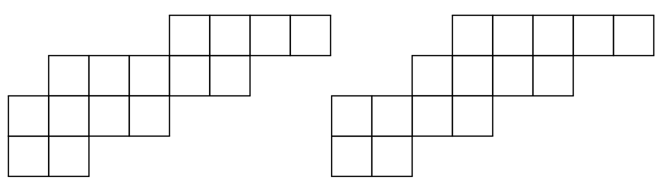
Given a set-valued tableaux, define its *column word* to be its entries read in a bottom-to-top, left-to-right order, where each square is read in increasing order. This expansion is given by the following Littlewood-Richardson rule:

Theorem 4.7. [2] $G_{\lambda/\mu} = \sum_{\nu \text{ a partition}} a_{\lambda/\mu,\nu} G_\nu$ *where the coefficient $a_{\lambda/\mu,\nu}$ is the number of set-valued tableaux whose column word is a reverse lattice word of weight ν.*

Since the column word coincides with the row word used in the original Littlewood-Richardson rule, the coefficients $a_{\alpha,\nu}$ for $|\nu| = |\alpha|$ are exactly the Littlewood-Richardson coefficients for α. While the counting set-valued tableaux giving reverse lattice words is more restricted than simply counting set-valued tableaux, based on initial attempts it seems still difficult to enumerate these coefficients.

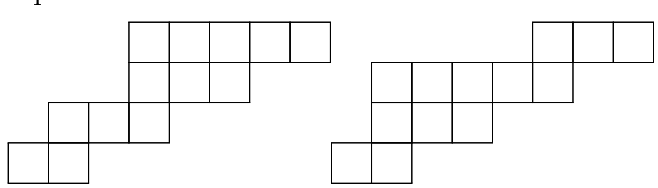
4.4. **Relation Between g equivalence and G equivalence.** It is natural to ask whether $g_A = g_B$ for two skew shapes A and B gives information about whether $G_A = G_B$, and vice versa. The following examples show that in general, neither equality implies the other.

Example 4.8. Based on computation, the shapes



are g-equivalent but not G equivalent. For example, the coefficients of $x_1^6 x_2^6 x_3^3 x_4$ in G are -353 and -354 respectively.

Example 4.9. The shapes



are G equivalent but not g equivalent. For example, we can see $b_4 + b_5 = 1$ for the shape on the left and $b_4 + b_5 = 0$ for the shape on the right.

5. Future Explorations

For skew shapes A, B it follows immediately from the Jacobi-Trudi identity that if $s_A = s_B$ then $s_{A^T} = s_{B^T}$. Since it remains to show an analogue of Jacobi-Trudi for skew Grothendieck polynomials, the answer to the analogous question for g and G is less obvious.

Question 5.1. Suppose $g_A = g_B$. Does it follow that $g_{A^T} = g_{B^T}$? Similarly, suppose $G_A = G_B$. Does it follow that $G_{A^T} = G_{B^T}$?

If conjugation does preserve g equivalence, then we immediately get another necessary condition on g equivalence by taking a transposed version of Theorem 3.10.

5.1. Ribbon Staircases.

Theorem 7.30 of [5] describes a class of nontrivial skew equivalences. A *nesting* is a word consisting of the symbols left parenthesis "(," right parenthesis ")," dot "." and vertical slash "|" where the parentheses must be properly matched. Given a skew shape which may be decomposed into a ribbon α in a certain manner as described in [5], one may attain a corresponding nesting. Theorem 7.30 states that shapes which may be decomposed with the same ribbon α such that the nestings are reverses of each other are Schur equivalent.

It is interesting to consider whether these equivalences hold for g and G as well. For example, Corollary 7.32 of [5] states for any diagram μ contained in the staircase partition $\delta_n = (n-1, n-2, \ldots, 1)$ it holds that $s_{\delta_n/\mu} = s_{\delta_n/\mu^t}$. Computation strongly suggests the same holds true for the Grothendieck polynomials as well.

Conjecture 5.2. Let μ be a diagram contained in the staircase partition $\delta_n = (n-1, n-2, \ldots, 1)$. Then $g_{\delta_n/\mu} = g_{\delta_n/\mu^t}$ and $G_{\delta_n/\mu} = G_{\delta_n/\mu^t}$.

However, not all equivalences described by Theorem 7.30 match. Further, the equivalences for g and G are different. For example, let $\alpha = (2, 3)$ and take the nesting

$$\begin{array}{cc} | & . \\ 1 & 2 \end{array}$$

and its reverse

$$\begin{array}{cc} . & | \\ 1 & 2 \end{array}$$

Then the corresponding skew shapes are exactly those given in Example 4.9; they match for G but not for g. Based on computation, for $\alpha = (2, 3)$ it appears that the skew shapes are G equivalent if and only if the nesting contains only vertical slashes and dots. However, this does not hold for all ribbons α. For example, take $\alpha = (1, 3)$. Then the shapes given by the nesting

$$\begin{array}{cc} | & . \\ 1 & 2 \end{array}$$

and its reverse do not give G equivalent skew shapes. It also remains to find any examples of equivalences for g besides those given in Conjecture 5.2. This leads to the following question.

Question 5.3. For which ribbons α and nestings $\mathcal{N}$ are the corresponding shapes g equivalent or G equivalent?

5.2. **Littlewood-Richardson Rule.** The stable Grothendieck polynomials on straight shapes form a basis for the ring of symmetric power series. Therefore, we can compare skew stable Grothendieck polynomials by comparing their expansions as a sum of straight stable Grothendieck polynomials.

Given a set-valued tableaux, define its *column word* to be its entries read in a bottom-to-top, left-to-right order, where each square is read in increasing order. This expansion is given by the following Littlewood-Richardson rule:

Theorem 5.4. [2] $G_{\lambda/\mu} = \sum_{\nu \text{ a partition}} a_{\lambda/\mu,\nu} G_\nu$ *where the coefficient* $a_{\lambda/\mu,\nu}$ *is the number of set-valued tableaux whose column word is a reverse lattice word of weight* ν.

While the counting set-valued tableaux giving reverse lattice words is more restricted than simply counting set-valued tableaux, it is still difficult to enumerate these coefficients. Since the column word coincides with the row word used in the original Littlewood-Richardson rule, the coefficients $a_{\alpha,\nu}$ for $|\nu| = |\alpha|$ are exactly the Littlewood-Richardson coefficients for α.

5.3. G **positivity.** We observed in many examples of ribbons α, β such that $s_\alpha = s_\beta$ and $G_\alpha \neq G_\beta$ that not only are the Littlewood-Richardson coefficients different, the coefficients of one shape are dominated by the coefficients of the other. This leads us to define the following analog of Schur positivity.

Definition 5.5. A symmetric function is G *positive* if the coefficients for its expansion in the basis $\{G_\lambda\}$ are all nonnegative. For example, the Littlewood-Richardson rule given previously shows G_A for a skew shape A is G positive. We define a partial order on G equivalence classes of skew shapes by $A \leq B$ if and only if $a_{A,\nu} \leq a_{B,\nu}$ for all ν. Equivalently, $A \leq B$ if and only if $G_B - G_A$ is G positive.

This partial order is a coarsening of the order $A \leq B$ if and only if $s_B - s_A$ is Schur positive. Our poset inherits many of the properties of the more well-studied poset given by Schur positivty. For example, when restricted to ribbons, the connected components of the poset are the sets of ribbons with a fixed number of rows and columns. Each component has a least element, the unique hook shape with the given number of rows and columns. Since the poset is a coarsening of the Schur positivity order, the same argument in Proposition 3.10 of [4] shows ribbons that are permutations of some fixed partition λ form a convex subposet of the poset of all ribbons. In some cases, the poset is highly structured.

We pose the following conjectures and questions about our G positivity order.

Conjecture 5.6. In addition to a least element, the set of ribbons with a fixed number of rows and columns has a greatest element.

Conjecture 5.7. For fixed λ, the set of ribbons which are permutations of λ has both a least and a greatest element.

Question 5.8. Computationally, the set of ribbons which are permutations of some fixed λ seems the follow the pattern that ribbons with the larger rows in the middle of the shape are larger in the poset. Is there a general rule that governs this phenomenon?

Conjecture 5.9. Like in the Schur positivity order, conjugation gives acts as an automorphism.

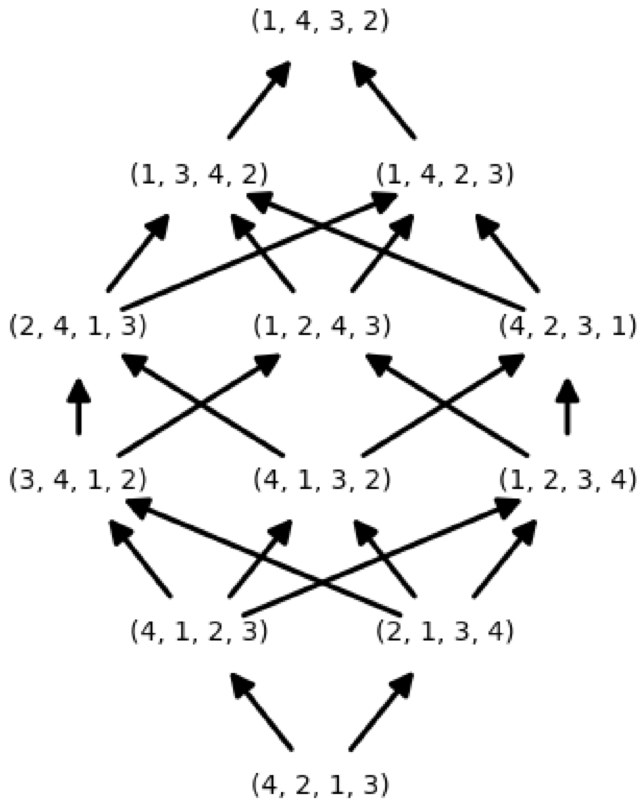


FIGURE 12. The Hasse diagram for all ribbons given by permutations of $(4, 3, 2, 1)$. It has 12 vertices, since each ribbon is equivalent to its 180 degree rotation. They correspond to distinct G_α.

Question 5.10. Are there ribbons α and β such that $s_\alpha = s_\beta$ and $G_\alpha \neq G_\beta$ but α and β are incomparable?

6. Acknowledgments

This research was carried out as part of the 2016 summer REU program at the School of Mathematics, University of Minnesota, Twin Cities, and was supported by NSF RTG grant DMS-1148634 and by NSF grant DMS-1351590. We would like to thank Rebecca Patrias and Sunita Chepuri for their mentorship and support.

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