Parametrizations of k-Nonnegative Matrices

Anna Brosowsky, Neeraja Kulkarni, Alex Mason, Joe Suk, Ewin Tang*

August 20, 2017

Abstract

Totally nonnegative (positive) matrices are matrices whose minors are all nonnegative (positive). In this paper, we search for parametrizations of k-nonnegative and k-positive matrices. A k-nonnegative (resp. k-positive) matrix has all minors of size k or less nonnegative (resp. positive). We give a generating set for these k-nonnegative matrices, as well as relations for certain special cases, i.e. the $k = n-1$ and $k = n-2$ unitriangular cases. In the above two cases, we find that the set of k-nonnegative matrices can be partitioned into cells, analoguous to the Bruhat cells of totally nonnegative matrices, based on their factorizations into generators. We will show that cells, like the Bruhat cells, are homeomorphic to open balls, and we prove some results about the topological structure of the closure of these cells. In addition, we have a set of k-positivity tests, which imply that the sets of k-positive matrices are in bijection with 1-positive matrices for all k. These tests give rise to cluster algebras which live in the total positivity cluster algebra.

1 Introduction

A totally nonnegative (respectively totally positive) matrix is a square matrix whose minors are all nonnegative (respectively positive). Total positivity and nonnegativity are well-studied phenomena and arise in areas such as planar networks, combinatorics, dynamics, statistics and probability. The study of total positivity and total nonnegativity admits of many and varied applications, some of which are explored in “Totally Nonnegative Matrices” by Fallat and Johnson [3].

In this report, we generalize the notion of total nonnegativity and positivity as follows. A *k-nonnegative* (resp. *k-positive*) matrix is a matrix where all minors of order k or less are nonnegative (resp. positive). With a view to producing results for k-nonnegative and

*This research was carried out as part of the 2017 REU program at the School of Mathematics at University of Minnesota, Twin Cities. Anna Brosowsky, Alex Mason, Joe Suk and Ewin Tang are grateful for the support of NSF RTG grant DMS-1148634, and Neeraja Kulkarni for the support of NSF grant DMS-1351590. The authors would like to thank Sunita Chepuri, Pavlo Pylyavskyy, and Victor Reiner for their guidance and encouragement. They are also grateful to Elizabeth Kelley for her many helpful comments on the manuscript and Connor Simpson for sharing his insight on the problem.

k-positive matrices such as those that already exist for totally nonnegative and totally positive matrices, we will consider all matrices in this report to be invertible.

Following the lead of Fomin and Zelevinsky in [9], we consider the following two questions:

(1) How can k-nonnegative matrices be parametrized?

(2) What are positivity tests for k-positive matrices?

To answer these questions. at least to some extent, we provide factorizations and relations for certain k, describe Bruhat cells for k-nonnegativity and give a cluster algebra framework for finding k-positivity tests.

In Section 2 we detail our most general results on factorizations of k-nonegative matrices, after describing some relevant background. We describe partial factorizations of k-nonnegative matrices and partly characterize the location of zero minors in the matrix.

In Section 3 we explore two special cases: $(n-1)$-nonnegative $n \times n$ matrices and $(n-2)$-nonnegative unitriangular $n \times n$ matrices, giving a specific generating set for the semigroup of k-nonnegative $n \times n$ real matrices as well as a set of relations. It is possible to define analogues of Bruhat cells, which share many properties of the standard cells of totally nonegative matrices, for k-nonnegative matrices in these special cases. Section 3 concludes by detailing some progress towards understanding the topology of the cells of k-nonnegative matrices.

In Section 4, we describe the k-positivity cluster algebras. These are sub-cluster algebras of the well-known total positivity cluster algebra. We show a method of deriving k-positivity tests from a family of these cluster algebras as well as a convenient indexing of this family in terms of Young diagrams. A representative double wiring diagram is also given for each member of the family, which can be mutated according to a subset of the typical rules to get other k-positivity tests.

Further questions to be explored with regard to factorizations include both questions on the topological structure of cells of k-nonnegative matrices for our special cases (i.e. $k = n - 1$ and $k = n - 2$) and questions on the locations of zero minors in k-nonnegative matrices for general k. Approaches to give generating sets for all k have also not been explored.

Further work to be pursued with regard to cluster algebras includes characterizing which minors are included in every k-positivity test (in particular resolving Conjecture 4.2), determining whether there are additional sub-cluster algebras outside of our known family which also give tests, and finding a concise characterization of which double wiring diagrams can be modified to give k-positivity tests.

2 Preliminaries

2.1 Background

We begin by establishing some notation that will be used throughout the paper. For any matrix X, $X_{I,J}$ refers to the submatrix of X indexed by a subset of its rows I and a subset of its columns J, and $|X_{I,J}|$ will refer to the minor, i.e. the determinant of this submatrix. We say a minor $|X_{I,J}|$ is of *order* k if $|I| = |J| = k$. We also use $[n]$ to refer to the set $\{1, \ldots, n\}$. Thus, $X_{[m],[m]}$ is the submatrix formed by taking the first m rows and columns.

The set of all totally nonnegative (resp. totally positive) matrices with real entries is closed under multiplication and, thus, forms a semigroup. This can be seen from the following identity:

Theorem 2.1 (Cauchy-Binet Identity). *Let X be an $n \times m$ matrix and let Y be a $m \times n$ matrix with $n \leq m$. Then, we have*

$$\det(XY) = \sum_{J \subseteq [m], |J|=n} |X_{[n],J}||Y_{J,[n]}|$$

In particular, a consequence of the Cauchy-Binet Identity is that minors corresponding to submatrices of XY of a given order can be related to the minors corresponding to submatrices of X and Y of the same order. If we restrict our attention to invertible square matrices, then it follows that the set of invertible totally nonnegative (resp. totally positive) matrices also form a semigroup. Similarly, the set of all upper unitriangular and the set of all lower unitriangular totally nonnegative matrices are both semigroups. Knowing this, two natural questions arise:

(1) How do we parametrize the set of all totally nonnegative invertible matrices?

(2) How do we test a given matrix in $\mathrm{GL}_n(\mathbb{R})$ for total positivity?

We first summarize the known answers to these two questions and then, in the remainder of the paper, discuss the answers to these questions in the context of k-nonnegative and k-positive invertible matrices. Most of the following can be found in [9]. We start by discussing the relationship between a generic totally nonnegative invertible matrix and totally nonnegative unitriangular matrices.

Theorem 2.2 (LDU Factorization). *Let X be an invertible $n \times n$ totally nonnegative matrix. Then, one can write*

$$X = LDU$$

where L is a lower unitriangular matrix, D is a diagonal matrix, U is an upper unitriangular matrix, and L, D, U are totally nonnegative.

In order to answer (1), we would first like to find the generators of the semigroup of totally nonnegative matrices. A *Chevalley generator* is a particular type of matrix which differs from the identity matrix in precisely one entry. We use $e_i(a)$ to denote matrices that only differ by an $a > 0$ placed in the $(i, i+1)$-st entry and $f_i(a)$ to denote matrices that differ by an $a > 0$ placed in the $(i+1, i)$-st entry. More generally, *elementary Jacobi matrices* differ from the identity in exactly one entry either on, directly above, or directly below the main diagonal. Thus, an elementary Jacobi matrix is a Chevalley generator or a diagonal matrix which differs from the identity in one entry on the main diagonal. The following result is from Loewner and Whitney:

Theorem 2.3 (Whitney's Theorem; Theorem 2.2.2 of [3])**.** *Any invertible totally nonnegative matrix is a product of elementary Jacobi matrices with nonnegative matrix entries.*

In fact, using Theorem 2.2, we can say more about the factorization given in Theorem 2.3 using the following.

Corollary 2.4. *An upper unitriangular matrix X can be factored into Chevalley generators $e_i(a)$ with nonnegative parameters $a \geq 0$. Similarly, a lower unitriangular matrix X can be factored into Chevalley generators $f_i(a)$ with nonnegative parameters $a \geq 0$.*

Next, in order to discuss parametrizations of totally nonnegative matrices, we first need to discuss the relations between these generators. Let $h_k(a)$ be a square matrix which differs from an identity matrix at the single entry (k, k) where it takes on the value a. We have

Theorem 2.5 (Theorems 1.9 and 4.9 of [8])**.** *The following identities hold:*

- $h_{k+1}(a)e_k(b) = e_k(b/a)h_{k+1}(a)$
- $h_k(a)e_k(b) = e_k(ab)h_k(a)$
- $h_k(a)e_j(b) = e_j(b)h_k(a)$ *if* $k \neq j, j+1$
- $h_{k+1}(a)f_k(b) = f_k(ab)h_{k+1}(a)$
- $h_k(a)f_k(b) = f_k(b/a)h_k(a)$
- $h_k(a)f_j(b) = f_j(b)h_k(a)$ *if* $k \neq j, j+1$
- $e_i(a)f_j(b) = f_j(b)e_i(a)$ *if* $i \neq j$
- $e_i(a)f_i(b) = f_i(b/(1+ab))h_i(1+ab)h_{i+1}(1/(1+ab))e_i(a/(1+ab))$

For the semigroup of unitriangular totally nonnegative matrices, the relations are much simpler:

Theorem 2.6 (Section 2.2 of [8])**.** *These following identities hold:*

- $e_i(a)e_{i+1}(b)e_i(c) = e_{i+1}\left(\frac{bc}{a+c}\right)e_i(a+c)e_{i+1}\left(\frac{ab}{a+c}\right)$
- $e_i(a)e_j(b) = e_j(b)e_i(a)$ *if* $|i-j| > 1$

They also hold if we replace $e_i(\cdot)$ with $f_i(\cdot)$.

The relations obeyed by the e_i's are the same as the braid relations between adjacent transpositions in the Coxeter presentation of the symmetric group. The strong Bruhat order of the symmetric group determined by these relations is deeply connected to parametrizations of totally nonnegative matrices and totally nonnegative unitriangular matrices. More information about this can be found in Section 3 - Factorizations 3.

Next, we list known results about positivity tests for $\mathrm{GL}_n(\mathbb{R})$ matrices along with the necessary definitions. Most of these can be found in “Totally Positive Matrices” [12].

Lemma 2.7 (Fekete [5])**.** *Assume X is an $n \times m$ matrix with $n \geq m$ such that all minors of order $m-1$ with columns $[m-1]$ are positive and all minors of size m with consecutive rows are positive. Then all minors of X of order m are positive.*

A minor $|X_{I,J}|$ is called *solid* if both I and J consist of several consecutive indices. $|X_{I,J}|$ is called *initial* if it is solid and $\{1\} \in I \cup J$. $|X_{I,J}|$ is called *column-solid* if J consists of several consecutive indices and *row-solid* if I consists of several consecutive indices.

Theorem 2.8. *Assume all solid minors of X are positive. Then X is totally positive.*

In fact, checking a smaller set of these minors will suffice.

Theorem 2.9. *Assume all solid minors of X with rows $[k]$ and also all solid minors of X with columns $[k]$ are positive for $k = 1, 2, \ldots$. Then X is totally positive.*

A minor $X_{I,J}$ is called a principal leading minor of order k if $I = J = [k]$. Although these minors do not give a positivity test for $\mathrm{GL}_n(\mathbb{R})$ matrices, they satisfy another strong condition.

Theorem 2.10 (Lemma 15 of [9])**.** *The leading principal minors $X_{[k],[k]}$ of an invertible totally nonnegative matrix X are positive for $k = 1, \ldots, n$.*

2.2 Equivalent Conditions and Elementary Generalizations

A natural question that arises when discussing k-nonnegative matrices (or, more generally, when discussing any condition on a matrix's minors) is whether we need to check all minors (usually an intractable computation), or just some subset of minors. For example, a well-known result, from [9], is that only column-solid minors are necessary to determine total nonnegativity.

The following three statements from [3] provide satisfactory answers to this question. While we independently proved these results, our proofs differ insignificantly from the above source, and so are not presented here.

We will generally assume all matrices are pulled from $GL_n(\mathbb{R})$; that is, they are invertible and square. Note that we sometimes abbreviate totally nonnegative as TNN and k-nonnegative as kNN. However, the following three statements hold true for matrices in the space of all $m \times n$ matrices as well as matrices in $GL_n(\mathbb{R})$.

Theorem 2.11 ([3] 3.1.6)**.** *If all solid minors of X of order at most k are positive, then X is k-positive.*

Theorem 2.12 ([4] 2.5)**.** *k-positive matrices are dense in the class of k-nonnegative matrices.*

Notice that this holds in the invertible case because invertible matrices are an open subspace in the space of all matrices.

Theorem 2.13 ([4] 2.3)**.** *If all initial minors of X of order at most $k-1$ are positive and all solid order k minors of X are positive, then X is k-positive.*

We will reword the above theorem in a way that will prove useful in Section 4.

Definition. The *k-initial minor matrix M* of a matrix X is defined as follows:

$$M_{ij} = \left|X_{[i-\ell+1,i],[j-\ell+1,j]}\right| \qquad \text{where } \ell = \min(k,i,j)$$

In other words, the value at position (i,j) is the value of the solid minor of largest order not exceeding k, such that (i,j) is the lower right corner of the corresponding submatrix.

For example, the n-initial minor matrix (also referred to as just the initial minors matrix) contains all of the initial minors of X, and a 1-initial minor matrix contains all of the entries of X.

Notice that the k-initial minor matrix gives us exactly the minors for the above necessary condition of k-positivity.

Corollary 2.14. *Let X be a matrix. Then X is k-positive if and only if the k-initial minor matrix has all positive entries.*

Remark 2.15. By a slight modification of the proof of Lemma 7 of [9], any choice of positive k-initial minors uniquely determines a matrix. This gives us an explicit bijection between k-positive matrices and 1-positive matrices, and therefore we have bijections between k-positive matrices and j-positive matrices.

Now, we present the following k-nonnegativity test. We have not found this in the literature, but it follows from a known proof technique, presented here.

Theorem 2.16. *For X a k-nonnegative and invertible matrix, if all column-solid minors of X of order at most k are nonnegative, then X is k-nonnegative. Same for row-solid minors.*

Proof. Let $Q_n(q) = (q^{(i-j)^2})_{i,j=1}^n$ for $q \in (0,1)$. This matrix has the two nice properties that it is totally positive and $\lim_{q\to 0^+} Q_n(q) = I_n$. Let $X_q = Q_n(q)X$. Apply Cauchy-Binet on an order $r \leq k$ column-solid minor:

$$|(X_q)_{I,J}| = \sum_{\substack{S\subset[n] \\ |S|=r}} |Q_n(q)_{I,S}|\,|X_{S,J}|$$

This must be positive, from column-solid minors of X being nonnegative and invertibility of X. By 2.11 X_q must be k-positive. Taking limit $q \to 0^+$ we conclude that X is k-nonnegative. To get the analogous statement for row-solid minors, use $X_q = XQ_n(q)$. □

We will also occasionally use these matrix maps, which have been shown to preserve k-nonnegativity.

Proposition 2.17 ([3] 1.4.1)**.** *The following (linear) maps preserve k-nonnegativity and k-positivity:*

- $A \mapsto A^T$
- $A \mapsto \tilde{T}A\tilde{T}$ *for* $\tilde{T}$ *the permutation matrix of* $w_0 = (n\; n{-}1 \cdots 2\; 1)$ *(that is,* $\tilde{T}\vec{e}_i = \vec{e}_{n-i+1}$*).*

Now, we will discuss some more complex results. First, we give some findings regarding the locations of zero minors, minors evaluating to zero, in k-nonnegative matrices. Again, recall that we are only working with invertible matrices. The following generalizes 1.6.5 from [3]:

Theorem 2.18. *Let M be a k-nonnegative matrix, and let $M_{I,J}$ be a rank $c-1$, size c solid submatrix for $c < k$, with $I = [i, i+c-1]$ and $J = \{j, j+c-1\}$. Then either:*

- $|M_{I',J'}| = 0$ *for* $I' \subset [1, i+c-1]$*,* $J' \subset [j, n]$*, and* $|I'| = |J'| = c$*; or*
- $|M_{I',J'}| = 0$ *for* $I' \subset [i, n]$*,* $J' \subset [1, j+c-1]$*, and* $|I'| = |J'| = c$*.*

In other words, if we have a zero minor of the form described, all of the minors of the same order, consisting only of rows and columns *further* from the diagonal than our zero minor, are zero.

Proof. Consider the sets of submatrices:

$$\begin{aligned}\mathcal{A} &= \{M_{i'\cup I, j'\cup J} \mid i' < i, j' < j\}\\ \mathcal{B} &= \{M_{I\cup i', J\cup j'} \mid i+c-1 < i', j+c-1 < j'\}\end{aligned}$$

Notice that $|M_{I,J}|$ is a principal minor for any $A \in \mathcal{A}$ and the antidiagonal flip of a principal minor for $B \in \mathcal{B}$. Because all matrices in $\mathcal{A} \cup \mathcal{B}$ are TNN (after all, size $c+1 \leq k$ submatrices of a kNN matrix), and $M_{I,J} = 0$, by 2.10 all matrices in $\mathcal{A} \cup \mathcal{B}$ are singular.

For some $A \in \mathcal{A}$, consider the submatrix formed by removing the first row (call this A_r) and the one formed by removing the first column (call this A_c). Then if $M_{I,J}$ is rank $c-1$, A_r and A_c are either rank $c-1$ or rank c.

If A_r and A_c are both rank c, then A is rank $c+1$. This is a contradiction, since A is not invertible. Further, suppose there exists some $\alpha, \beta \in \mathcal{A}$, corresponding to adding the rows and columns $i_\alpha, j_\alpha, i_\beta, j_\beta$, such that α_r and β_c are both rank c. Then the element $\gamma \in \mathcal{A}$ corresponding to i_β and j_α has both γ_r and γ_c rank c, giving us a contradiction (since γ is therefore invertible).

Thus, for all $A \in \mathcal{A}$, either A_r or A_c is rank $c-1$. The same applies for $B \in \mathcal{B}$. If a_c and b_c are both always rank $c-1$, then M is not invertible (since the submatrix $M_{[n],J}$ is of rank $c-1$), giving us a contradiction. By the same logic, we cannot have both a_r and b_r rank $c-1$. So it either a_c and b_r or a_r and b_c are always rank $c-1$. These are symmetric cases corresponding to the two cases in the theorem.

Without loss of generality, consider a_c and b_r. Then we know that $M_{k,J}$ for $k < i$ can be expressed as a linear combination of the $c-1$ linearly independent rows of $M_{I,J}$, and correspondingly for $M_{I,\ell}$ with $\ell > j+c-1$ and the $c-1$ linearly independent columns of $M_{I,J}$. Thus, if we consider any submatrix using those rows and J or the columns and I, it has zero determinant.

Using the same argument as above, we can get that any minor using those rows and those columns are 0, as desired. ☐

We can generalize the above statement to some extent.

Definition. Consider an $n \times n$ matrix M. Let a $c \times d$ submatrix with rank r have the *bound property* if for the corresponding indices $I = \{i_1 < \cdots < i_c\}$ and $J = \{j_1 < \cdots < j_c\}$, one of the following is true:

- $M_{[1,i_1-1]\cup I, J\cup[j_c+1,n]}$ has rank r (call this submatrix a NE-bound);
- $M_{I\cup[i_c+1,n],[1,j_1-1]\cup J}$ has rank r (call this submatrix a SW-bound).

Notice that 2.18 precisely proves the bound property for a set of submatrices:

Corollary 2.19. *For M an $n \times n$ k-nonnegative matrix, solid submatrices of size c and rank $c-1$ have the bound property.*

And we can say more, using this more general framework:

Proposition 2.20. *For M an $n \times n$ k-nonnegative matrix, all solid submatrices that are not full rank (that is, whose rank are not maximal) have the bound property.*

Further, for invertible M, solid submatrices not of full rank have the bound property if and only if solid square submatrices of size c and rank $c-1$ have the bound property.

Proof. Consider a $c \times d$ solid submatrix A without full rank, with indices $I = [i, i+c-1]$ and $J = [j, j+d-1]$. Without loss of generality we assume $c \leq d$, so we know the rank of A is some $r < c$. Induct on c. Clearly, when $c = 1$, all A is a zero matrix. We know that the submatrix $[0]$ has the bound property. Thus, for any d, if it did not satisfy the bound property, this would mean that our zeros in A satisfy the bound property, but in different directions. This breaks invertibility of M.

Now assume the statement for $c-1$. We split into two cases: when $d = c$ and when $d > c$. In the first case, when $r = c-1$ we are done by 2.18. So suppose $r < c-1$; then consider the two submatrices of M (and of A) $A_0 = M_{[i,i_0],J}$ and $A_1 = M_{[i_1,i+c-1],J}$, where i_0 and i_1 are minimal and maximal, respectively, such that A_0 and A_1 is not of full rank. We know that the ranks of A_0 and A_1 (call these r_0 and r_1) are both at most r. Further, this means that A_0 is a $r_0+1 \times d$ matrix, and has the bound property by the inductive hypothesis. Similarly for A_1. Notice that if they have the bound property in the same direction, then A has the bound property (with the interesting consequence that either A_0 or A_1 has rank r). They cannot have the bound property in different directions, since otherwise $M_{[n]\times J}$ has rank less than d, and so M is singular.

Now consider the next case, when $d > c$. Consider the submatrices of A $A_k = M_{I,[j+k,j+k+c-1]}$ for $0 \leq k \leq d-c$, which are the maximal size square solid submatrices of A. Because A is not of full rank, A_k is also not of full rank. By the above, these have the bound property. If any of these have the bound property in different directions, then M is not invertible. Thus, they have the bound property in the same direction, and A must have the bound property.

This proof does not rely on k-nonnegativity, and the reverse direction of the implication is obvious. Thus, we have the if and only if statement as well. □

And further, we can say more about the bound property.

Proposition 2.21. *Let M be an invertible matrix. Consider a $c \times d$ solid submatrix A of M, with rank r less than the maximal rank $s = \min(c, d)$, with the bound property. Then none of the kth superdiagonals and subdiagonals of M intersect with A maximally, for $k < s - r$.*

Further, if the diagonals where M intersect with A maximally are subdiagonals, then A is a SW-bound. Otherwise, the diagonals are superdiagonals, and A is a NE-bound.

Proof. If any of the above conditions fail to hold, then we have that row operations can bring an element on the diagonal to zero, along with all of the entries to the NW or the SE. This implies that M is singular. □

This gives a nicer condition than the bound property for solid submatrices. However, we define the bound property as we do because we suspect that even nonsolid submatrices have it for kNN matrices, or at least some subset:

Conjecture 2.22. *For a k-nonnegative matrix M, all submatrices not of full rank have the bound property.*

We have proved this only for solid submatrices. It seems reasonable that Plucker relations provides a path for proof for the above. If the above conjecture turns out to be true, dependent only on solid submatrices having the bound property, then we can characterize the shape of the locations of nonzero minors solely by considering solid, square, rank $c - 1$, order c minors. With what we have proved it is easy to see that the shapes must be staircase patterns (or using the terminology of [3], in double echelon form), but it is less clear whether these determine the locations of all zero minors.

We can actually use this to prove a generalization of Theorem 3.1.10 from [3]. Call a minor a *corner minor* if it is indexed either by $[1, k]$ and $[n - k + 1, n]$ or $[n - k + 1, n]$ and $[1, k]$ for some k.

Corollary 2.23. *Suppose X is a $k + 1$-nonnegative $n \times n$ matrix. Then X is k-positive if and only if all corner minors of order $\leq k$ are positive.*

Further, X is $k + 1$-positive if and only if all corner minors of order $\leq k$ and all solid minors of order $k + 1$ are positive.

Proof. The forward implication is obvious. Prove the contrapositive of the backwards implication: suppose X is not k-positive. Then 2.11 gives us that some solid minor of order at most k is non-positive. By 2.20 the corresponding submatrix has the bound property, so a corner minor of the corresponding order must take value zero.

The final statement is from 2.11. $\square$

The description of generators for TNN matrices given by the Loewner-Whitney theorem has no obvious generalization for kNN matrices. However, these generators arise from the context of an LDU decomposition. We consider the algorithm for decomposition at an elementary level, and this provides a generalization of the proof that gives us a statement about how close we can get to completion of an LDU decomposition.

First, it will be useful to explicitly state the following technical lemma. It immediately follows from Cauchy-Binet.

Lemma 2.24. *Suppose X' is a matrix X, transformed through a Chevalley operation. Then minors of X' are equal to minors of X except in the following cases:*

1. *If $X' = Xe_k(a)$ (adding a copies of column k to column $k+1$) then $\left|X'_{I,J}\right| = |X_{I,J}| + a\left|X_{I,J\setminus k+1\cup k}\right|$ when J contains $k+1$ but not k;*
2. *If $X' = e_k(a)X$ (adding a copies of row $k+1$ to row k) then $\left|X'_{I,J}\right| = |X_{I,J}| + a\left|X_{I\setminus k\cup k+1,J}\right|$ when I contains k but not $k+1$.*

For f_k, swap $k+1$ and k in the above statements.

Now, we give a definition that will prove helpful in describing the LDU decomposition process.

Definition. Call a k-nonnegative matrix M k-irreducible if $M = RS$ in the semigroup of invertible kNN matrices implies $R, S \notin \{f_i(a), e_i(a) \mid a > 0\}$.

Theorem 2.25. *Every k-nonnegative matrix X can be factored into a product of finitely many Chevalley generators and a k-irreducible matrix.*

Proof. Suppose X is not k-irreducible. Then there is some inverse Chevalley operation we can perform to X, maintaining nonnegativity. Without loss of generality suppose $e_i(a)^{-1}X$ is k-nonnegative for some $i \in [n]$ and $a \in \mathbb{R}_{>0}$ (corresponding to removing a copies of row $i+1$ to row i). We claim it is possible to choose $b > 0$ so that $e_i(b+\delta)^{-1}X$ is not k-nonnegative for any $\delta > 0$.

We want to determine when $e_i(x)^{-1}X$ is k-nonnegative in terms of x. It suffices to consider row-solid order $\leq k$ minors containing row i and not row $i+1$. These determinants are linear functions d_γ in x of the form $A - xB$ for some minors A, B of X. Thus, $d^{-1}([0,\infty))$ is closed for any d and the intersection $\cap_\gamma d_\gamma^{-1}([0,\infty))$ is also closed and compact; there must be some d with an upper bound, since otherwise we would break invertibility of X. We know this set is nonempty because a is in it. Thus, there is a maximal b in the intersection and applying an inverse Chevalley with any greater value will make the quotient matrix not k-nonnegative. It is also clear that this maximal b is of the form A/B (i.e. the minimum such A/B). Thus, $e_i(b+\delta)^{-1}X$ is not k-nonnegative for any $\delta > 0$.

So, in this way, we factor out a Chevalley generator, leaving a matrix with one more zero minor of order at most k. We can iterate this process, which must stop eventually because the number of minors of size at most k is finite. The resulting matrix must be k-irreducible. $\square$

Note that while the above states that these k-irreducible matrices act "nicely", these will not give our desired minimal set of generators. In fact, since Chevalley generators are not commutative or normal (in the sense that multiplying a matrix on the left by a Chevalley is not equivalent to multiplying by a Chevalley on the right), we get cases where k-irreducible matrices can be factored into Xe_iY. Such a case is seen in the $k = n-1$ section.

Now, we describe the extent to which we can factor Chevalley matrices from a generic kNN matrix.

Theorem 2.26. *If a matrix A is k-nonnegative, we can express it as a product of Chevalley matrices (specifically, only e_is) and a single kNN matrix where the ij-th entry is zero when $|j-i| > n-k$.*

That is, if a matrix is k-irreducible, the ij-th entry is zero when $|j-i| > n-k$.

Proof. We use the following lemma:

Lemma 2.27. *Let A be a kNN matrix. Then for a_{ij} such that the following hold, either $a_{ij} = 0$ or $e_i(-a_{ij}/a_{i+1\,j})A$ is kNN.*

(1) $i < j$;

(2) $a_{xy} = 0$ for $x \leq i$ and $y \geq j$, not including a_{ij} itself;

(3) $i < k$.

So we can reduce our matrix to one where a_{ij} is zero by factoring out a Chevalley matrix.

Proof. First, notice that our row operation is well-defined, since $a_{i+1\,j} = 0 \implies a_{ij} = 0$ from 2.20 and (1). Further, notice that from 2.16 and 2.24 the only minors we need to worry about are those row-solid minors containing row i but not row $i+1$. From (2), this means that any I, J to check for nonnegativity has $I = [h, i]$ for some $h > i - x$ and J having no indices greater than or equal to j. Let I, J define such a minor. Then using 2.24,

$$\begin{aligned}
|e_i(-a_{ij}/a_{i+1\,j})A_{I,J}| &= |A_{I,J}| - \frac{a_{ij}}{a_{i+1\,j}} \left|A_{I \setminus i \cup i+1, J}\right| \\
&= \frac{1}{a_{i+1\,j}} \left(a_{i+1\,j} \left|A_{I,J}\right| - a_{ij} \left|A_{I \setminus i \cup i+1, J}\right|\right) \\
&= \frac{1}{a_{i+1\,j}} \left|A_{I \cup i+1, J \cup j}\right|
\end{aligned}$$

And because the minor is of order one greater than the order of the original minor, when we only have minors of order less than k, the resulting matrix must be k-nonnegative. This is true by (3). $\square$

We can consider iterating this factorization, using the criteria to find another entry to eliminate. The top-right corner satisfies the criterion for the lemma, and for a matrix where that entry is zero, the entry directly below satisfies the criterion, and so on. We can eliminate

$k-1$ entries in the last column, one by one top-down, then $k-2$ entries in the second-to-last, and continue until we all entries desired to zero. Take the transpose of everything in the above argument to get the zeros in the bottom-left corner. $\square$

Note that if we set $k = n$, we get the Loewner-Whitney theorem, so we have simply generalized an elementary proof for this. This particular elementary proof has not been found in the literature, and so may be of mild interest.

We can actually say slightly more. The following results from a simple application of 2.27.

Lemma 2.28. *Let M be a k-nonnegative k-irreducible matrix for $k > 2$. Then*

$$m_{ij} = 0 \implies \begin{cases} m_{i-1,j-1} = 0 & \text{if } i \leq k \text{ or } j \leq k \\ m_{i+1,j+1} = 0 & \text{if } i > n-k \text{ or } j > n-k \end{cases}$$

The other question about irreducibility is describing the minors that prevent Chevalley matrices from dividing k-nonnegative matrices. For the following small case we can do so as follows:

Lemma 2.29. *If a matrix M is 2-irreducible and invertible, then there is a solid minor of order 2 equal to 0 in every pair of consecutive rows and every pair of consecutive columns.*

Proof. Suppose that there are two columns $\vec{a}$ and $\vec{b}$ which have no solid 2-minor that is 0, i.e. $\det \begin{bmatrix} a_i & b_i \\ a_{i+1} & b_{i+1} \end{bmatrix} \neq 0$ for all i. Suppose that $\vec{c}$ is the column immediately to the right of $\vec{b}$. We now define two sets, S and T, as follows:

$$S = \{\det \begin{bmatrix} a_i & b_i \\ a_{i+1} & b_{i+1} \end{bmatrix} / \det \begin{bmatrix} a_i & c_i \\ a_{i+1} & c_{i+1} \end{bmatrix} : \det \begin{bmatrix} a_i & c_i \\ a_{i+1} & c_{i+1} \end{bmatrix} \neq 0\}.$$
$$T = \{b_i/c_i : c_i \neq 0\}.$$

We now define ϵ to be the minimum value of $S \cup T$. (Note that since M is invertible, both S and T are nonempty, so this definition makes sense.) Observe that the column operation $\vec{b} - \epsilon\vec{c}$ does one of the following, while preserving 2-nonnegativity: (a) it makes at least one solid 2-minor 0, or (b) it makes one entry b_i 0.

We note now that if (b) were true, then the minor $\begin{bmatrix} b_i - \epsilon c_i & c_i \\ b_{i+1} - \epsilon c_{i+1} & c_{i+1} \end{bmatrix}$ would be negative unless either $b_{i+1} - \epsilon c_{i+1}$ or c_i were also 0. The latter case cannot hold, as if c_i were 0, b_i/c_i would not have been included in the set T and the column operation specified above would not have made b_i 0. If the former case holds, then the minor $\begin{bmatrix} a_i & b_i - \epsilon c_i \\ a_{i+1} & b_{i+1} - \epsilon c_{i+1} \end{bmatrix}$ is zero. Thus, there is now at least one solid 2-minor with value 0 in columns $\vec{a}$ and $\vec{b}$. $\square$

Theorem 2.30. *If a matrix M is k-irreducible and invertible, then there is a zero k-initial minor in every set of k consecutive columns or rows.*

Proof. Suppose that there are k consecutive columns that have no zero k-initial minor. Label these columns, left to right, by $\vec{a_1} \dots \vec{a_k}$, and label the column immediately to the right of $\vec{a_k}$ by $\vec{b}$. We will subtract ε times $\vec{b}$ from $\vec{a_k}$, choosing ε such that it preserves the k-nonnegativity of the matrix while making at least one column-solid minor of order at most k zero. (If there already exists such a zero minor of size less than k, then we would take ε to be 0.)

We will first show that at least one solid minor of size at most k is then 0. We already have a zero column-solid minor $|C|$, say of order c. Consider the minor $|C'|$ obtained by exchanging the bottom row of the submatrix C with another row, not in C, but between the top and bottom rows in C. It is enough to show that the minor of the new submatrix C' must be 0, than we can continue this procedure to obtain a solid minor. To show this, we will use Lewis Caroll's identity, stated below. For $i \leq i'$ and $j \leq j'$,

$$\Delta^{i,j}(x)\Delta^{i',j'}(x) - \Delta^{i,j'}(x)\Delta^{i',j}(x) = \det(x)\Delta^{i,i',j,j'}(x).$$

Here x represents a submatrix that contains the rows and columns of C as well as the column immediately to the left of C and the row that we added in C'. Take i' to be the bottom row in C and i to be the row in C' that replaced row i'. Take j' to be the rightmost column in C (this is the column $\vec{a_k}$) and take j to be the column immediately to the left of C. Then we know that $\Delta^{i,j}$, the determinant of C, is zero. Since the right-hand side of the equation above is nonnegative, we know that either $\Delta^{i,j'}(x)$ or $\Delta^{i',j}(x)$, which is the determinant of C', must be 0. But $\Delta^{i,j'}(x)$ does not include j', so if it is 0, it was so even before the row operation.

Now let $a_{i,k}$ be the bottom entry of the zero solid minor in the column $\vec{a_k}$. Then the zero solid minor is the antidiagonal flip of a leading principal minor of a j-nonnegative $j \times j$ submatrix, where $j = \min(i, k)$. Since leading principal minors of an invertible, totally nonnegative matrix are always positive, as are their antidiagonal flips, we infer that the $j \times j$ submatrix cannot be invertible. Thus, the k-initial minor corresponding to the matrix entry $a_{i,k}$ is 0. ☐

3 Factorizations

In this section we will give specific generators and relations for $(n-1)$-nonnegative matrices and $(n-2)$-nonnegative unitriangular matrices. We also specify the generators for k-nonegative matrices of specific forms, namely, tridiagonal matrices and pentadiagonal unitriangular matrices.

3.1 $k = n - 1$

Using Theorem 2.26, we already know that an $(n-1)$-nonnegative matrix can be reduced to the following form while still preserving $(n-1)$-nonnegativity.

$$X = \begin{bmatrix} a_1 & b_1 & \dots & \dots & \dots \\ c_1 & a_2 & b_2 & \dots & \dots \\ \dots & \ddots & \ddots & \ddots & \dots \\ \dots & \dots & c_{n-2} & a_{n-1} & b_{n-1} \\ \dots & \dots & \dots & c_{n-1} & a_n \end{bmatrix}$$

Here all the entries except those of the diagonal, the superdiagonal and the subdiagonal are 0s. We will refer to matrices of this form as tridiagonal matrices. Observe that by dividing each column by a constant, we can have $c_i = 1$ for all i.

We will show that such matrices can be generated using the Chevalley generators and generators of the following form:

$$K(\vec{a}, \vec{c}) = \begin{bmatrix} a_1 & a_1c_1 & 0 & \dots & \dots & 0 \\ 1 & a_2 + c_1 & a_2c_2 & \ddots & \dots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & a_{n-3} + c_{n-4} & a_{n-3}c_{n-3} & 0 \\ \vdots & \dots & \ddots & 1 & c_{n-3} & c_{n-2}Y \\ 0 & \dots & \dots & 0 & 1 & X \end{bmatrix},$$

where $a_1, \dots, a_{n-3}, c_1, \dots, c_{n-2}$ are positive numbers, $Y = c_1c_2\cdots c_{n-3}$ and $X = c_{n-2} \cdot \left|K_{[2,n-3],[3,n-2]}\right|$. All entries except those on the diagonal, superdiagonal and subdiagonal are 0s.

Observe that $K(\vec{a}, \vec{c})$ is $(n-1)$-nonnegative but has negative determinant. This can be seen by direct computation of the solid minors of $K(\vec{a}, \vec{c})$, each of which has one of the following forms.

1. $$\left|\begin{bmatrix} a_1 & a_1c_1 & \dots & \dots & \dots \\ 1 & a_2 + c_1 & a_2c_2 & \dots & \dots \\ \dots & \ddots & \ddots & \ddots & \dots \\ \dots & \dots & 1 & a_{i-1} + c_{i-2} & a_{i-1}c_{i-1} \\ \dots & \dots & \dots & 1 & a_i + c_{i-1} \end{bmatrix}\right| = a_1a_2\dots a_i$$

2. $$\left|\begin{bmatrix} a_i + c_{i-1} & a_ic_i & \dots & \dots & \dots \\ 1 & a_{i+1} + c_i & a_{i+1}c_{i+1} & \dots & \dots \\ \dots & \ddots & \ddots & \ddots & \dots \\ \dots & \dots & 1 & a_{n-3} + c_{n-4} & a_{n-3}c_{n-3} \\ \dots & \dots & \dots & 1 & c_{n-3} \end{bmatrix}\right| = c_{n-3}c_{n-4}\dots c_{i-1}$$

3.

$$\left|\begin{bmatrix} a_ic_i & \dots & \dots & \dots \\ \ddots & \ddots & \dots & \dots \\ \ddots & \ddots & \ddots & \dots \\ \dots & 1 & a_j+c_{j-1} & a_jc_j \end{bmatrix}\right| = a_ic_i \dots a_jc_j \text{ and } \left|\begin{bmatrix} 1 & a_i+c_{i-1} & a_ic_i & \dots \\ \dots & \ddots & \ddots & \dots \\ \dots & \dots & 1 & a_j+c_{j-1} \\ \dots & \dots & \dots & 1 \end{bmatrix}\right| = 1$$

4.

$$\left|\begin{bmatrix} a_i+c_{i-1} & a_ic_i & \dots & \dots \\ 1 & a_{i+1}+c_i & a_{i+1}c_{i+1} & \dots \\ \dots & \ddots & \ddots & \ddots \\ \dots & \dots & 1 & a_j+c_{j-1} \end{bmatrix}\right|$$

$$= a_ia_{i+1}\cdots a_j + c_{i-1}a_{i+1}\cdots a_j + c_{i-1}c_ia_{i+2}\cdots a_j + \dots + c_{i-1}\cdots c_{j-1}.$$

5.

$$\left|\begin{bmatrix} a_i+c_{i-1} & a_ic_i & \dots & \dots & \dots \\ 1 & a_{i+1}+c_i & a_{i+1}c_{i+1} & \dots & \dots \\ \dots & \ddots & \ddots & \ddots & \dots \\ \dots & \dots & 1 & c_{n-3} & c_{n-2}Y \\ \dots & \dots & \dots & 1 & X \end{bmatrix}\right|$$

$$= (c_{i-1}\dots c_{n-2})(a_2\dots a_{i-1} + c_1a_3\dots a_{i-1} + \dots + c_1\dots c_{i-3}a_{i-1})(a_i\dots a_{n-3}).$$

Note that the equation in (5) above only holds for $i \neq 1$.

To show that matrices of the form $K(\vec{a}, \vec{c})$, along with the Chevalley matrices, generate all $(n-1)$-nonnegative matrices, we will use the following two lemmas.

Lemma 3.1. *A tridiagonal matrix X that is $(n-1)$-nonnegative is $(n-1)$-irreducible if and only if the two minors corresponding to entries $(n-1, n-1)$ and (n, n) in its $(n-1)$-initial minor matrix are zero.*

Proof.

$$A = \begin{bmatrix} x_{1,1} & x_{1,2} & \dots \\ \dots & \ddots & \ddots \\ \dots & 1 & x_{n-1,n-1} \end{bmatrix}, B = \begin{bmatrix} x_{2,2} & x_{2,3} & \dots \\ \dots & \ddots & \ddots \\ \dots & 1 & x_{n,n} \end{bmatrix}$$

Suppose that the minor A is positive. As a result, the principal leading minors of A are all positive. Label the principal leading minors by $C_1, C_2 \dots C_{n-2}$, i.e. $C_1 = x_{1,1}$, $C_2 = \begin{bmatrix} x_{1,1} & x_{1,2} \\ 1 & x_{2,2} \end{bmatrix}$, etc. Then we observe that C_{n-1} (which is the determinant of A) is equal to $x_{n-1,n-1}C_{n-2} - x_{n-2,n-1}C_{n-3}$.
We now perform a row operation in the matrix X. Take ε to be equal to C_{n-1}/C_{n-2} and subtract ε times row n from row $n-1$ in X. This reduces $x_{n-1,n-1}$ to $a_{n-1} - \varepsilon$ and makes the minor corresponding to the $(n-1, n-1)$ entry of the $n-1$-initial minor matrix 0. A similar procedure can make the minor B 0 if it is positive.

It needs to be verified that the minors of size less than $n-1$ are still nonnegative. By Theorem 2.11 we only need to consider solid minors, and a quick glance at X will show that the only solid minors that could be negative are the ones that have a string of $x_{i,i}$'s, ending in $x_{n-1,n-1}-\varepsilon$, along the diagonal. It is not difficult to see that if any of these minors except C_{n-2} were nonpositive, then C_{n-2} would be negative. But we know C_{n-2} is zero, therefore X is reducible.

We now show that X is $(n-1)$-irreducible if the two key minors specified are zero. A quick glance at a tridiagonal matrix will show that subtracting a multiple of one row or column from another will almost always introduce a negative element in the matrix. This means that the only Chevalley generators that could be factors of X are those that do one of the following: (a) subtract a multiple of column 1 from column 2, or (b) subtract a multiple of row n from row $n-1$.

If the minor corresponding to matrix B is 0, it is clear that row operation (a) is impossible, as any such operation would reduce the value of $x_{2,2}$, causing the determinant of B to become negative. Similarly, if matrix A has determinant 0, row operation (b), which reduces the value of $x_{n-1,n-1}$, makes the determinant of A negative. Thus, X is $(n-1)$-irreducible.

□

Lemma 3.2. *A matrix has the form of $K(\vec{a},\vec{c})$ above if and only if it is $(n-1)$-irreducible and maximally $(n-1)$-nonnegative.*

Proof. It is easy to verify by computation that in K, the two key minors specified in Lemma 3.1 are equal to 0. This shows that G is $(n-1)$-irreducible. For the other direction, we have already shown that any $(n-1)$-irreducible matrix is tridiagonal and has the two key minors equal to 0. It is again easy to verify that the entries of the matrix can be parametrized in terms of some $\vec{a}$ and $\vec{c}$ where all a_i's and c_i's are greater than 0, so that the matrix is equal to $K(\vec{a},\vec{c})$. □

Together with Theorem 2.25, this is sufficient to show that the Chevalley generators and matrices of the form $K(\vec{a},\vec{c})$ generate the set of $(n-1)$-nonnegative matrices.

We now turn our interest to relations involving generators of the form $K(\vec{a},\vec{c})$. It can be seen by direct computation that the following relations hold:

$$e_i(x)\cdot K(\vec{a},\vec{c}) = K(\vec{A},\vec{C})\cdot e_{i+1}(X), \text{ where } 1 \leq i \leq n-2. \tag{3.1.1}$$

In these relations $\vec{A}$, $\vec{C}$ and X are expressed in terms of $\vec{a}$, $\vec{c}$ and x, as specified below.

For all $j \leq i-1$ and all $j \geq i+2$, $a_j = A_j$ and $c_j = C_j$.

$$\begin{aligned}
A_i &= a_i + x. \\
C_i &= c_i + \frac{x \cdot a_{i+1}}{a_i + x}. \\
A_{i+1} &= \frac{a_i \cdot a_{i+1}}{a_i + x}. \\
C_{i+1} &= \frac{c_i \cdot c_{i+1} \cdot (a_i + x)}{c_i \cdot (a_i + x) + x a_{i+1}}. \\
X &= \frac{c_{i+1} \cdot a_{i+1} \cdot x}{c_i \cdot (a_i + x) + x a_{i+1}}.
\end{aligned}$$

Again by direct computation, it can be found that the following three relations (i.e. equations 3.1.2, 3.1.3 and 3.1.4) involving the generator K also hold. We will use X_1 and X_2 to denote the (n,n)th entry of $K(\vec{a},\vec{c})$ and $K(\vec{A},\vec{C})$ respectively.

$$e_{n-1}(x) \cdot K(\vec{a},\vec{c}) = h_n(1/(1+x'X_2)) \cdot K(\vec{A},\vec{C}) \cdot f_{n-1}(x'), \tag{3.1.2}$$

where the following equalities hold.

1. $a_i = A_i$ and $c_i = C_i$ for all $1 \leq i \leq n-3$.
2. $x = x'C_1 \ldots C_{n-2}$.
3. $$c_{n-2} + X_1 \cdot x'C_{n-2} = C_{n-2} \Rightarrow c_{n-2} + c_{n-2}X_2 \cdot x' = C_{n-2} \Rightarrow c_{n-2} = \frac{C_{n-2}}{1+x'X_2}.$$
4. $X_1 = X_2/(1+x'X_2)$. (This is implied by the previous equation.)

The third relation involving K is given by the following equation.

$$h_{i+2}(w) \cdot f_{i+1}(x) \cdot K(\vec{a},\vec{c}) = K(\vec{A},\vec{C}) \cdot f_i(x) \cdot h_i(w), \text{ where } 1 \leq i \leq n-2 \tag{3.1.3}$$

where the following equalities hold:

1. For all $1 \leq j \leq i-2$ and all $i+3 \leq j \leq n-3$, $a_j = A_j$ and $c_j = C_j$.
2. $C_{i+2} = c_{i+2}$.
3. $A_{i+2} = a_{i+2}/(xa_{i+1} + xc_{i+1} + 1)$.
4. $C_{i+1} = c_{i+1} \cdot (1 + xa_{i+1})/(xa_{i+1} + xc_i + 1)$.
5. $A_{i+1} = a_{i+1}(xa_{i+1} + xc_i + 1)/(1 + xa_{i+1})$.
6. $A_{i-1} = a_{i-1}$.
7. $C_{i-1} = c_{i-1}(xa_{i+1} + xc_i + 1)$.
8. $A_i = a_i(xa_{i+1} + 1)$.
9. $C_i = c_i/(xa_{i+1} + 1)$

10. $c_{n-2} = C_{n-2}$.
11. $w = 1/(1 + a_i + c_{i-1})$.

The fourth relation involving K is given by the following equation.

$$f_1(x) \cdot K(\vec{a}, \vec{c}) \cdot h_1(1 + xa_1) = K(\vec{A}, \vec{C}) \cdot e_1(x'), \tag{3.1.4}$$

where the following equalities hold:

1. $A_1 = a_1/(1 + xa_1)$.
2. $A_i = a_i$ for all $2 \leq i \leq n-3$ and $C_i = c_i$ for all $1 \leq i \leq n-2$.
3. $a_1c_1 = A_1C_1 + x'A_1 = A_1c_1 + x'A_1 \Rightarrow x' = xc_1a_1$.

3.2 $k = n - 2$

Using Theorem 2.26, we already know that an $(n-2)$-nonnegative unitriangular matrix can be reduced to the following form while still preserving $(n-2)$-nonnegativity.

$$X = \begin{bmatrix} 1 & a_1 & b_1 & \dots & \dots \\ \dots & \ddots & \ddots & \ddots & \dots \\ \dots & \dots & 1 & a_{n-2} & b_{n-2} \\ \dots & \dots & \dots & 1 & a_{n-1} \\ \dots & \dots & \dots & \dots & 1 \end{bmatrix}$$

In the matrix X, we have $x_{i,j} = 0$ for all $i > j$ and for all $j - i > 2$. In addition, $x_{i,j} = 1$ for all j. We will refer to such a matrix as a *pentadiagonal unitriangular* matrix.

We will show that pentadiagonal unitriangular matrices can be generated using the Chevalley generators and generators of the following form:

$$T(\vec{a}, \vec{c}) = \begin{bmatrix} 1 & a_1 & a_1c_1 & 0 & \dots & \dots & 0 \\ 0 & 1 & a_2 + c_1 & a_2c_2 & \ddots & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \dots & \vdots \\ \vdots & \dots & \ddots & 1 & a_{n-3} + c_{n-4} & a_{n-3}c_{n-3} & 0 \\ \vdots & \dots & \dots & \ddots & 1 & c_{n-3} & c_{n-2}Y \\ \vdots & \dots & \dots & \dots & \ddots & 1 & X \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 \end{bmatrix},$$

where $a_1, \dots a_{n-3}, c_1 \dots c_{n-2}$ are positive numbers, $Y = c_1 \cdot c_2 \dots \cdot c_{n-3}$ and $X = c_{n-2} \cdot \left|G_{[2,n-3],[3,n-2]}\right|$. To show that these matrices generate all $(n-2)$-nonnegative unitriangular matrices, we will use the following lemmas to characterize $(n-2)$-irreducible matrices. The lemmas use proofs that are very similar to the ones used in the previous section for $(n-1)$-nonnegative matrices.

Lemma 3.3. *A pentadiagonal unitriangular matrix X that is $(n-2)$-nonnegative is $(n-2)$-irreducible if and only if the two minors corresponding to entries $(n-2, n-1)$ and $(n-1, n)$ of its k-initial minor matrix are zero.*

Proof. First we will show that X is reducible if either of the two minors specified above are nonzero. Note that if they are nonzero, they are positive, and the two matrices below are totally nonnegative and invertible.

$$A = \begin{bmatrix} a_1 & b_1 & \dots \\ \dots & \ddots & \ddots \\ \dots & 1 & a_{n-2} \end{bmatrix}, B = \begin{bmatrix} a_2 & b_2 & \dots \\ \dots & \ddots & \ddots \\ \dots & 1 & a_{n-1} \end{bmatrix}$$

As a result, the principal leading minors of these matrices are all positive. Label the principal leading minors of the second submatrix above $C_1, C_2 \dots C_{n-2}$, i.e. $C_1 = a_2$, $C_2 = \begin{bmatrix} a_2 & b_2 \\ 1 & a_3 \end{bmatrix}$, etc. Then we observe that C_{n-2} (which is the determinant of B above) is equal to $a_{n-1}C_{n-3} - b_{n-2}C_{n-4}$.

We now perform a row operation in the matrix X. Take ε to be equal to C_{n-2}/C_{n-3} and subtract ε times row n from row $n-1$ in X. This reduces a_{n-1} to $a_{n-1} - \varepsilon$ and makes the minor corresponding to the $(n-1, n)$ entry of the k-initial minor matrix 0.

It needs to be verified that the minors of size less than k are still nonnegative. By Theorem 2.11 we only need to consider solid minors, and a quick glance at X will show that the only solid minors that could be negative are the ones that have a string of a_i's, ending in $a_{n-1} - \varepsilon$, along the diagonal. It is not difficult to see that if any of these minors, except C_{n-2}, were not positive, then C_{n-2} would be negative, while we know it is 0. Therefore X is reducible.

We now show the other direction, proving that X is $(n-2)$-irreducible if the two key minors specified are zero. Observe that the only Chevalley generators that could be factors of X are those that do one of the following: (a) subtract a multiple of column 1 from column 2, (b) subtract a multiple of column 2 from column 3, (c) subtract a multiple of row n from row $n-1$, or (d) subtract a multiple of row $n-1$ from row $n-2$.

It is sufficient to show that (c) and (d) are impossible, and similar arguments will apply to (a) and (b). If the minor corresponding to matrix B is 0, it is clear that row operation (c) is impossible, as any such operation would cause the determinant of B to become negative. Similarly, if matrix A has determinant 0, row operation (d), which reduces the value of a_{n-2}, makes the determinant of A negative. Thus, X is $(n-2)$-irreducible. □

Lemma 3.4. *A matrix has the form of $T(\vec{a}, \vec{c})$ above if and only if it is $(n-2)$-irreducible and maximally $(n-2)$-nonnegative.*

Proof. It is, again, easy to verify by computation that in T, the two key minors specified in Lemma 3.3 are equal to 0. This shows that T is $(n-2)$-irreducible. For the other direction, we have already shown that any $n-2$-irreducible unitriangular matrix is pentadiagonal and has the two key minors equal to 0. Given that these two minors are zero, it is easy to see that the entries of the matrix can be parameterized in terms of some $\vec{a}$ and $\vec{c}$ where all a_i's and c_i's are greater than 0, so that the matrix is equal to $T(\vec{a}, \vec{c})$. □

Together with Theorem 2.25, this is again sufficient to show that the Chevalley generators and matrices of the form T generate the set of $(n-2)$-nonnegative unitriangular matrices.

We now turn our interest to relations involving generators of the form $T(\vec{a}, \vec{c})$. It can be seen by direct computation that the following relations hold:

$$e_i(x) \cdot T(\vec{a}, \vec{c}) = T(\vec{A}, \vec{C}) \cdot e_{i+2}(X), \text{ where } 1 \le i \le n-3. \tag{3.2.1}$$

In these relations $\vec{A}$, $\vec{C}$ and X are expressed in terms of $\vec{a}$, $\vec{c}$ and x, as specified below.

For all $j \le i-1$ and all $j \ge i+2$, $a_j = A_j$ and $c_j = C_j$.

$$\begin{aligned}
A_i &= a_i + x. \\
C_i &= c_i + \frac{x \cdot a_{i+1}}{a_i + x}. \\
A_{i+1} &= \frac{a_i \cdot a_{i+1}}{a_i + x}. \\
C_{i+1} &= \frac{c_i \cdot c_{i+1} \cdot (a_i + x)}{c_i \cdot (a_i + x) + x a_{i+1}}. \\
X &= \frac{c_{i+1} \cdot a_{i+1} \cdot x}{c_i \cdot (a_i + x) + x a_{i+1}}.
\end{aligned}$$

Again by direct computation, it can be found that the following two relations (i.e. equations 3.2.2 and 3.2.3) involving the generator T also hold.

$$e_{n-2}(x) \cdot T(\vec{a}, \vec{c}) = T(\vec{A}, \vec{C}) \cdot e_1(X), \tag{3.2.2}$$

where the following equalities hold.

1. $C_{n-3} = c_{n-3} + x$,
2. $A_i = (a_i \cdot c_i)/C_i$, for $1 \le i \le n-3$.
3. $C_i = a_{i+1} + c_i - A_{i+1}$, for $1 \le n-4$.

   (Note that $C_{n-3} > c_{n-3}$, and consequently $A_{n-3} < a_{n-3}$. In turn, $C_{n-2} > c_{n-2}$, etc, so that in general $C_i > c_i$ and $A_i < a_i$.)
4. $X = a_1 - A_1$.

The third relation involving t is given by the following equation.

$$e_{n-1}(x) \cdot T(\vec{a}, \vec{c}) = T(\vec{A}, \vec{C}) \cdot e_2(X), \tag{3.2.3}$$

where the following equalities hold:

1. $a_1 = A_1$ and $c_1 = C_1 + X$
2. $a_i = A_i$ for $2 \le i \le n-3$ and $c_i = C_i$ for $2 \le i \le n-3$.
3. $c_{n-2} = (C_{n-2} \cdot C_1)/(C_1 + X)$.
4. $x = X \cdot \left| G(\vec{A}, \vec{C})_{[3,n-3],[4,n-2]} \right|$.

3.3 Bruhat Cells

The semigroup of $(n-1)$-nonnegative invertible matrices and the semigroup of $(n-2)$-nonnegative unitriangular invertible matrices can both be partitioned into *cells* based on their factorizations. In this section, we will describe these cells by *reduced words* and study their topology.

3.3.1 Cells of $(n-2)$-nonnegative matrices

Given any $n-2$-nonnegative unitriangular matrix M, we can write it as a product of Chevalley generators and possibly a T-generator.

$$M = e_{i_1}(a_1) \dots e_{i_l}(a_l) T(x_1 \dots x_{n-3}, y_1 \dots y_{n-2}),$$

where $a_1 \dots a_l, \vec{x}, \vec{y}$ are positive parameters. Let k be the number of parameters in the product (in the formulation above $k = l + 2n - 5$). If k is minimal among all such expressions for M, then we say that the word $w = i_1 \dots i_l T$ in the alphabet $\mathcal{A} = \{1, 2 \dots n-1, T\}$ is a *reduced word* for M and k is the *length* of M.

Two words $u = u_1 \dots u_k$ and $v = v_1 \dots v_m$ are said to be *equal* if their equality can be deduced from the relations 3.2.1, 3.2.2 and 3.2.3, in addition to the known relations on Chevalley generators, which are listed in the introduction. This implies that if $u = v$, then for any set of parameters $a_1, \dots a_k$, the matrix given by $e_{u_1}(a_1) \dots e_{u_k}(a_k)$ can also be written as a product $e_{v_1}(b_1) \dots e_{v_m}(b_m)$ for some parameters $v_1, \dots, v_m$, and vice versa. As its name suggests, this is an equivalence relation and allows us to make the following definition.

Definition. For some word $w = w_1 \dots w_k$ in the alphabet $\{1, 2, \dots, n-1, T\}$, the *cell* $V(w)$ is defined as the set of matrices with the form $e_{w_1}(a_1) e_{w_2}(a_2) \cdots e_{w_k}(a_k)$ for any parameters $a_1 \dots a_k$. The letter T in the word represents a T-generator and has $2n-5$ parameters.

As is clear by definition, if $u = w$ according to our relations, then $V(u) = V(w)$. It is also clear that any $(n-2)$-nonnegative unitriangular matrix belongs to a cell $V(w)$ for some word w. We will further show that within a particular subset of reduced words, if $u \neq w$ the cells corresponding to u and w are actually disjoint. For this we need to describe the reduced words of the cells we want to include.

To do this, we first need to introduce the *Bruhat order.* This is a partial order structure that arises in many places in algebra and geometry, but it plays an especially interesting role in the study of Coxeter groups. More details about this can be found in [1]. Here we will give a very brief description of the Bruhat order that relies on the subword property, not on the original definition that arose in the context of Coxeter groups and cell decompositions.

Let $w = w_1 \dots w_q$ be a reduced word. Then $u \leq w$ in the strong Bruhat ordering if and only if there exists a reduced word $u = w_{i_1} \dots w_{i_r}$ where $1 \leq i_1 < \dots < i_r \leq q$. We say $u \leq w$ in the weak Bruhat ordering if $u \leq w$ in the strong order and in addition $i_k = i_{k+1} - 1$ for all k and either $i_1 = 1$ or $i_r = q$. We will eventually extend this ordering slightly to include some relations between words involving T and words not involving T, as this order cannot be described wholly on the basis of subwords. However, in following lemmas we will only use the Bruhat order on words that do not involve T.

We are now ready to enumerate the cells that, as we will later show, partition the set of $n-2$-nonnegative unitriangular matrices.

Theorem 3.5. *Let $\alpha = (n-2)\cdots(1)(n-1)\cdots(1)$. Let $\beta = (n-2)\cdots(1)(n-1)\cdots(2)$. The* coarse *cells are $V(w)$, where*

$$w \in \begin{cases} w'K & w'\alpha \text{ is reduced, } K \in \{G, (n-1)G, (n-2)G, (n-1)(n-2)G\} \\ w' & w' \not> \beta \end{cases}$$

The fine *cells are $V(w)$, where*

$$w \in \begin{cases} w'K & w'\alpha \text{ is reduced, } K \in \{G, (n-1)G, (n-2)G, (n-1)(n-2)G\} \\ w' & \end{cases}$$

where w' does not involve T.

Proof. First, notice that for words with a G, $n-1$ and $n-2$ commute with everything except for each other. Further, consider the following lemma:

Lemma 3.6. *Let $\alpha = n-2 \ \cdots \ 1 \ n-1 \ \cdots \ 1$, representing e_is (so $e_i^2 = e_i$). Then the following is true:*

- *$k\ \alpha = \alpha\ (k+2 \mod n-1)$ for all k.*
- *If $k \neq j$ then $k\ \alpha \neq j\ \alpha$ unless $k, j = n-2, n-1$, in which case they are all equal to α.*
- *$k\ \alpha$ is a reduced word unless $k = n-2, n-1$, in which case we have α as a reduced word.*

Proof. Examine the number of inversions of all of these. When the number of inversions increases, we know we have a reduced word, and so our permutations can be distinguished simply by value. Using this, showing all of the above requires minimal computation. □

Lemma 3.7. *Let φ_K be the map taking reduced words $w\alpha$ to wK, where $K \in \{T, (n-1)T, (n-2)T, (n-1)(n-2)T, (n-2)(n-1)T\}$, and w does not include $n-1$ or $n-2$. Then φ_K is a bijection.*

Proof. First, notice that this is a well-defined map. Second, this map is bijective, since the relations of $w\alpha$ and wK are exactly equal when we restrict to words without $n-1$ or $n-2$. □

Finally, notice that anything with more than one $(n-2)$ and $(n-1)$ is not a reduced word. This gives us the theorem. □

Theorem 3.8. *For reduced words u and w as qualified by Theorem* 3.5*, if $u \neq w$ then $V(u)$ and $V(w)$ are disjoint.*

Proof. If the cell does not have a G this is a known result from [13]. If it has a G and an $n-2$ followed by an $n-1$, then we know that this cell can be decomposed into three cells. Two of them are totally nonnegative, and so are known to be distinct. Notice that $n-1$ and $n-2$ commute with everything. Further, G, $n-1$ G, $n-2$ G, and $n-2$ $n-1$ G all have the same relations properties as α provided there are no $n-1$s or $n-2$s. Thus, the fact that $V(w')K \subset B_w$ gives us that all of our cells are distinct, up to containing $n-2$ and $n-1$. But we have exactly found the distinctions between these. So we are done, and all cells are distinct. ☐

3.3.2 Topology of cells

Each of the cells $V(w)$ is homeomorphic to an open ball, as can easily be seen from the following lemma.

Lemma 3.9. *Given a word $w = w_1 \ldots w_k$, the map from the parameters $(a_1 \ldots a_k)$ to $e_{w_1}a_1 \ldots e_{w_k}(a_k)$ is a bijection when its image is restricted to $V(w)$. (As always, one of the w_i's could be a T, in which case it has $2n-5$ parameters.)*

Proof. Suppose that the map is not injective. Then we cannot have the Gs be different with the es the same, as otherwise the superdiagonal values are guaranteed to be different, since they are nondegenerate in G's parameters. So there must be an e that is different. We can take its inverse, giving us two different reduced words for the same element. By Theorem 3.8 this is a contradiction. ☐

In fact the map is a homeomorphism, so the cell $V(w)$ is homeomorphic to an open ball. A conjecture of Fomin, proved by Hersh, is that the closure of the cells $U(w)$ where the w consists of Chevalley generators alone, is homeomorphic to a closed ball. In this section we will try to understand the structure of the closure of $V(w)$. Most of the following results follow very closely from propositions and proofs in Pylavskyy's lecture notes [13].

We now introduce some preliminary notation and lemmas which are necessary for the following theorems. Let $\omega \in S_n$ be represented as a permutation matrix (that is, the matrix such that $\omega e_i = e_{\sigma(i)}$). Let B^+ and B^- be the upper and lower triangular Borel subgroups of $GL_n(\mathbb{R})$, respectively. Let $B^-_\omega = B^-\omega B^-$ be the Bruhat cell associated to ω, and similarly for $B^+_\omega = B^+\omega B^+$. Define double Bruhat cells $B_{u,w} = B^+_u \cap B^-_w$. If we restrict to a kind of cell, any element of $GL_n(\mathbb{R})$ is in one of these cells.

Definition. For an $\omega \in S_n$, let $X[I,J]$ be a ω-NE-ideal if $I = \omega(J)$ and $(\omega(i), i) \in (I,J) \implies (\omega(j), j) \in (I,J)$ for j such that $j > i$ and $\omega(j) < \omega(i)$.

Call $X[I,J]$ a shifted ω-NE-ideal if $I \leq I'$ and $J' \leq J$ in termwise order for some ω-NE-ideal (I', J') where $I, J \neq I', J'$.

Essentially we choose some set of entries that have ones in the permutation matrix ω, and have our ideal be those rows and columns, along with the rows and columns of any ones to the NE of any of our existing ones. Shifted ideals are submatrices that are further to the NE than the ideals.

Definition. Call a matrix X ω-NE-bounded if the following two conditions hold:

- $X[I,J] \neq 0$ for I,J ω-NE-ideal.
- $X[I,J] = 0$ for I,J shifted ω-NE-ideal.

Lemma 3.10. *M is in B_w^- iff it is w-NE-bounded.*

Proof. From inspection the w is w-NE-bounded. Further, Cauchy-Binet gives us that multiplying by elements in B^- preserve this. □

Lemma 3.11. *Let w be some reduced word of $\sigma \in S_n$ suffixed by some word α (that is, such that $\alpha \leq w$ in the weak Bruhat order).Then, for $M \in B_\alpha$, $U(w \setminus \alpha)M \subset B_w$.*

Proof. Proof by induction. The base case is obviously true, since $M \in B_\alpha$. Now suppose we are taking some $N \in B_\beta$ and considering $e_i(c)N$ such that $s_i\beta$ is a reduced word. This occurs precisely when $s_i\beta$ has more inversions than β. Considering β in terms of one-line notation, we see this can only happen when we have i before $i+1$.

Now, we consider the $s_i\beta$-NE-ideals, and compare them to the β-NE-ideals. The ideals that do not contain rows i and $i+1$ are exactly the same, as are the ideals that contain both. In both cases, the corresponding minors are unaffected by the e_i. When a $s_i\beta$-NE-ideal contains row $i+1$, it must contain row i, so the only remaining case are the $s_i\beta$-NE-ideals that contain only i. These are in bijection to the β-NE-ideals containing i but not $i+1$. We know from an above lemma that for an $s_i\beta$-NE-ideal I,J,

$$\det(e_i(c)M)_{I,J} = \det M_{I,J} + cM_{I \setminus i \cup i+1, J} > 0$$

since the right hand side is the sum of a β-NE-ideal and a shifted β-NE-ideal.

Now, consider a shifted $s_i\beta$-NE-ideal I,J. We consider the I',J' from the definition (that is, the $s_i\beta$-NE-ideal such that $I \leq I'$ and $J' \leq J$). If I' does contains neither i nor $i+1$ or contains both, then I',J' is a β-NE-ideal as well, and I,J must be a sum of shifted ideals which I',J' apply for. If I' contains i but not $i+1$, then the ideal swapping out i for $i+1$ is a β-NE-ideal. This ideal shows that I,J can be expressed as a sum of shifted β-NE-ideal minors. □

Proposition 3.12. *The closure $\overline{V(w)}$ consists of all $V(u)$ for all $u \leq w$ in the Bruhat order.*

Proof. If the reduced word w does not contain $\mathcal{T}$, this is a known result, so suppose that w has a $\mathcal{T}$. The closure of such a a word contains all subwords of w involving T, all words obtained by setting exactly one x_i or y_i in T to 0 as well as subwords of these. By the definition of our ordering, all these words are less than w.
The non-trivial part of the proof is to show that there is no $V(u)$, where $u \not\leq w$, in the closure of $V(w)$. We use the following formululation of the Bruhat order from Bjorner and Brenti [1]. For each $(i,j) \in [n] \times [n]$, let $N_w(i,j) = \mid \{k \mid k \leq i, w(k) \geq w(i)\} \mid$. Then $u \leq w$ if and only if for every (i,j), we have $N_u(i,j) \leq N_w(i,j)$. Thus, if $u \not\leq w$, there exists (i,j) with $N_u(i,j) > N_w(i,j)$. Consider the minimal u-NE-ideal X_c containing cell (i,j). Then $|X_C| = \det X_C \neq 0$ for $X \in V(u)$, by Lemma 3.10. But if $X_C \in V(w)$, then X_C is not of

full rank, because it is obtained by performing row operations on a matrix of rank less than $N_u(i,j)$. Thus $\det X_C = 0$, which means $X_C \notin V(w)$. □

Thus we see that the Bruhat order represents something "natural" going on in these matrices. We can characterize the new poset in the following way:

Notice that we can make the choice of whether to take $n-2\ n-1$ or $n-1\ n-2$. If we choose the coarse option at a certain level, it must be consistent with all the options above. Thus, we have $\ell(\alpha, w_0)$ refinements.

Theorem 3.13. *The k-nonnegative matrices consist of two path-connected components, those with negative determinant and those with positive determinant.*

Proof. Take some k-nonnegative matrix M. Consider some minor that is 0. If we cannot affect this minor with Chevalleys then it must be the case that that M is not invertible. Thus, we can always make minors nonzero. Thus, we can always bring M to a totally positive matrix, except for the determinant itself which must have fixed sign. We know that totally positive matrices are homeomorphic to a ball, so we are done for that component. We can reduce any $k-1$-nonnegative matrix to a G, which is a ball by the above lemma. Thus, this is path-connected as well. □

3.3.3 Further Comments on the Poset

In this section we will prove that the poset on cells $V(w)$ is a graded poset.

Lemma 3.14 (Exchange Property for new relations)**.** *If w is a word in $\Lambda = \{1, \ldots, n-1\}$ subject to Chevalley relations; that is, the following relations hold:*

- *$i\ i \leftrightarrow i$ (the shortening relation)*
- *$i\ j\ i \leftrightarrow j\ i\ j$ if $|i-j| = 1$ (the adjacent relation)*
- *$i\ j \leftrightarrow j\ i$ if $|i-j| > 1$ (the nonadjacent relation)*

Then if w is a reduced word, for $t \in \Lambda$, exactly one of the following is true:

- *tw is reduced, so $\ell(tw) = \ell(w) + 1$;*
- *$tw = w$, so $\ell(tw) = \ell(w)$.*

Proof. Suppose tw is not reduced. Let $M = \{m_1, m_2, \ldots, m_r\}$ be a sequence with $tw = m_1$, m_r a reduced word, each one at most one local move away from the previous, with no $i \to i\ i$ moves.

To see that there always exists such a sequence, consider this. We know such a sequence exists for Coxeter groups from [1] (see section 3.3.1). Mimic this sequence until we hit a shortening relation. Use this relation, and continue with the resulting smaller word.

Define $\varphi : M \to [\ell(w)+1] \cup \varnothing$ recursively in the following way (we want it to indicate a sort of location for our t when we are performing the local moves):

$$\varphi(m_1) = 1$$
$$\varphi(m_i) = \begin{cases} \varnothing & \text{in shortening relation or } \varphi(m_{i-1}) = \varnothing \\ \varphi(m_{i-1}) \pm 1 & \text{in left/right position of nonadjacent relation} \\ \varphi(m_{i-1}) \pm 2 & \text{in left/right position of adjacent relation} \\ \varphi(m_{i-1}) - 1 & \text{shortening relation earlier in word} \\ \varphi(m_{i-1}) & \text{otherwise} \end{cases}$$

Lemma 3.15. *The following properties are true:*

(a) φ *is well-defined.*

(b) There are no length-shortening moves that don't involve t

(c) $m_r = w$

Proof.

(a) Because we chose the sequence such that we never get a longer word than tw, our function remains in the codomain. Thus, the only thing to check for well-definedness is whether $\varphi(m_{i-1})$ can ever be in the middle of an adjacent relation.

Let $n_i = m_i \setminus m_{i,\varphi(m_i)}$, where we take out nothing if $\varphi(m_i) = \varnothing$; that is, we take out the t from the word, and $\varnothing$ signifies that the t no longer exists. Then notice that $w = n_1$, and each n_i is at most one local move away from n_{i-1}. The reason for this is that removing t does not affect any local moves not involving t, and the local moves that do involve t don't affect anything except t. Suppose we do have a move where the location of the t is in the center. Then if we consider the n_i up to that point, we get that there is an n_i with two adjacent identical letters:

$$m_i = \cdots i\ t\ i \cdots \implies n_i = \cdots i\ i \cdots$$

However, this would imply that w can be reduced to something of smaller length. This is a contradiction.

(b) Same reasoning; consider the n_i. If there was a length-shortening move then obviously we would get that n_i is a series of moves that shortens w, which is not possible.

(c) We must have a shortening relation to get a reduced word. This relation must contain t, so there must be exactly one. Notice that once $\varphi(m_i) = \varnothing$, $m_i = n_i$. We know that $w = n_i$ for all i. Thus, $m_i = w$.

□

This gives us the statement. □

Lemma 3.16. *The order restricted to* $w\,T$ *is graded.*

Proof. Though we call it the weak Bruhat order, it can be shown that the order is just the strong Bruhat order interval. So it works by rankedness of strong Bruhat order. □

Theorem 3.17. *The Bruhat poset is graded.*

Proof. For anything not containing a T, this is known from [13]. For T, this is known by the lemma. We must consider words $w\,T$.

Now, suppose that wT is reduced but reducing T to t makes wt not reduced. We want to show that there is a chain between wT and wt that behaves correctly with respect to our rank function.

Let $w = w_1 \cdots w_a$. Then consider $w_i \cdots w_a t$, starting from $i = a$ to $i = 1$. Using the exchange property, we can see that this reduces to some $w't$, where w' is a subword of t. Thus, this has the intermediary $w'T$, and from the lemma we get intermediaries as desired. □

3.4 Miscellaneous Cases

We will discuss some easy or edge miscellaneous cases.

There are two easy unitriangular cases. First, when $k = 1$, then it is easy to see that we can generate the semigroup with $e_{ij}(a) = I + \delta_{ij}(a)$, for $i < j$. The group of unitriangular matrices are generated by these as well, when a is arbitrary ([11] §5). However, when we restrict a to be positive, the relations still maintain closure, so we have a complete list of generators and relations. Second, when $k = n - 1$, it is easy to see that $(n-1)$-nonnegative matrices are TNN.

We are able to give generators for k-nonnegative tridiagonal and pentadiagonal unitriangular matrices in general. The two cases are fairly similar.

Lemma 3.18. *For M a k-nonnegative tridiagonal matrix, we can write M as a product of Chevalley generators, diagonal matrices, and matrices of the form*

$$\begin{bmatrix} I_p & & \\ & H_q & \\ & & I_{n-p-q} \end{bmatrix}$$

Where H_q is a $q \times q$ invertible tridiagonal k-nonnegative matrix with ones on the subdiagonal, no zeros on the superdiagonal, and $q > k$.

Proof. Induct on n. Clearly we have the statement for $n = 1$.

Now, consider a matrix M. By factoring out diagonal matrices we can assume that all of the entries in the subdiagonal are one. Note that via 2.21, no diagonal entries can be zero.

If there are no zeros off the diagonal then M is of the desired form. Suppose $m_{i,i+1} = 0$. Define $K_{i,j}(x)$ to be a zero matrix except for the bottom-left corner, whose value is x. Then as block diagonals we have

$$M = \begin{bmatrix} M_{[i],[i]} & K_{i,n-i}(m_{i,i+1}) \\ & M_{[i+1,n],[i+1,n]} \end{bmatrix} = \begin{bmatrix} M_{[i],[i]} & \\ & I_{n-i} \end{bmatrix} e_i\left(\frac{m_{i,i+1}}{m_{i,i}m_{i+1,i+1}}\right)\begin{bmatrix} I_i & \\ & M_{[i+1,n],[i+1,n]} \end{bmatrix}$$

It is easy to see that the matrices resulting from the factorization are k-nonnegative; the minors of the factors can easily be related to minors of M. Further, these matrices are obviously invertible.

Thus, we are left with one generator and two subcases, which we can further decompose by the inductive hypothesis if there are further zeros on the off-diagonals. The analogous case where $m_{i+1,i} = 0$ is given by the transpose of the above, and if both values $m_{i+1,i} = m_{i,i+1} = 0$, then we get the same formula as above, but without the Chevalley matrix in the middle.

If we end up with factors with tridiagonal blocks of size less than k, then k-nonnegativity implies TNN in these cases, and said factors can be decomposed into Chevalley generators by 2.3. □

Now, we only need to worry about classifying H_q matrices as described in the above theorem. We consider an $n \times n$ matrix J of such a form. Because J must be tridiagonal matrix with ones on the subdiagonal, there are only $2n - 1$ entries that are unknown. Let $a_i = j_{i,i}$ and $b_i = j_{i,i+1}$.

We first observe that a minor in a tridiagonal matrix (where non-diagonal entries are units) can be expressed in terms of a continued fraction. We will notate continued fractions in the following way:

$$[a_0; a_1, \ldots, a_m; b_1, \ldots, b_m] := a_0 - \frac{b_1}{a_1 - \frac{b_2}{a_2 - \cdots}}$$

This is different from the standard notation, which adds recursively rather than subtracts.

From observation we can see the following:

Lemma 3.19. *Let $C_i(j) = \left|M_{[i,i+j-1],[i,i+j-1]}\right|$. Then the following recursive relation is satisfied:*

$$C_i(0) = 1,\ C_i(1) = a_i,\ C_i(r) = a_{i+r-1}C_i(r-1) - b_{i+r-2}C_i(r-2)$$

We can give another relation by re-writing the above:

Lemma 3.20. $C_i(r) = C_i(r-1)[a_{i+r-1}; \ldots a_i; b_{i+r-2}, \ldots, b_i]$ *when $C_i(k) \neq 0$ for $k < r$.*

Proof. It is obviously true for the base cases of the recurrence. Rewrite the equation as follows:

$$\begin{aligned} \frac{C_i(r)}{C_i(r-1)} &= a_{i+r-1} - b_{i+r-2}\frac{C_i(r-2)}{C_i(r-1)} \\ &= a_{i+r-1} - \frac{b_{i+r-2}}{[a_{i+r-2}; a_{i+r-3}, \ldots, a_i; b_{i+r-3}, \ldots, b_i]} \\ &= [a_{i+r-1}; \ldots a_i; b_{i+r-2}, \ldots, b_i] \end{aligned}$$

$\square$

Theorem 3.21. *Let J be an invertible tridiagonal matrix with 1s on the subdiagonal and nonzero entries on the superdiagonal. Then J is k-nonnegative and k-irreducible if and only if the following hold:*

$$\begin{aligned} a_i, b_i &> 0 \\ [a_{k+1}; \dots a_2; b_k, \dots, b_2] &= 0 \\ [a_{n-1}; \dots a_{n-k}; b_{n-2}, \dots, b_{n-k}] &= 0 \\ [a_x; \dots a_{x-k+1}; b_{x-1}, \dots, b_{x-k+1}] &\geq 0 \qquad \textit{if } x \geq k \textit{ and not } n-1, k+1 \\ [a_x; \dots a_1; b_{x-1}, \dots, b_1] &> 0 \qquad \textit{if } x < k \end{aligned}$$

Proof. First, notice that M is kNN if and only if column-solid minors of order at most k are nonnegative by 2.16. From the matrix being tridiagonal, this is equivalent to all minors of the form $C_i(j)$ being nonnegative for $j \leq k$, and all of the b_is being nonnegative (that is, positive, since we know they are nonzero by assumption). For $j < k$, we cannot have $C_i(j) = 0$: it breaks invertibility in the $j = 1$ case, and we get that $C_i(j) \neq 0$ by induction; if not, then $C_i(j+1)$ is negative. Thus, we can use 3.20, and say that $C_i(j)$ are all nonnegative precisely when the base case, a_i, are positive, as well as all of the corresponding continued fractions. Among these continued fractions, notice that if $[a_k; \dots a_i; b_{k-1}, \dots, b_i] > 0$, then so are the continued fractions achieved by truncating at any $j \leq k$.

This gives us a necessary and sufficient condition for k-nonnegativity:

$$\begin{aligned} b_i &> 0 \\ [a_x; \dots a_{x-k+1}; b_{x-1}, \dots, b_{x-k+1}] &\geq 0 \qquad \text{if } x \geq k \\ [a_x; \dots a_1; b_{x-1}, \dots, b_1] &> 0 \qquad \text{if } x < k \end{aligned}$$

Second, to guarantee k-irreducibility, notice that the only Chevalley generators we need to worry about factoring out are e_{n-1} and f_1 (from the left), and e_1 and f_{n-1} (from the right). If we cannot factor any of these from J, then the order k principal minors $C_2(k)$ and $C_{n-k}(k)$ are zero (if not, then we break the form being correct or we break k-nonnegativity). This gives us the criteria in the statement. $\square$

Remark 3.22. Invertibility is an issue. The matrix:

$$\begin{bmatrix} 1 & 1 & & & \\ 1 & 1 & 1 & & \\ & 1 & 1 & 1 & \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \end{bmatrix}$$

satisfies all of the criterion for $n = 5$ and $k = 2$, but is not invertible. It is not obvious what more restrictions need to be added to give us invertibility.

Theorem 3.23. *The above criterion can be simplified into a $(2n-3)$-parameter family. Thus, the subset of invertible matrices in the family, along with Chevalley generators and diagonal matrices, generates all tridiagonal invertible kNN matrices.*

Proof. The criterion only gives lower bounds for our a_i and b_i, so we can turn them into parametrizations very easily.

Let our parameters be α_i for $i \in [1, n] \setminus \{k+1, n-1\}$, and β_i for $i \in [1, n-1]$. Let $b_i = \beta_i$. We specify $\beta_i \in \mathbb{R}_{>0}$.

Let $a_i = \alpha_i + [a_{x-1}, \ldots, a_1; b_{x-1}, \ldots, b_1]$, for $i < k$. We specify that $\alpha_i \in \mathbb{R}_{>0}$. Let $a_i = \alpha_i + [a_{x-1}, \ldots, a_{x-k+1}; b_{x-1}, \ldots, b_{x-k+1}]$, for $i \geq k$ and where α_i is defined. We specify that $\alpha_i \in \mathbb{R}_{\geq 0}$.

Finally, we cannot choose a_{k+1} and a_{n-1}, only solve using the above equations. Notice that because each inequality "tells us" a lower bound for each parameter, this simple family gives us precisely what we want. Showing this is just a matter of computation. □

Remark 3.24. We can actually reduce this to an $(n-3)$-parameter family, just by scaling the superdiagonal to ones via diagonal matrices. However, this proves more natural in the cases we have seen, so we present it in this manner.

Remark 3.25. If we imagine the $C_i(j)$ as a triangle indexed by i and j, then the k-initial minor matrix's diagonal entries are equal to $C_1(i)$ when $i \leq k$ and $C_{i-k+1}(k)$ when $i \geq k$. Further, the superdiagonal of the minor matrix gives us the value of the b_is. Thus, if we know our matrix is from our parameter family, the minor matrix determines the element from the family.

Theorem 3.26. *Let M be a matrix of the above form. Then if $RS = M$ in the semigroup of invertible k-nonnegative $n \times n$ matrices, one of R or S is a diagonal matrix.*

That is, any generating set of the semigroup must include all elements from the parameter family, up to scaling.

Proof. Suppose we have $RS = M$. From 2.21 we know that R and S have nonzero diagonals. Thus, we know that $r_{i,i+2}$, $s_{i,i+2}$ and their transpose analogues are all 0 from the formula for matrix multiplication. Further, we know that one of $r_{i,i+1}$ and $s_{i+1,i+2}$ are 0, and one of $r_{i,i+1}$ and $s_{i,i+1}$ is positive. Together, these show that R and S can only be as described above. □

And now a similar analysis for pentadiagonal unitriangular matrices. These are very similar, since a pentadiagonal unitriangular matrix is a tridiagonal matrix with ones on the subdiagonal, with an additional row and column added.

Lemma 3.27. *For M a k-nonnegative invertible pentadiagonal unitriangular matrix, we can write M as a product of Chevalleys, diagonal matrices, and matrices of the form*

$$\begin{bmatrix} I_p & & \\ & H_q & \\ & & I_{n-p-q} \end{bmatrix}$$

Where H_q is a $q \times q$ invertible pentadiagonal unitriangular k-nonnegative matrix with all entries nonzero that can be nonzero, and $q > k$.

We notate similarly as before: entries on the superdiagonal are a_is, and entries on the super-superdiagonal are b_is.

Theorem 3.28. *Let S be a pentadiagonal unitriangular matrix with all entries nonzero that can be nonzero. Then S is k-nonnegative and k-irreducible if and only if the following hold:*

$$\begin{aligned} a_i, b_i &> 0 \\ [a_x; \dots a_{x-k+1}; b_{x-1}, \dots, b_{x-k+1}] &= 0 \qquad \text{if } x \in \{k, k+1, n-1, n-2\} \\ [a_x; \dots a_{x-k+1}; b_{x-1}, \dots, b_{x-k+1}] &\geq 0 \qquad \text{if } x \geq k \text{ and not listed above} \\ [a_x; \dots a_1; b_{x-1}, \dots, b_1] &> 0 \qquad \text{if } x < k \end{aligned}$$

The proof is very similar; the only difference is that more minors get set to zero from k-irreducibility, two more than for the tridiagonal case. Notice that we do not get the same issue with invertibility as before.

Theorem 3.29. *The above criterion can be simplified into a $(2n-7)$-parameter family. Thus, the family, along with Chevalley generators, generates all pentadiagonal unitriangular kNN matrices.*

Theorem 3.30. *Let M be a matrix of the above form. Then if $RS = M$ in the semigroup of invertible k-nonnegative $n \times n$ matrices, one of R or S is the identity.*

That is, any generating set of the semigroup must include all elements from the parameter family, up to scaling.

Finally, notice that most of the results from Section 2 we discuss apply only very weakly to the $k = 1$ case. Generally, the smaller the k, the less TNN structure the semigroup seems to have. Thus, in some sense this is the "worst" case. We have essentially no results for this case, but we present some smaller items of note.

First, notice that *row-operation* generators, $e_{ij}(a) = I + \delta_{ij}(a)$ for $i \neq j$, are in this semigroup.

Thus, instead of considering 1-irreducible matrices, we can consider the following definition:

Definition. Consider a matrix M. Let $R_i = \{k \mid M_{ik} \neq 0\}$ and $C_i = \{k \mid M_{ki} \neq 0\}$ (so the indices representing nonzero elements in that row or column). Then M is *op-irreducible* if for all i, j, $R_i \subset R_j$ implies $i = j$ and $C_i \subset C_j$ implies $i = j$. In other words, no two R_i are comparable with each other, and no two C_i are comparable with each other.

Equivalently, a matrix M is op-irreducible when $M = RS$ in the semigroup of 1-nonnegative invertible $n \times n$ matrices implies that neither R nor S is a row-operation generator.

This definition has an analogous theorem to 2.25, with a similar proof.

Theorem 3.31. *If M is not op-irreducible, it can be expressed as a product of row-operation generators and a single op-irreducible matrix.*

Proof. Without loss of generality M has two rows m_i and m_j such that $\{k \mid m_{ik} \neq 0\} \subseteq \{k \mid m_{jk} \neq 0\}$. Let α be the largest ratio between any m_{ia} and m_{ja} (where a is an index where the two rows are both nonzero). Then the row operation sending m_j to $m_j - m_i/\alpha$ results in a matrix with one more zero than before.

If the resulting matrix is not op-irreducible, we can continue with this algorithm. We add one zero each time, so this algorithm eventually terminates. What we are left with must be op-irreducible. $\square$

Remark 3.32. Non-diagonal op-irreducible matrices are not TNN. Further, adding all op-irreducible matrices to the generators of TNN matrices gives a generating set for 1-nonnegative matrices.

Because the definition of an op-irreducible matrix is not dependent on the values in the nonzero entries (meaning replacing changing the values in nonzero entries does not affect op-irreducibility as long as they are being changed to other nonzero values), a natural question becomes what shapes (what patterns of nonzero entries and zero entries) of a matrix are op-irreducible. This question is hard.

Remark 3.33. The number of shapes of op-irreducible $n \times n$ matrices up to permutation of rows and columns form a sequence. The first six elements of this sequence are $2, 1, 2, 5, 20, 296$. If we specify that there must be an invertible matrix with that shape, the only difference becomes that the first element of the sequence is 1.

Example 3.34. For $n = 4$, the shapes are as follows (the asterisks mark nonzero entries):

$$\begin{bmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{bmatrix} \begin{bmatrix} * & & & \\ & & * & * \\ & * & & * \\ & * & * & \end{bmatrix} \begin{bmatrix} * & * & & \\ * & & * & \\ & * & & * \\ & & * & * \end{bmatrix} \begin{bmatrix} & * & * & * \\ * & * & & \\ * & & * & \\ * & & & * \end{bmatrix}$$

This example gives the false impression that an induction argument may be able to enumerate all possible op-irreducible shapes. This is mostly a coincidence of small numbers; for larger cases, there are no obvious patterns among op-irreducible shapes. See Appendix B for op-irreducible 5×5 shapes that are not obviously derived from original cases.

Remark 3.35. Once we have the shapes of op-irreducible matrices, we immediately get parameter families for our generators, when we add Chevalley matrices, diagonal matrices, and permutation matrices. These can be given for n up to five: for $n = 2$ we need to add nothing more, for $n = 3$ [2] has a generating set, and for $n = 4, 5$ we have presented the op-irreducible matrices sufficient to add. However, even for cases larger than three, we suspect that this is far from minimal; we may not even need an op-irreducible matrix of every shape in a minimal generating set.

Finally, since the problem is tied to numerous other topics like Sperner families and matrices with fixed row and column sums, we can rewrite the problem of counting op-irreducible matrix shapes in a number of ways. We give one here. A *clutter* is a hypergraph where no edge properly contains another.

Remark 3.36. The number of shapes of op-irreducible matrices up to permutations is equal to the number of clutters with n vertices and n edges whose duals are also clutters (up to isomorphism).

4 Cluster Algebras

4.1 Definitions

We start by giving a brief overview of relevant background on cluster algebras. This background is given for the sake of completeness, but we only use some of the combinatorial properties and don't need the additional algebraic structure. For more detailed and general discussion, see [7], [10], and [6]. These definitions are reproduced in a slightly modified form below.

Definition. A *quiver* is a directed multigraph with no loops or two-cycles. The vertices are labeled with elements of $[m]$. A directed edge (i, j) will be denoted $i \to j$. A *mutation* of a quiver Q at vertex j is a process, defined as follows, that produces another quiver $\mu_j(Q)$.

1. For all pairs of vertices i, k such that $i \to j \to k$, create an arrow $i \to k$.
2. Reverse all arrows adjacent to j.
3. Delete all two cycles.

If two quivers are related by a sequence of mutations, we say they are *mutation equivalent.*

Definition. Let $\mathcal{F} = \mathbb{C}(x_1, \ldots, x_m)$ be the field of rational functions over $\mathbb{C}$ in m independent variables (this is our *ambient field*). A *labeled seed* of geometric type in $\mathcal{F}$ is a pair $(\tilde{\boldsymbol{x}}, Q)$ where $\tilde{\boldsymbol{x}} = (x_1, \ldots, x_m)$ is an algebraically independent set of $\mathcal{F}$ and Q is a quiver on m vertices such that vertices in $[n]$ are *mutable* and vertices in $[n + 1, m]$ are *frozen* (unable to be mutated at). We call $\tilde{\boldsymbol{x}}$ the labeled *extended cluster* of the seed; $\boldsymbol{x} = (x_1, \ldots, x_n)$ the *cluster* with elements $x_1, \ldots, x_n$ the *cluster variables*; and remaining elements $x_{n+1}, \ldots, x_m$ the *frozen variables.*

Definition. A *seed mutation* at index $j \in [n]$ satisfies $\mu_j((\tilde{\boldsymbol{x}}, Q)) = (\tilde{\boldsymbol{x}}', \mu_j(Q))$, where $x'_i = x_i$ if $i \neq j$ and x'_j satisfies the following *exchange relation*:

$$x_j x'_j = \prod_{i \to j} x_i + \prod_{j \to k} x_k,$$

where arrows are counted with multiplicity. The right hand side is also referred to as the *exchange polynomial.*

Definition. For some starting seed $(\boldsymbol{x}, Q)$, let χ be the union of all cluster variables over seeds which are mutation equivalent. Let $R = \mathbb{C}[x_{n+1}, \ldots, x_m]$. Then the *cluster algebra* of rank n over R is $\mathcal{A} = R[\chi]$ together with the seeds generating it.

We do not care about the algebraic structure of a cluster algebra so much as the combinatorial objects behind it: the clusters and seeds themselves.

Definition. We consider two clusters *equivalent* if they share the same variables, up to permutation. Then the *exchange graph* has equivalence classes of clusters as vertices and an edge if two clusters are connected via a quiver mutation.

Total positivity tests for $n \times n$ matrices form a cluster algebra of rank $(n-1)^2$. All cluster variables correspond to rational functions in the matrix entries of a matrix of indeterminates $X = (x_{ij})_{i,j \in [n]}$. The *initial minors quiver*, $Q_i(n)$, has n^2 vertices, labeled by $(i,j) \in [n] \times [n]$. The variable associated with (i,j) is $\left|X_{[i-m+1,i],[j-m+1,j]}\right|$ where $m = \min(i,j)$. In other words, the variables are the entries of the initial minor matrix for X. There are arrows $(i,j+1) \to (i,j)$, $(i+1,j) \to (i,j)$, and $(i,j) \to (i+1,j+1)$. The vertices (n,j) and (i,n) are frozen for all $i,j \in [n]$. Below is an example for $n=3$. On the left is the initial minors quiver, where frozen vertices are denoted with $*$. Note that edges between frozen vertices never affect any exchange relations or mutations and hence can be disregarded. On the right is a table containing the variables corresponding to each vertex.

$$\begin{tikzcd}
\bullet \arrow[dr] & \bullet \arrow[l] \arrow[dr] & * \arrow[l] \\
\bullet \arrow[u] \arrow[dr] & \bullet \arrow[l] \arrow[u] \arrow[dr] & * \arrow[l] \\
* \arrow[u] & * \arrow[u] & *
\end{tikzcd}$$

	col 1	col 2	col 3
row 1	x_{11}	x_{12}	x_{13}
row 2	x_{21}	$x_{11}x_{22} - x_{12}x_{21}$	$x_{12}x_{23} - x_{13}x_{22}$
row 3	x_{31}	$x_{21}x_{32} - x_{22}x_{31}$	det

This framework leads to a natural set of sub-cluster algebras when looking for k-positivity tests. Because every exchange polynomial is subtraction free, using an exchange relation preserves positivity of the old cluster variable as long as all variables used in the mutation are positive. So for any quiver corresponding to a cluster in the totally positive cluster algebra whose variables are all minors, we take the sub-cluster algebra generated by the subquiver formed by freezing all vertices adjacent to minors of order greater than k, and then deleting all vertices whose variables are minors of order greater than k. These deleted vertices will sometimes be referred to as *dead vertices*. This ensures that mutations will preserve positivity. It will also be interesting at some points to consider the quiver this comes from, as it shows how the sub-cluster algebra embeds into the total positivity one, so we will move freely between these interpretations. When restricted, we will refer to the subquiver as the *k-quiver*, and when looking at how it's embedded we will refer to it as the *full quiver*. In the following example of the full quiver for $n = 3$, $k = 2$ with the initial minors quiver, frozen vertices are represented with $*$ and dead vertices are represented with $\ominus$. The k-quiver is depicted on the right.

$$\begin{tikzcd}
\bullet \arrow[dr] & \bullet \arrow[l] \arrow[dr] & * \arrow[l] \\
\bullet \arrow[u] \arrow[dr] & * \arrow[l] \arrow[u] \arrow[dr] & * \arrow[l] \\
* \arrow[u] & * \arrow[u] & \ominus
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[dr] & \bullet \arrow[l] \arrow[dr] & * \arrow[l] \\
\bullet \arrow[u] \arrow[dr] & * \arrow[l] \arrow[u] & * \arrow[l] \\
* \arrow[u] & * \arrow[u] &
\end{tikzcd}$$

This still does not quite give a k-positivity test: for general k the minimal size of a test is n^2 (which follows from the corresponding result for $k = n$ in Example 3.1.8 of [3]). The solution is to define a *test cluster*: we append to the extended cluster polynomials in the matrix entries until the size is n^2; these variables will be a potential k-positivity test and stay constant across the entire sub-cluster algebra. These extra variables will sometimes be referred to as *test variables*. For example, all test clusters for $k = n$ are in fact extended clusters. The cluster for the k-quiver initial minors quiver can be extended by adding all the missing solid minors of order k as test variables, giving the k-initial minors test. Not all choices of extra variables will give a valid k-positivity test, and in fact not all clusters can even be extended to k-positivity test of minimal size, as we shall discuss in Section 4.2. Although we do know which to add in specific cases (see Sections 4.3 and 4.4), as of now we lack a proof for the general method. With this setup, proving that a single test cluster in such a sub-cluster algebra is a k-positivity test proves that all clusters are: we can go between the variables in the extended clusters using subtraction-free rational expressions, and the rest of the variables in the test cluster stay the same.

For any k, the exchange graphs for these restricted sub-cluster algebras break the total positivity exchange graph into connected components. This is because the freezing of a vertex corresponds to deleting all edges corresponding to mutation there from the graph, and likewise for marking a vertex as dead. We can relate these components by looking at quivers for the sub-cluster algebras.

Definition. Two clusters from different sub-cluster algebras have a *bridge* between them if they have the same test cluster and there is a quiver mutation connecting them which occurs at a vertex which is frozen in the k-quiver.

See Figure 2 for an example. We can think of a bridge as swapping a cluster variable for a test variable. Mutation at dead vertices provides another method of jumping between sub-cluster algebras, as both the resulting test cluster and k-quiver are identical. In fact, mutation at a dead vertex produces a completely identical sub-cluster algebra, but one which is embedded differently in the total positivity cluster algebra. If one sub-cluster algebra provides k-positivity tests, then so do any connected via a bridge, as the "starting" test cluster which is bridged to is the same as one in the old sub-cluster algebra and hence also provides a k-positivity test.

4.2 The $n = 3$, $k = 2$ case

For 3×3 matrices, we'll label the entries as shown below:

$$M := \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}$$

An uppercase letter will denote the 2×2 minor formed by picking rows and columns that do not contain the lowercase version of the letter. For example, $A := ej - fh$. We further define $K := aA - \det M$ and $L := jJ - \det M$. From Exercise 1.4.4 of [6], we know that the only possible cluster variables correspond to minors and the two extra polynomials K and L. For a matrix which is totally positive, K and L must also be positive since they occur in clusters (and hence can be written as subtraction-free rational expressions in the initial minors). For a matrix which is maximally 2-positive, K and L are also positive as they are both differences of a positive term and a negative one. Therefore we can further generalize the natural sub-cluster algebras to starting quivers which also contain K or L,

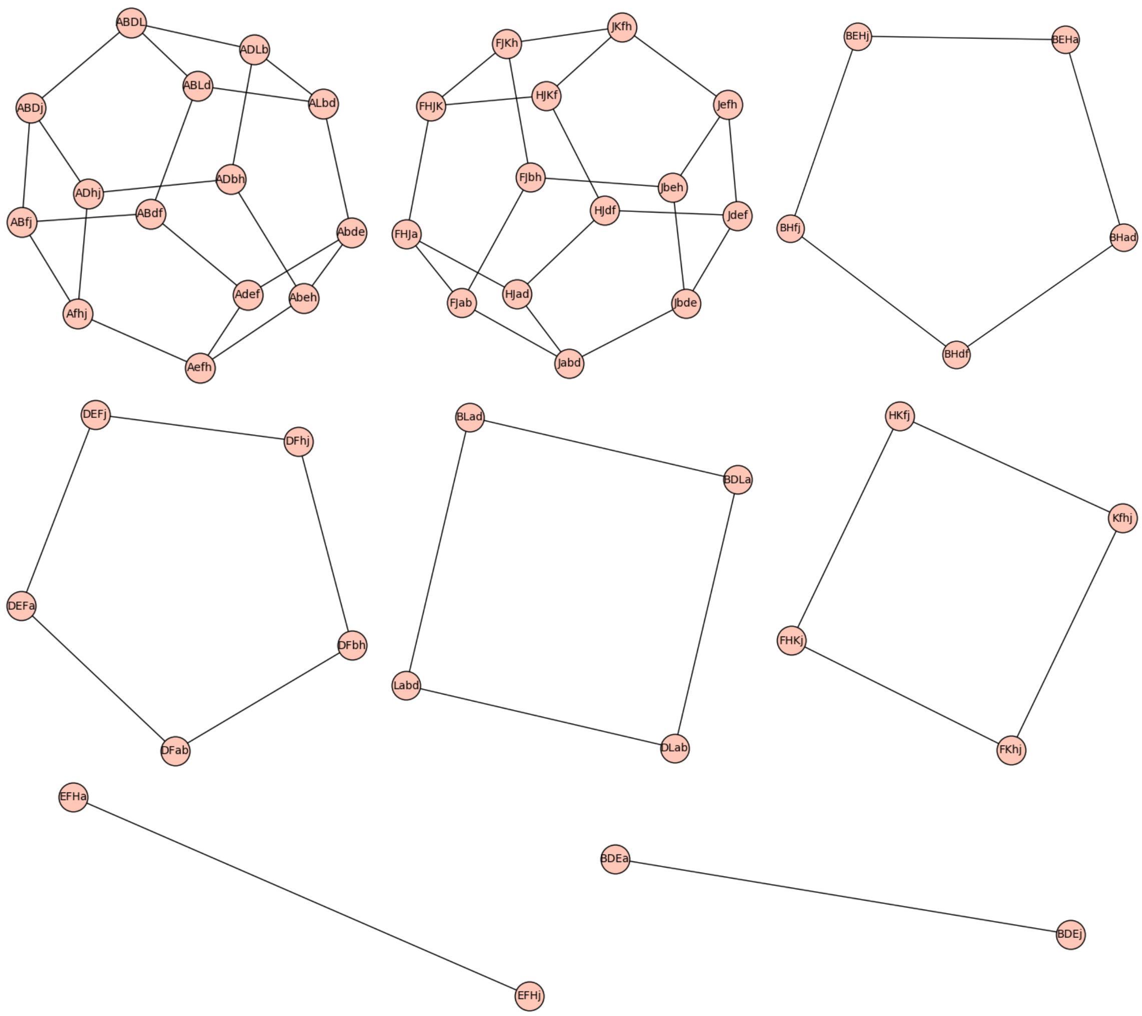


Figure 1: The connected components of a 2-positivity test graph derived from the 3×3 exchange graph.

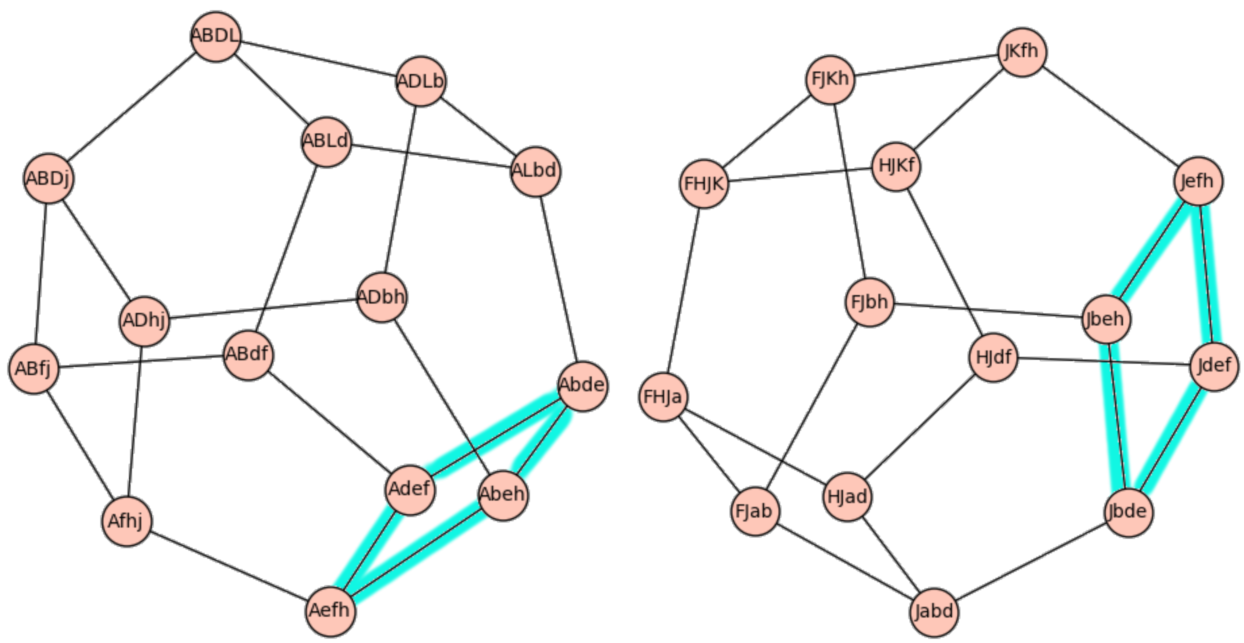


Figure 2: The bridges between the two associahedra. The left has test variable J and the right has test variable A. The two cyan squares pair up to give 4 bridges, by matching $Adef$-$Jdef$, $Aefh$-$Jefh$, $Abeh$-$Jbeh$, and $Abde$-$Jbde$ (i.e. those with the same test cluster).

where still only vertices adjacent to the determinant are frozen. The exchange graphs for the 8 sub-cluster algebras are depicted in Figure 1. The vertices are labeled by the cluster variables corresponding to vertices of the quiver which are mutable in the total positivity algebra, so that the extended cluster contains the listed variables plus $cgCG$. The two large associahedra both generate 2-positivity tests: the left contains $Afhj$ and so extending the test cluster with J creates the anti-diagonal k-initial test; the right contains $Jabd$ and so extending the test cluster with A creates the k-initial test. None of the other components can give 2-positivity tests (at least, not of size n^2): all are missing both of the minors A and J, but the extended cluster can only have one test variable added to it. Why must the test cluster contain both A and J? Consider the matrix

$$\begin{bmatrix} \epsilon & 1 & \epsilon^2 \\ 1 & \epsilon & 1 \\ \epsilon^2 & 1 & \epsilon^{-2} \end{bmatrix}$$

,

for some small positive constant ϵ. Note that every minor except J is positive, as well as the non-minors K and L. Thus, the positivity of J is not implied by the positivity of any other cluster variables involved in this example, and thus it must appear in every 2-positivity test. The same applies to A, with a slightly different matrix.

The bridging is depicted in Figure 2. The two cyan squares have the same test clusters (though different extended clusters), and we get 4 bridges between them by swapping A and J in and out of the clusters.

Note that in general, the subquiver induced by mutable vertices of the 2-quiver of the initial minors quiver is an orientation of the Dynkin diagram A_{2n-3}. From Theorem 5.1.3 of [10] and the discussion in Chapter 6, the exchange graph is then the corresponding associahedron of Cartan type A_{2n-3}.

4.3 k-essential minors

To help determine which of these components provide tests, we define the following: a minor $|X_{I,J}|$ is *k-essential* if there exists a matrix X such that $|X_{I,J}| \leq 0$, but $\forall (I', J') \neq (I, J)$, $|I'| = |J'| \leq k$, we have $|X_{I',J'}| > 0$. That is to say, a k-essential minor appears in all possible tests for k-positivity consisting only of minors, although based on general behavior we expect them to be present universally.

Proposition 4.1. *Solid k-minors are k-essential for $k \leq 3$.*

Proof. The $k = 1$ case is trivial, as there is exactly one test for 1-positivity: that consisting of all n^2 elements of the matrix. Explicitly, we can let $x_{i,j} = -1, x_{i',j'} = 1$ for $(i, j) \neq (i', j')$.

For $k = 2$, consider the matrix

$$M_2 := \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & . \\ \ldots & \epsilon^{-5} & \epsilon^{-3} & 1 & \epsilon^3 & \epsilon^7 & \epsilon^{10} & \ldots \\ \ldots & \epsilon^{-3} & \epsilon^{-2} & 1 & \epsilon^2 & \epsilon^5 & \epsilon^7 & \ldots \\ \ldots & 1 & 1 & \epsilon & 1 & \epsilon^2 & \epsilon^3 & \ldots \\ \ldots & \epsilon^3 & \epsilon^2 & 1 & \epsilon & 1 & 1 & \ldots \\ \ldots & \epsilon^7 & \epsilon^5 & \epsilon^2 & 1 & \epsilon^{-2} & \epsilon^{-3} & \ldots \\ \ldots & \epsilon^{10} & \epsilon^7 & \epsilon^3 & 1 & \epsilon^{-3} & \epsilon^{-5} & \ldots \\ . & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where ϵ is a sufficiently small positive constant, and the powers of ϵ throughout the rest of the matrix are inductively chosen so that all minors (that do not cross the rows or columns of the central minor) with that entry in the outermost (i.e. farthest from the center) position are positive. To show that this matrix is 2-positive everywhere except at the central minor, first note that we already know that all the 1-minors (entries of the matrix) are positive. Furthermore, if we look at the 2-positive matrix formed by switching the 1s and ϵs in the central minor, we see that we only need to look at minors whose values are smaller in M_2, i.e. those with a 1 from the central minor off the diagonal, or an ϵ from the central minor on the diagonal. These are positive by construction, as none of these cross the central minor.

The $k = 3$ case has more cases, but is roughly analogous. Let

$$M_3 := \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & . \\ ... & \epsilon^{-1} & 1 & \epsilon^2 & \epsilon^4 & \epsilon^8 & ... \\ ... & 1 & 1+\epsilon & 1+\epsilon & \epsilon & \epsilon^4 & ... \\ ... & \epsilon^2 & 1+\epsilon & 1+2\epsilon & 1+\epsilon & \epsilon^2 & ... \\ ... & \epsilon^4 & \epsilon & 1+\epsilon & 1+\epsilon & 1 & ... \\ ... & \epsilon^8 & \epsilon^4 & \epsilon^2 & 1 & \epsilon^{-1} & ... \\ . & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

again making all minors of size 3 or less not crossing the central minor positive by construction. As before, we can make the whole matrix 3-positive by replacing the central minor with

$$\begin{bmatrix} 1 & \epsilon & \epsilon^2 \\ \epsilon & 1 & \epsilon \\ \epsilon^2 & \epsilon & 1 \end{bmatrix}$$

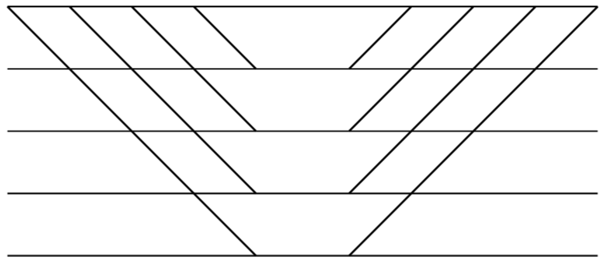
so minors not intersecting the central minor need not be considered either. This covers all 1- and 2-minors. 3-minors that cross the central 3 rows but not the central 3 columns (or the reverse) have a 1 in the upper left and the lower right, and term of order 1 in the center. All other terms in such a minor are smaller than this one by a factor of ϵ, so the minor is positive. Now consider a minor crossing both the center rows and columns. The upper left and lower right entries are negative powers of ϵ, while the upper right and lower left entries are positive powers of ϵ. The middle entry is on the scale of either 1 or ϵ, and all other terms are of order 1 at most. Thus, all other terms aside from the main diagonal term are smaller by a factor of ϵ, and thus the minor is positive. $\square$

Providing a constructive proof for the general case has proved difficult, as the central minors were constructed from maximally k-positive matrices consisting only of 1s and 0s, which do not exist for $k \geq 3$. Nevertheless, this is expected to generalize:

Conjecture 4.2.

- *Solid k-minors are k-essential.*
- *k-essential minors are present in every k-positivity test.*

By the combinatorial proof of Theorem 3.1.10 of [3] and the discussion following it, all corner minors are n-essential. That proof motivates an interesting classification of totally nonnegative k-positive matrices in terms of planar networks. For convenience, the following definitions are repeated from [9]. The planar network Γ_0 is



where all edges are directed rightwards and sources on the left side and sinks on the right side are labeled top to bottom from n to 1. We can think of this as being composed of n "tracks", where the i-th track is the path of horizontal edges connecting source i and sink i. The *weight matrix* has (i, j)-th entry the sum of weights of all paths from source i to sink j, where the weight of a path is the product of the weights of its edges. An edge is *essential* if it is slanted or is one of the n horizontal edges in the middle. A weighting is *semi-essential* if every essential edge has weight ≥ 0 and every other edge has weight 1. Then any invertible totally nonnegative matrix can be written as the weight matrix of some semi-essential weighting of Γ_0 (though perhaps not uniquely).

Proposition 4.3. *A semi-essential weighting yields a k-positive weight matrix if and only if every horizontal edge in the lowest k tracks is > 0, and the first k downward slanted and last k upward slanted edges between tracks i and $i + 1$ are > 0.*

Proof. Suppose we have such a weighting. Then for any $I, J \subset [n]$, $|I| = |J| \leq k$, there is a vertex disjoint path connecting sources in I to sinks in J which doesn't go through any 0 edges. Specifically, the ℓ-th source in I takes the ℓ-th downward slant path to the ℓ-th track, then takes the ℓ-th from the right upward slant path until the ℓ-th sink in J. Thus the appropriate minor is > 0 by Lindström's Lemma.

Conversely, suppose the weight matrix is k-positive. Then all the corner minors are positive up to order k. The only path from n to 1 goes down all the first downward slants, and so by positivity of this corner, all are positive. In general, the only collection of vertex disjoint paths from $[n - \ell + 1, n]$ to $[\ell]$ takes the first ℓ downward slants all the way down and then goes across. By positivity of that corner minor, all of these edges must be positive. The same argument applies to paths from $[\ell]$ to $[n - \ell + 1, n]$ and upward slanted edges. $\square$

This has as a corollary a weaker version of Corollary 2.23, where $k + 1$-nonnegativity is replaced with total nonnegativity.

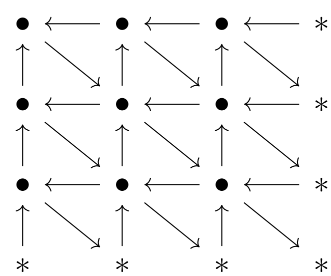
4.4 Path between Tests

In the general case, we would like to use our two known tests to find more. We do this by explicitly constructing a path between the initial minors quiver and its opposite quiver (the same quiver with all arrows reversed) which corresponds to the anti-diagonal flip test. A path here means a sequence of mutations such that every seed found corresponds to a valid k-positivity test.

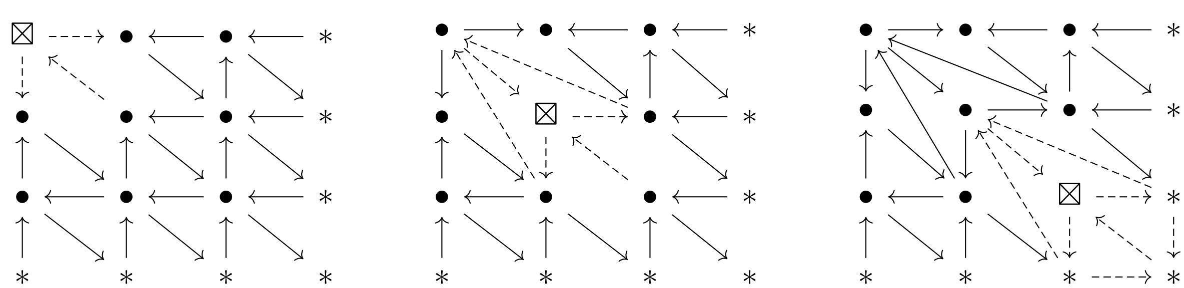
Proposition 4.4. *The following path connects the quivers corresponding to the initial test of Theorem 2.14 and its antidiagonal flip. Every edge in the path is a valid determinant-avoiding mutation, with the exception of a set of bridges and mutations at dead vertices. The path is as follows: mutate down the main diagonal, then along each sub- and superdiagonal (always skipping the last element, which lies in the last row or column). Repeat in the top left $m \times m$ submatrix as m ranges from $n-1$ to 1. Or more algorithmically:*

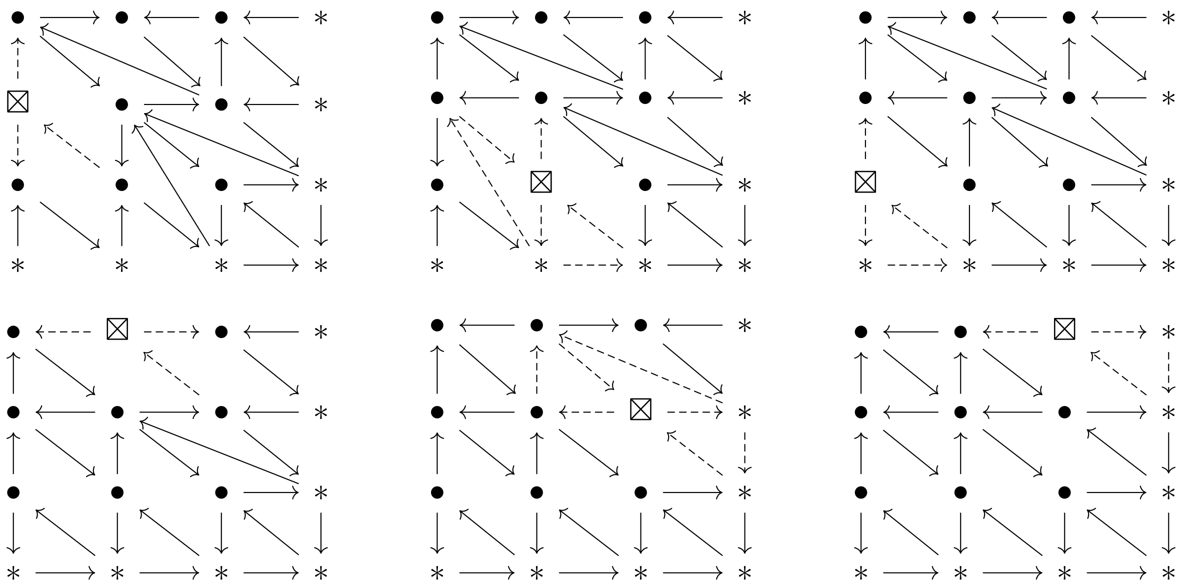
```
Q := Q_i(n)
For m in n, n-1, ..., 1:
    For i in [m-1]: # mutate down the diagonal
        Q := μ_(i,i)(Q)
    For r in [2,m-1]: # mutate down subdiagonals
        For i in [0,m-r-1]:
            Q := μ_(r+i,i+1)(Q)
    For c in [2,m-1]: # mutate down superdiagonals
        For i in [0,m-r-1]:
            Q := μ_(r+i,i+1)(Q)
```

Note that this requires $O(n^3)$ mutations. We now work through an example. The initial minors quiver is depicted below, with $*$ marking the frozen vertices.



Below is the first round of mutations, arranged in normal reading order. The $\boxtimes$ represents the vertex which was mutated at to get from the previous diagram, and arrows are dashed if they have changed.



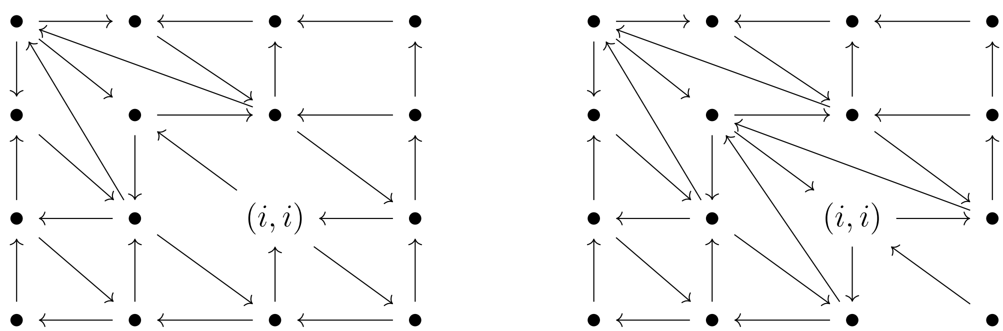


At the end of this round, we see that the upper right 3×3 subquiver is in fact the $n = 3$ initial minors quiver, and so the next round commences. We also see that the arrows not in that 3×3 subquiver have reversed directions, setting the outer portion up to be the antidiagonal flip of the initial minors quiver.

Proof. We do this via induction. After the ℓ-th round of this algorithm (which occurs in the $(n - \ell + 1) \times (n - \ell + 1)$ submatrix), we show that the variable at (i, j) corresponds to minor $\left|X_{[i-m+\ell+1,i+\ell],[j-m+\ell+1,j+\ell]}\right|$ where $m = \min(i, j)$. and that the subquiver obtained from the top left $(n - \ell) \times (n - \ell)$ submatrix (the "square" subquiver) has the form of the initial minors quiver, the subquiver obtained by eliminating that (the "L" subquiver) has the form of the opposite quiver, except that any edges on the boundary of these two subquivers are missing. Note in particular that each vertex always corresponds to a solid minor, the size of the submatrix to which it corresponds never changes, and the bottom left corner of the submatrix shifts one down its subdiagonal each time. Additionally, the test variables are always all of the solid minors of order k which are not already present in the extended cluster

Initially, we have $\ell = 0$ and the variables are in fact $\left|X_{[i-m+1,i],[j-m+1,j]}\right|$, the L subquiver is only the outer right and left edges which do have the horizontal and vertical arrows missing, and the test variables are the missing order k solid minors, as this is the k-initial minors test.

We now address what happens during the ℓ-th round. First we check the mutation down the diagonal. Mutating at (i, i) for $i > 1$ transforms the quiver as in the before and after images below (which each depict a subquiver). Note that if $i = n - \ell$, then the last row and column of arrows are not present in the initial image, and the after image has an up and right arrow in place of the missing ones.



In the base case of the upper left corner, one can check the forms of the quiver, and the exchange polynomial is

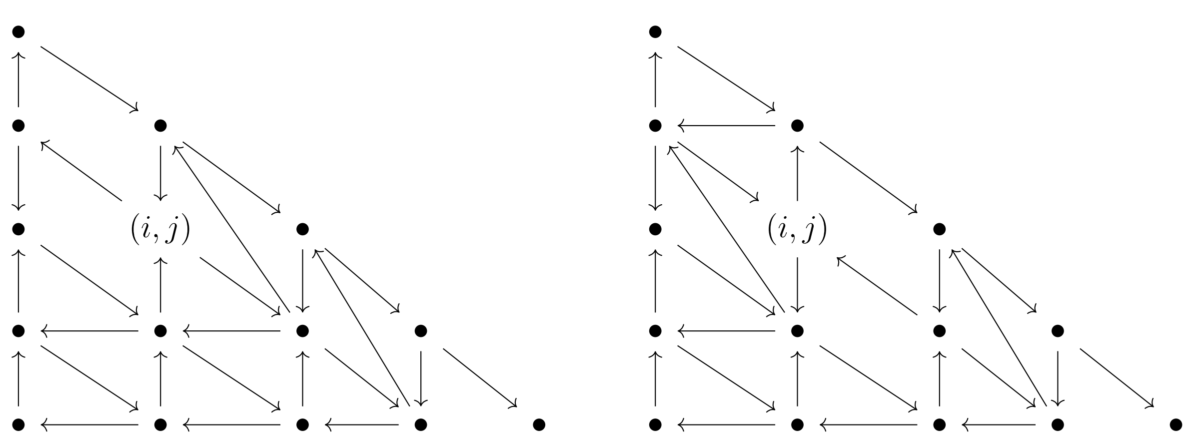
$$x' \cdot x_{\ell,\ell} = x_{\ell+1,\ell} \cdot x_{\ell,\ell+1} + \left|X_{[\ell,\ell+1],[\ell,\ell+1]}\right|$$

and thus $x' = x_{\ell+1,\ell+1}$. When mutating at (i,i) for $i > 1$, the exchange polynomial is

$$\begin{aligned} x' \cdot \left|X_{[\ell,i+\ell-1],[\ell,i+\ell-1]}\right| = \left|X_{[\ell+1,i+\ell-1],[\ell+1,i+\ell-1]}\right| \cdot \left|X_{[\ell,i+\ell],[\ell,i+\ell]}\right| \\ + \left|X_{[\ell+1,i+\ell],[\ell,i+\ell-1]}\right| \cdot \left|X_{[\ell,i+\ell-1],[\ell+1,i+\ell]}\right|. \end{aligned}$$

By Lewis Carroll's identity, this gives $x' = \left|X_{[\ell+1,i+\ell],[\ell+1,i+\ell]}\right|$. It is not hard to check the new quiver has the correct form as well. In particular, mutating at (k,k) exchanges $\left|X_{[\ell,k+\ell-1],[\ell,k+\ell-1]}\right|$ for $\left|X_{[\ell+1,k+\ell],[\ell+1,k+\ell]}\right|$. But the latter was already in the test cluster since it's a k-initial minor, and so this mutation is a bridge. Mutating at (i,i) for $i > k$ doesn't actually affect the test cluster at all, since these are dead vertices.

The previous paragraph gives the form of the quiver after the diagonal mutations. Now restrict to the subquiver using vertices on the diagonal and below (the case of above diagonal is symmetric). Inducting down the subdiagonal, one can check that mutating at (i,j) takes the quiver between the before and after subquivers depicted below. The case when $j = 1$ or $i = n - \ell$ have slightly different form, but one can check that mutating at $j = 1$ gives the correct setup for the general case (and clears the "extra" arrow), and that mutating at $i = \ell - 1$ also leaves the correct form, particularly in the arrows coming into and out of (i,j) from below, satisfying that part of the inductive hypothesis.



We now address the variables, again by induction on round. The diagonal case is addressed above. <ithout loss of generality, we travel down a subdiagonal (so that for any (i, j) we have $j = \min(i, j)$). As the formula for diagonal variables is in the same form as subdiagnal variables, the base case of the longest subdiagonal behaves the same as the general case, so we candeal with them together. When inducting down a particular subdiagonal, for $j = 1$, the exchange equation gives

$$x' \cdot x_{i+\ell-1,\ell} = x_{i-1+\ell,\ell+1} \cdot x_{i+\ell,\ell} + \left|X_{[i+\ell-1,i+\ell],[\ell,\ell+1]}\right|$$

and thus

$$x' = x_{i+\ell,\ell+1}.$$

Otherwise we then get an exchange equation of the form

$$\begin{aligned} x' \cdot \left|X_{[i-j+\ell,i+\ell-1],[\ell,j+\ell-1]}\right| &= \left|X_{[i-j+\ell,(i-1)+\ell],[\ell+1,j+\ell]}\right| \cdot \left|X_{[(i+1)-j+\ell,i+\ell],[\ell,j+\ell-1]}\right| \\ &\quad + \left|X_{[i-j+\ell+1,(i-1)+\ell],[\ell+1,(j-1)+\ell]}\right| \cdot \left|X_{[i-j+\ell,i+\ell],[\ell,j+\ell]}\right|. \end{aligned}$$

Using Lewis Carroll's identity on the submatrix with rows $[i - j + \ell, i + \ell]$ and columns $[\ell, j + \ell]$, this gives

$$x' = \left|X_{[i-j+\ell+1,i+\ell],[\ell+1,j+\ell]}\right|.$$

This proves the form of the variables.

Now we confirm the validity of these mutations in preserving k-positivity tests. Mutating at (i, k) turns $\left|X_{[i-k+\ell,i+\ell-1],[\ell,k+\ell-1]}\right|$ into $\left|X_{[i-k+\ell+1,i+\ell],[\ell+1,k+\ell]}\right|$. The latter is a k-initial minor from the test cluster and so this is an allowed exchange. As before, (i, j) for $j > k$ is a dead vertex and such mutations don't affect the test. Based on the form of the quiver, no other mutations go through larger submatrices. These are the only mutations we need to worry about, as one can check that any arrow added by an arbitrary mutation along the path only goes to the previous row and/or column. □

From the proof of this proposition, we can also easily prove the following fact:

Proposition 4.5. *Each sub-cluster algebra found along the path described in Proposition 4.4 has rank $(n - 1)^2 - (n - k)^2$.*

Proof. The rank of the subcluster algebra is the number of active vertices in its quivers. The initial quiver has $(n - 1)^2 - (n - k)^2$ active vertices: the bottom right $(n - k)^2$ are ignored as they correspond to minors of size $> k$, and the W of frozen vertices adjacent to this square contains $2n - 1$ elements. This gives the correct rank. As discussed in the above proof, no mutation at a dead vertex in the ignored square affects any of the active vertices, and a mutation at a frozen vertex (which occurs when jumping between subalgebras) never adds edges between active and dead vertices, and always keeps the frozen vertex adjacent to a dead one. Therefore the number of active vertices is the same. □

In fact, there is some choice in the order to do these mutations. Consider permutations of the path in which mutations on any particular sub- or super diagonal occur sequentially. Just as in the original path, the ℓ-th time mutating on a particular diagonal does not mutate at the last ℓ vertices. In other words, the only change is how these (sub-/super)diagonals are interleaved with each other. We use words in an alphabet $A = \{d, b_i, p_i \mid i \in [n]\}$ to keep track of this interleaving, where the letter in the j-th position denotes the j-th series of mutations: a d is mutating the diagonal, a b_i is mutating the i-th subdiagonal, and p_i is mutating the i-th superdiagonal. A word is turned into a diagram as follows: attach all x's (for any fixed letter x) into a diagonal chain of boxes with that many elements. Then attach them so that the first d box is anchored in the upper left corner, the first p_1 box (if any) is to the right, and the first p_i box is to the right of the first p_{i-1} box (if any, leaving a gap if it is not present). The same rule holds for the b_i, but these get attached to the bottom instead. For example, the word $dp_1p_2dp_1b_2$ becomes

$$\begin{array}{|c|c|c|} \hline d & p_1 & p_2 \\ \hline \multicolumn{1}{c|}{} & d & p_1 \\ \hline b_2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

A *valid path variant* is one which at every step looks locally like the original path, both in shape of quiver and variables. Specifically, after the ℓ-th mutation at any vertex, the variable and local quiver are the same as in the original path. These diagram transformations determine when some variants are valid.

Lemma 4.6. *Let w be a word with $n-1$ d's, $n-1-i$ b_i's, and $n-1-i$ p_i's. If the diagram formed from every initial subword of w is a Young diagram, then this sequence of mutations gives a valid path variant.*

For example, the word $dp_1p_2b_1dp_1b_2b_1d$ is valid for $n = 4$, but $dp_1p_2db_1p_1b_2b_1d$ is not because the initial subword dp_1p_2d has diagram

$$\begin{array}{|c|c|c|} \hline d & p_1 & p_2 \\ \hline \multicolumn{1}{c|}{} & d & \multicolumn{1}{|c}{} \\ \cline{2-2} \end{array}$$

Proof. Since mutations on sub- and superdiagonals are isolated from each other, we can freely commute mutations above and mutations below. We note that at any point along the path, any vertex in any quiver can only be adjacent to a subset of 8 different vertices: those above, below, left, right, above-left, below-right, as well as two more for the "extra slanted" edges (with specific direction depending on whether the vertex is diagonal, above, or below). Thus the mutation is indeed only affected by the variables and shape of the quiver locally, and we now proceed to confirm that the details of the proof of the variables in the path applies for such words.

All of the following statements can be verified inductively. For the local area to have the right shape to apply the ℓ-th mutation at a vertex on the main diagonal, it's good enough to have cleared the "extra" edges by mutating above and to the left $\ell - 1$ times. To get the same exchange relation, the vertices below and to the right must also have been mutated at exactly $\ell - 1$ times. The equivalent condition for the constructed diagram is that when placing the ℓ-th d box, it has a b_1 to the left and a p_1 above, and no extra b_1 or p_1 boxes to the right or below. Next look at the ℓ-th mutation of subdiagonal vertices (superdiagonal are symmetric). For the exchange relation to have the right form, the higher diagonal must have been done ℓ times, and the lower diagonal only $\ell - 1$. This same condition gives the quivers the right form. This corresponds to having a b_{i-1} above and a b_{i+1} to the right, but no extra below or to the left. Therefore since w gives a Young diagram at every step, the proof of Proposition 4.4 extends and for any k, these mutations preserve k-positivity tests. $\square$

Such Young diagram words give valid choices of paths between components of the exchange graph. Observe that two different such words which give the same final Young diagram both end in the same component. Observe also that any boxes outside of an $(n-k) \times (n-k)$ square are all mutations at mutable vertices in the k-quiver (since the length down which we mutate the diagonal decreases by one each round). Thus one gets a correspondence between Young diagrams contained in an $(n-k) \times (n-k)$ square and these components found along the path. Using such a Young diagram, one can also recover an explicit test cluster as discussed below.

In order to catalog the connected components of the exchange graph found along this family of paths, we define the *bridge graph* as $G_b = (V_b, E_b)$, where the elements of V_b are test cluster sets of the connected components on the path and we assign an edge between two components if and only if there exists a test cluster shared by those two components such that the corresponding extended clusters differ by exactly one element. We then label these edges by the elements that differ between them. If the lower-right corners of the exchanged minors lie on the center diagonal, we use the label d; if it lies on the ith superdiagonal (resp. subdiagonal) we use p_i (resp. b_i).

Given a young diagram Y contained in a $(n-k) \times (n-k)$ box, we can give its corresponding test cluster, arranged as entries of an $n \times n$ matrix M. First we construct a related young diagram Y' by taking Y and for every row of length $n-k$, appending $k-1$ boxes the right, and for every column of length $n-k$, appending $k-1$ boxes below. Note that Y' is now contained in a $(n-1) \times (n-1)$ box. Additionally, if Y filled its entire box, add boxes to Y' until it does as well. Now split Y' into components: D, all boxes on the diagonal, B_i, all boxes on the i-th subdiagonal, and P_i, all boxes n the i-th superdiagonal. For example, if $n = 7$ and $k = 2$, and $Y = (5, 5, 3, 2)$, then $Y' = (7, 7, 3, 2)$ and the following diagram labels each box according to which component it belongs to.

$$Y = \ydiagram{5,5,3,2} \qquad\qquad Y' = \begin{ytableau} D & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \\ B_1 & D & P_1 & P_2 & P_3 & P_4 & P_5 \\ B_2 & B_1 & D \\ B_3 & B_2 \end{ytableau}$$

The entries of M are similar to the entries of the k-initial minor matrix, except that the lower right corner of each minor is shifted down the appropriate sub/super diagonal by the number of boxes in that component, but the entry in (i, j) can be shifted at most $\min(n-i, n-j)$

places. More formally, if $i < j \le k$, the entries of M are as follows:

$$m_{ii} = \left|X_{[1+d,i+d],\,[1+d,i+d]}\right|$$
$$m_{ij} = \left|X_{[1+p_{j-i},i+p_{j-i}],\,[j-i+p_{j-i}+1,j+p_{j-i}]}\right|$$
$$m_{ji} = \left|X_{[j-i+b_{j-i}+1,j+b_{j-i}],\,[1+b_{j-i},i+b_{j-i}]}\right|$$

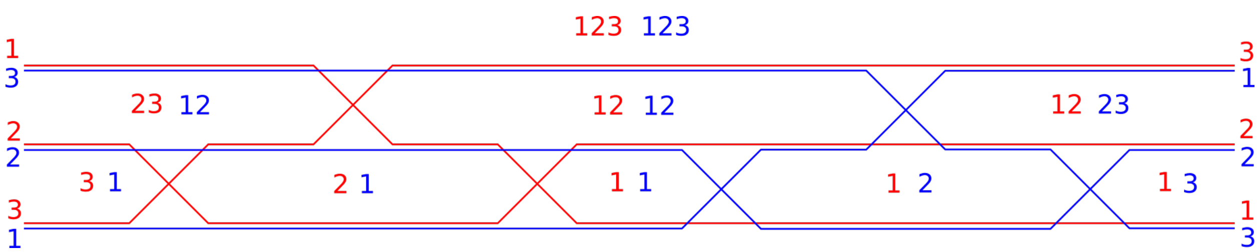
where $d = \min(n-i, |D|)$, $p_{j-i} = \min(n-j, |P_{j-i}|)$, and $b_{j-i} = \min(n-j, |B_{j-i}|)$ (and for the diagonal entry, we can take $i = k$ as well). Otherwise, if $k \le i < j$, the entries of M are all $k \times k$ minors. These can be filled in as desired, though it is convenient to have the lower right corners on the correct diagonal.

4.5 Double Wiring Diagrams

We start by recalling the appropriate definitions from [9].

Definition. A *wiring diagram* consists of a family of n piecewise straight lines, all of the same color, such that each line intersects every other line exactly once. A *double wiring diagram* is two wiring diagrams of different color which are overlaid.

We will color our diagrams red and blue, and number the strings such that the left endpoint of the reds go down, and the left endpoints of the blue go up. Each diagram has n^2 "chambers". We can label a chamber by the tuple (r, b), where r is the indices of all red strings passing below it, and b is the indices of all blue strings passing below it. For example,



We can associate each chamber with the minor of the correspondingly indexed submatrix $|X_{r,b}|$. With this correspondence, every double wiring diagram gives a total positivity test.Additionally, double wiring diagrams can be transformed into a quiver giving the corresponding test (see [6]) and there is also a method for transforming double wiring diagrams via braid relations (see [9]).

An alternative way to conceptualize this process is as follows: To describe a diagram, it is sufficient to describe the relative positions of all of the crossings. We can think of the diagram having n "tracks", where track i has all the chambers for $i \times i$ submatrices, and each crossing occurs in one of the first $n-1$ tracks. We label a red crossing in the i-th track as e_i, and a blue crossing in the i-th track as f_i. Then a sequence of crossings describing a double wiring diagram is a reduced word for the element (w_0, w_0) of the Coxeter group $S_n \times S_n$, where w_0 is the longest word, the order reversing permutation. This choice of variable names is not coincidental, double wiring diagrams and those with the weaker condition that every pair of same colored strings intersects at most once corresponds to factorizations, see [9].

The Young diagram correspondence from the previous section can now be extended one step further to result in a double wiring diagram. First, some useful notation. Let $r_i = e_{n-i} \cdots e_1$ for $1 \leq i < n-k$, and let $r_{n-k} = \prod_{i=1}^{k} e_i \cdots e_1$, ordered such that $i = 1$ is on the left and $i = k$ is on the right. Let $b_i = f_1 \cdots f_{n-i}$ for $1 \leq i < n-k$ and let $b_{n-k} = \prod_{i=k}^{1} f_1 \cdots f_i$, where now the concatenation runs in the opposite order: $i = k$ is on the left and $k = 1$ is on the right.For example, the lexicographically minimal diagram is the word $r_{n-k} \cdots r_1 b_1 \cdots b_{n-k}$, which corresponds to the initial minors test, and the word $b_1 \cdots b_{n-k} r_{n-k} \cdots r_1$ is the anti-diagonoal flip of the initial minors test. The order of the red wires between r_i and r_{i-1} (where r_0 and r_{n-k+1} correspond to no crossings) for $i \leq n-k$ is $i(i+1)\cdots(n-1)n(i-1)\cdots 21$, reading from bottom to top. This is proven via induction on i. If $i = 1$, the wires are already in order. In the general case, moving from the right of r_i to the left brings the bottom wire up to the $n-i$-th level, and all wires originally in the $(n-i)$-th row or lower are shifted down by one. To the left of r_{n-k}, the wires are ordered $n(n-1)\cdots 21$. The blue case behaves symmetrically, and the order of the blue wires between b_{i-1} and b_i for $i \leq n-k$ is $i(i+1)\cdots(n-1)n(i-1)\cdots 21$ and right of b_{n-k} is $n(n-1)\cdots 21$.

Theorem 4.7. *Let Y be a Young diagram which fits in a $(n-k)\times(n-k)$ box. Construct the corresponding double wiring diagram as follows: start with the word $b_1 \cdots b_{n-k}$. For $i \in [n-k]$, insert r_i between b_ℓ and $b_{\ell+1}$ where ℓ is the number of boxes in the i-th row of Y. If there is already an r_j in that position, insert r_i to the left if and only if $i > j$, otherwise insert it to the right. The result is an interleaving of the words $b_1 \cdots b_{n-k}$ and $r_{n-k} \cdots r_1$ which gives the total positivity test corresponding to that path component. To turn it into the correct k-positivity test, disregard all chambers above the k-th track and add in the remaining solid minors.*

Proof. We do prove this by showing that the full quiver given by this Young diagram corresponds to this double wiring diagram, and then the correct k-positivity test statement will follow. When doing this, we for simplicity work with Young diagrams contained in an $(n-1)\times(n-1)$ box, i.e. the $k = 1$ case. Here, r_{n-k} behaves exactly like the rest of the letters because it is only composed of a single decreasing chain of e_i; in fact $r_{n-1} = e_1$. The other cases will follow since grouping more decreasing chains together into r_{n-k} just corresponds to skipping the intermediate diagrams which go with those independent moves, and this grouping corresponds to the extra row or column which is appended in the Young diagram correspondence described at the end of Section 4.4. We proceed by induction on the number of boxes in the diagram. The base case is the lexicographically minimal diagram, already discussed. Assume the statement holds for diagrams with j boxes. Now add an ℓ-th box to the i-th row, where i is a row such that this is a valid addition. This changes the word from $\cdots r_i b_\ell \cdots$ to $\cdots b_\ell r_i \cdots$. The chambers which change are in tracks $\min(n-i, n-\ell)$ and lower. The chamber in track j goes from $([i, i+j-1], [\ell, \ell+j-1])$ to $([i+1, i+j], [\ell+1, \ell+j])$. The added box is on the main diagonal if $i = \ell$, is in component $P_{\ell-i}$ if $\ell > i$, and otherwise in $B_{i-\ell}$. The number of boxes in this component in the new diagram is $\min(i, \ell)$. By the inductive hypothesis and the original chambers, we see that if the minors are arranged in a matrix (as in the construction of the Young diagram test), it is the correct diagonal which is being changed, and the resulting chambers are correct as well. □

A Code

All code used can be found at https://github.com/ewin-t/k-nonnegativity. In particular, we have code for:

1. Generating shapes of op-irreducible matrices (through a somewhat-optimized brute force technique).
2. Generating k-nonnegative matrices (slowly and through brute force).
3. Generating the exchange graphs of the sub-cluster algebras for $k \leq 2$ or $n \leq 3$.

B Op-irreducible shapes for 5×5 matrices

We give enumerate all op-irreducible shapes for 5×5 matrices, from fewest nonzero entries to most nonzero entries. This was computed through brute-force code. In the matrices below, the asterisks denote the locations of nonzero entries. We only list the shapes up to twelve nonzero entries, since we can establish a bijection between shapes with k nonzero entries and shapes with $25 - k$ nonzero entries, by swapping the locations of zero and nonzero entries. Thus, we only need to list the shapes with at most half of the entries nonzero. These are permuted in what we perceive to be the neatest form; we have not found a consistent order that allows for clear patterns in the shapes.

The observant reader will note that if there is a row with 1 or $n-1$ nonzero entries, then there must be a column with 1 or $n-1$ nonzero entries, respectively. Removing the row and column gives us an op-irreducible size $n-1$ shape.

$$\begin{bmatrix} * & & & & \\ & * & & & \\ & & * & & \\ & & & * & \\ & & & & * \end{bmatrix} \begin{bmatrix} * & & & & \\ & * & & & \\ & & * & * & \\ & & * & & * \\ & & & * & * \end{bmatrix} \begin{bmatrix} * & & & & \\ & * & * & & \\ & * & & * & \\ & & * & & * \\ & & & * & * \end{bmatrix} \begin{bmatrix} * & & & & \\ & * & * & & \\ & * & & * & \\ & * & & & * \\ & & * & * & * \end{bmatrix} \begin{bmatrix} * & * & & & \\ * & & * & & \\ & * & & * & \\ & & * & & * \\ & & & * & * \end{bmatrix}$$

$$\begin{bmatrix} * & * & & & \\ * & & * & & \\ * & & & * & \\ & * & & & * \\ & & * & * & * \end{bmatrix} \begin{bmatrix} * & * & & & \\ * & & * & & \\ & * & & * & \\ & & * & & * \\ * & & & * & * \end{bmatrix} \begin{bmatrix} * & * & & & \\ * & & * & & \\ & * & & * & \\ & * & * & & * \\ * & & & * & * \end{bmatrix} \begin{bmatrix} * & * & & & \\ * & & * & & \\ & * & * & * & \\ & * & * & & * \\ & & & * & * \end{bmatrix} \begin{bmatrix} * & * & & & \\ * & & * & & \\ * & & & * & \\ * & & & & * \\ & * & * & * & * \end{bmatrix}$$

References

[1] A. Björner and F. Brenti. *Combinatorics of Coxeter Groups*. Springer, 2000.

[2] E. I. Bunina and V. V. Nemiro. “The Group of Fractions of the Semigroup of Invertible Nonnegative Matrices of Order Three Over a Field”. In: *Journal of Mathematical Sciences* 206.5 (2015), pp. 474–485. URL: https://link.springer.com/content/pdf/10.1007%2Fs10958-015-2326-4.pdf.

[3] Shaun M. Fallat and Charles R. Johnson. *Totally Nonnegative Matrices*. Princeton University Press, 2011. ISBN: 9780691121574. URL: http://www.jstor.org/stable/j.ctt7scff.

[4] Shaun Fallat, Charles R. Johnson, and Alan D. Sokal. “Total positivity of sums, Hadamard products and Hadamard powers: Results and counterexamples”. In: *Linear Algebra and its Applications* 520 (2017), pp. 242–259. ISSN: 0024-3795. URL: http://www.sciencedirect.com/science/article/pii/S0024379517300411.

[5] M. Fekete and G. Pólya. “Über ein problem von laguerre”. In: *Rendiconti del Circolo Matematico di Palermo (1884-1940)* 34.1 (Dec. 1912), pp. 89–120. ISSN: 0009-725X. DOI: 10.1007/BF03015009. URL: https://doi.org/10.1007/BF03015009.

[6] S. Fomin, L. Williams, and A. Zelevinsky. “Introduction to Cluster Algebras. Chapters 1-3”. In: *ArXiv e-prints* (Aug. 2016). arXiv: 1608.05735 [math.CO].

[7] S. Fomin and A. Zelevinsky. “Cluster algebras I: Foundations”. In: *J. Amer. Math. Soc.* 15 (2002), pp. 497–529. URL: https://doi.org/10.1090/S0894-0347-01-00385-X.

[8] Sergey Fomin and Andrei Zelevinsky. “Double Bruhat cells and total positivity”. In: *Journal of the American Mathematical Society* 12.2 (Apr. 1999), pp. 335–380. ISSN: 0894-0347.

[9] Sergey Fomin and Andrei Zelevinsky. “Total Positivity: Tests and Parametrizations”. In: *The Mathematical Intelligencer* 22.1 (2000), pp. 23–33.

[10] R.J. Marsh and European Mathematical Society. *Lecture Notes on Cluster Algebras*. Zurich lectures in advanced mathematics. European Mathematical Society, 2013. ISBN: 9783037191309. URL: https://books.google.com/books?id=ZmsdMj958ZcC.

[11] John Milnor. *Introduction to Algebraic K-Theory.(AM-72)*. Vol. 72. Princeton University Press, 2016.

[12] Allan Pinkus. *Totally Positive Matrices*. Cambridge Tracts in Mathematics. Cambridge University Press, 2009. DOI: 10.1017/CBO9780511691713.

[13] Pavlo Pylyavskyy. “Total Positivity Lecture Notes”. In: URL: https://149ec70c-a-62cb3a1a-s-sites.googlegroups.com/site/pylyavskyy/TPNotesPub.pdf?attachauth=ANoY7cpt_Ik2lw3JWOL-NSa6ZfTQcnNK_yQhn-O9Gx-7xzOi7OxNOHQOxdZPechEpc_BjT6s-J2kkE-obmb-BGkGcEiyIz0Ypxf_tvJhPZ3cQYus8zg2qOfuGecY5NJZmIUM2EF-CttaY5zRkNQuj1ui8gkPyiGsUk-M_G4zPIiAiIR1UNSnLZ8s4AYNJ4Tp-d3XTKgky8DIR_I1k86WP1AJloOsc4ul3Q%3D%3D&attredirects=0.