Cluster Algebras and k-positivity Tests

Anna Brosowsky[1], Sunita Chepuri[2], Alex Mason[3]

[1]Cornell University, [2]University of Minnesota, [3]Washington University in St. Louis

Introduction

An $n \times n$ matrix is *totally positive* if every minor is positive. Naïve testing is inefficient as there are $\binom{2n}{n} - 1$ minors. However, there are minimal tests of size n^2. We generalize the work of [1] to *k-positive* matrices, which only require that every minor of order at most k be positive. We find a family of cluster algebras embedded in the total positivity cluster algebra which give k-positivity tests, and give a combinatorial interpretation of some of these tests.

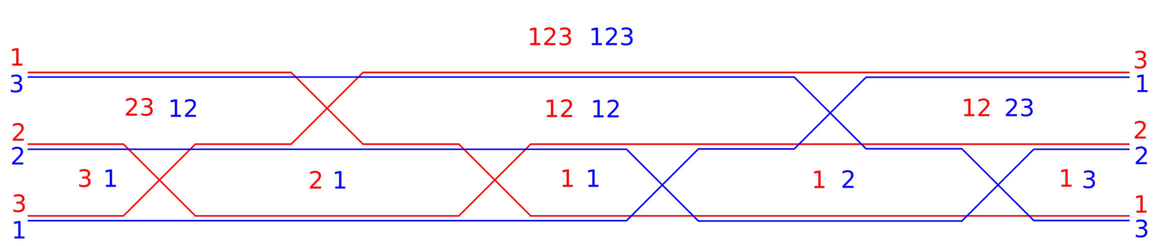


Figure 1: Double wiring diagram

Double Wiring Diagrams

A *double wiring diagram* is a family of n red and n blue numbered wires such that each pair of same-colored wires intersects exactly once. The *chambers* are spaces between the wires, labeled by wires which pass underneath. This turns diagrams into total positivity tests. Three types of *local moves* can "mutate" a diagram into a new one. Changed chamber satisfies subtraction-free *exchange relation* in surrounding chambers.

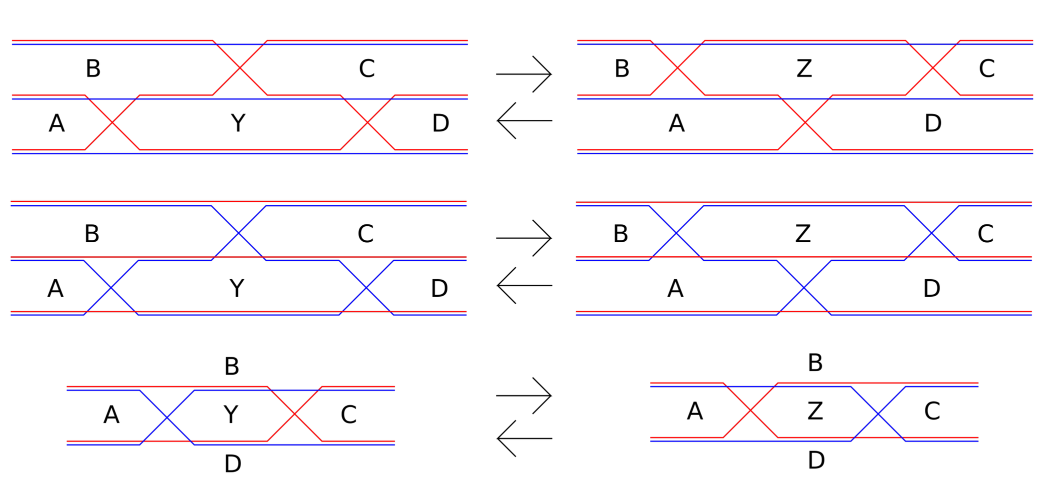


Figure 2: Local moves; exchange relation $YZ = AC + BD$

We make each chamber a vertex and overlay a quiver onto the diagram, so that arrows in and out of a vertex determine exchange relation.

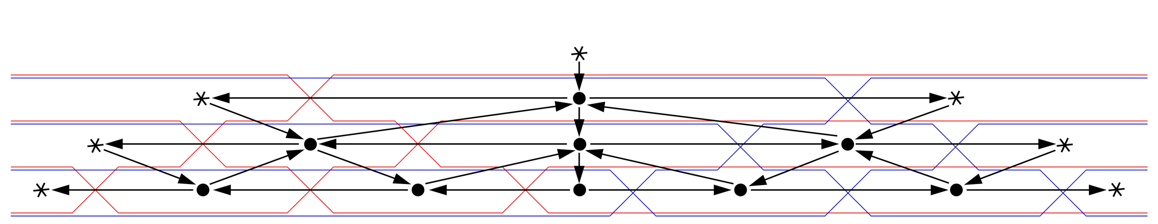


Figure 3: Quiver for double wiring diagram, $*$ = frozen.

Cluster Algebras

With quiver, can *mutate* at any vertex except *frozen* ones, even if no local move possible, and again have subtraction-free exchange relation. This makes TP tests a *cluster algebra*. Call test expressions *variables* (might not be matrix minors anymore). Subtraction-freeness means by proving a certain initial test works, any cluster of variables in the cluster algebra gives a test since we can write the initial test as a subtraction-free rational expression in the minors of this test.

Generalization

Does this argument hold for general k?

Problem: initial test has minors of every order. In general case, the large ones (e.g. the determinant) aren't guaranteed to be positive.

Solution: restrict the quivers. For an all-minors quiver, freeze any variable adjacent to a minor of order greater than k, then delete all minors which are too big. Call restricted cases *k-seeds.*

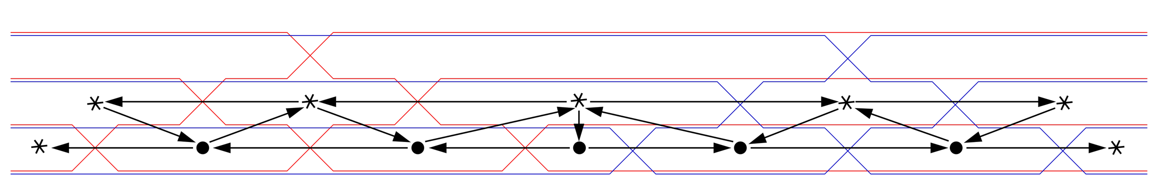


Figure 4: Restricted quiver for Figure 3 when $k = 2$.

Problem: now have fewer than n^2 minors; not enough to prove validity of an inital test.

Solution: add more variables. *Test variables* are a collection of expressions such that adding them to some initial k-seed gives k-positivity test of size n^2 (a *test seed*). Restricted initial test with missing *solid* minors (coming from contiguous rows and columns) of order k added gives k-positivity test [2].

Bridging

Restriction breaks total positivity cluster algebra into components—not all clusters can be connected by series of exchanges, since some might require too-large minors. Some components can be extended to give tests, others not. *Bridges* are restricted TP mutations which swap test variable for cluster variable.

Example

Let $n = 3$, $k = 2$. Define

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}$$

and let a capital letter denote the minor obtained by removing the row and column of the lowercase version, e.g. $A := ej - fh$. There are also two non-minor cluster variables, $K := aA - \det M$ and $L := jJ - \det M$. The restriction splits TP cluster algebra into 8 sub-cluster algebras; 2 can be extended to 2-positivity test cluster algebras. Below is graph showing clusters connected by mutations and bridges.

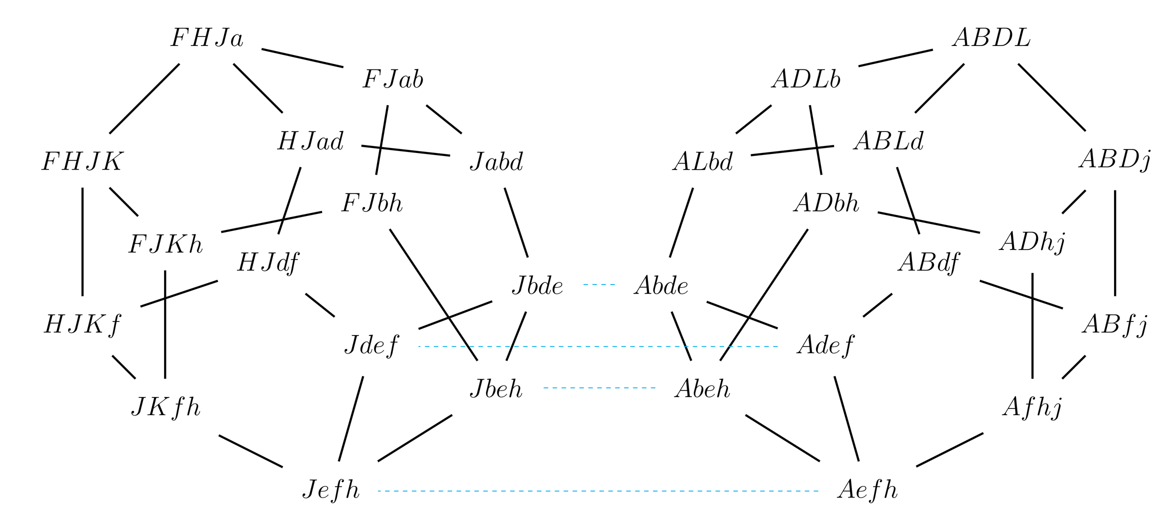
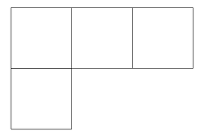


Figure 5: Bridges between two test components in $n = 3$, $k = 2$ case. The left has test variable A, the right has J. All have frozen variables c, g, C, G.

Young Diagrams

Describe double wiring diagrams by their crossings. r_i is downward-right diagonal of $n - i$ red crossings, ending with bottom wire. b_i is same but starts at bottom and is upward-right. Figure 1 is $r_2 r_1 b_1 b_2$. Given Young diagrams contained in $(n-1) \times (n-1)$ square, can construct double wiring diagram:

1. Start with $b_1 b_2 \cdots b_{n-1}$.
2. Let ℓ_i be the number of boxes in i^{th} row of Y.
3. For $i \in [n-1]$ insert r_i between b_{ℓ_i} and $b_{\ell_i + 1}$, preserving decreasing order of r_j's.

$Y =$ 

gives $r_3 b_1 r_2 b_2 b_3 r_1$, the double wiring diagram from Figures 3,4.

Fundamental Paths

Each Young diagram contained in an $(n-k) \times (n-k)$ square gives a k-positivity test in a different component by applying construction and adding missing solid k-minors as test variables.

Proof Sketch

Empty Young diagram gives initial test as base case. Adding one box swaps an r_i and b_j; a series of third type of local move applied increasingly high chambers. Lower swaps are sub-cluster algebra mutations; a swap on order k chamber is a bridge since original and exchanged minors both solid; higher swaps ignored by restriction. Once a box is outside of the $(n-k) \times (n-k)$ square, all swaps are low order and stay within same component.

k-essential Minors

Question: Which sub-cluster algebras give tests?

A minor of order $\leq k$ is *k-essential* if $\exists$ a matrix where it is the only non-positive minor of order $\leq k$. These must be in every all-minors test.

Conjecture: Solid k-minors are k-essential.

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