THE SUPER CATALAN NUMBERS $S(m, m+s)$ FOR $s \leq 3$ AND SOME INTEGER FACTORIAL RATIOS

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Abstract. We give a combinatorial interpretation for the super Catalan number $S(m, m+s)$ for $s \leq 3$ using lattice paths and make an attempt at a combinatorial interpretation for $s = 4$. We also examine the integrality of some factorial ratios.

1. Introduction

The Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}, \tag{1}$$

are known to be integers and have many combinatorial interpretations.

In 1874, E. Catalan [1] observed that the numbers

$$S(m, n) = \frac{(2m)!(2n)!}{m!n!(m+n)!}, \tag{2}$$

are also integers. Gessel [2] later referred to these numbers as the super Catalan numbers since $S(1, n)/2$ gives the Catalan number C_n. Gessel and Xin [3] presented combinatorial interpretations for $S(n, 2)$ and $S(n, 3)$, but in general it remains an intriguing open problem to find a combinatorial interpretation of the super Catalan numbers. In this paper, we give a combinatorial interpretation for $S(m, m+s)$ for $s \leq 3$.

2. A New Super Catalan Identity

We first derive another identity for the super Catalan numbers from the well-known Von Szily identity ([2, Section 6])

$$S(m, n) = \sum_k (-1)^k \binom{2n}{n-k}\binom{2m}{m+k}. \tag{3}$$

We provide both a combinatorial and an algebraic proof for this new identity.

Proposition 2.1. *For $m, s \geq 0$, the following identity for the super Catalan numbers holds:*

$$S(m, m+s) = \sum_k (-1)^k \binom{2m}{m-k}\binom{2s}{s+2k}. \tag{4}$$

Combinatorial Proof. We first interpret (3) in terms of lattice paths. We assume without loss of generality that $m \leq n$ and note that for any k, $\binom{2n}{n-k}\binom{2m}{m+k}$ counts the number of lattice paths from $(0, 0)$ to $(m+n, m+n)$ going though the point $(m+k, m-k)$ with unit right and up steps.

We now define a sign-reversing involution ϕ on the lattice paths

$$\bigcup_k ((0,0) \to (m+k, m-k) \to (m+n, m+n))$$

with the sign determined by the parity of k such that the number of fixed paths under this involution is the super Catalan number. We let $P = (s_1, s_2, \ldots, s_{2m+2n})$, an ordered set, be a path from the point $(0,0)$ to $(m+n, m+n)$ where each $s_i = R$ or U depending on whether it is a right step or an up step. We then find (if it exists) the least i, $1 \le i \le 2m$, such that $s_i \neq s_{2m+i}$ and switch these two steps. In other words, we have the map $\phi(P) = P' = (s'_1, s'_2, \ldots, s'_{2m+2n})$ such that

$$s'_j = \begin{cases} s_{2m+i}, & j = i \\ s_i, & j = 2m+i \\ s_j, & \text{otherwise.} \end{cases}$$

For such a path, since k is the number of right steps in the first $2m$ steps minus m, P and P' will have k's of opposite parity so ϕ is a sign-reversing involution. We notice that all the lattice paths with $s_k \neq s_{2m+k}$ for some $1 \le k \le 2m$ are cancelled under the involution ϕ. Therefore, the paths fixed under the involution are those from the point $(0,0)$ to $(m-k, m+k)$ to $(m+n, m+n)$ such that $s_i = s_{2m+i}$ for all $1 \le i \le 2m$. For any k, we have $\binom{2m}{m-k}\binom{2n-2m}{n-m+2k}$ such paths since by symmetry there are $\binom{2m}{m-k}$ ways to choose the first $4m$ steps and $\binom{2n-2m}{n-m+2k}$ ways to choose the last $2n-2m$ steps. Accounting for the parity of the k's, we then have that

$$S(m,n) = \sum_k (-1)^k \binom{2m}{m-k}\binom{2n-2m}{n-m+2k}.$$

Letting $s = n - m$ then gives us (4) as desired.

□

Algebraic Proof. Equating the coefficients of x^{m+n} in

$$(1+x)^{2n}(1-x)^{2m} = (1-x^2)^{2m}(1+x)^{2n-2m},$$

we have that

$$\sum_j (-1)^{m+j}\binom{2n}{n-j}\binom{2m}{m+j} = \sum_k (-1)^k \binom{2m}{k}\binom{2n-2m}{m+n-2k}.$$

Applying Von Szily's identity (3) on the left hand side and replacing k by $m-k$ on the right hand side, we get that

$$(-1)^m S(n,m) = \sum_k (-1)^{m-k}\binom{2m}{m-k}\binom{2n-2m}{n-m+2k}.$$

The result follows by letting $s = n - m$. □

3. A Combinatorial Interpretation of $S(m, m+s)$ for $s \le 3$

3.1. **The Main Theorem.** In this subsection, we use the identity in Proposition 2.1 to provide a combinatorial interpretation for $S(m, m+s)$ for $s \le 3$. To state our result, we first need to define four line segments, see Figure 1:

ℓ_1 connects $(0,1)$ and $(m-1/2, m+1/2)$,
ℓ_2 connects $(1,0)$ and $(m+1/2, m-1/2)$,
ℓ_3 connects $(m-1, m+1)$ and $(m+s-1, m+s+1)$, and
ℓ_4 connects $(m+1, m-1)$ and $(m+s+1, m+s-1)$.

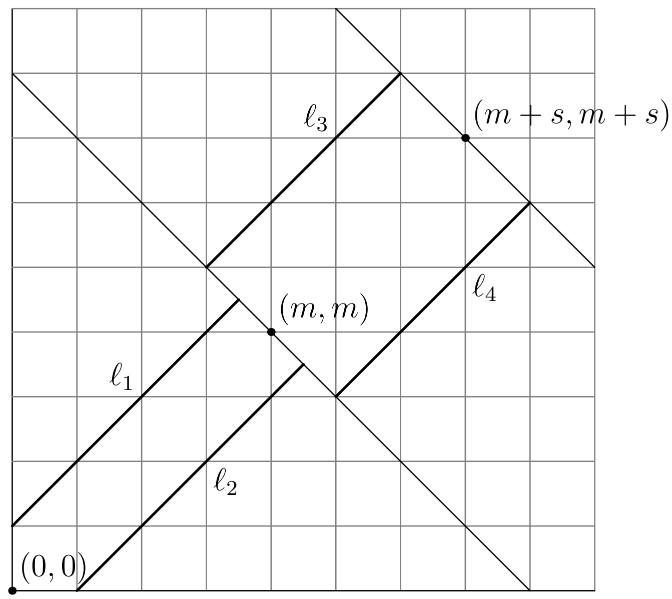

FIGURE 1. ℓ_1 through ℓ_4 for the $m = 4, s = 3$ case.

Theorem 3.1. *For $s \leq 3$, $S(m, m+s)$ counts the number of paths from $(0,0)$ to $(m+s, m+s)$ passing through (m, m) that do not intersect both lines ℓ_1 and ℓ_4 or both ℓ_2 and ℓ_3.*

Proof. From (4), we have that for $s \leq 3$,

$$S(m, m+s) = \binom{2m}{m}\binom{2s}{s} - \binom{2m}{m-1}\binom{2s}{s+2} - \binom{2m}{m+1}\binom{2s}{s-2}.$$

We notice that $\binom{2m}{m}\binom{2s}{s}$ counts the number of paths from the point $(0,0)$ to (m, m) to $(m+s, m+s)$ and denote this set of paths by Path_0. Similarly, $\binom{2m}{m-1}\binom{2s}{s+2}$ counts the number of paths from $(0,0)$ to $(m-1, m+1)$ to $(m+s+1, m+s-1)$ and $\binom{2m}{m+1}\binom{2s}{s-2}$ counts the number of paths from $(0,0)$ to $(m+1, m-1)$ to the point $(m+s-1, m+s+1)$. We denote these sets of paths by Path_{-1} and Path_1, respectively.

We define an injection from $\text{Path}_{-1} \cup \text{Path}_1$ to Path_0. To do so, first notice that any path $P \in \text{Path}_{-1}$ must intersect ℓ_1. We can find the last such intersection and reflect the tail of P's $(0,0)$ to $(m-1, m+1)$ segment over ℓ_1. This will give us a segment from $(0,0)$ to (m, m). We then translate the last $2s$ steps of P so that they start at (m, m) and end at $(m+s+2, m+s-2)$. This segment must then intersect ℓ_4. We can find the last such intersection and reflect the tail of this segment over ℓ_4. Combining the two new segments, we now have a path in Path_0, see Figure 2 for an example. Notice that this map has a well-defined inverse. We define a similar map for paths in Path_1, but reflect over lines ℓ_2 and ℓ_3 instead of ℓ_1 and ℓ_4.

We notice that paths in Path_{-1} cancel exactly with the paths in Path_0 that intersect both lines ℓ_1 and ℓ_4. Similarly, we find that paths in Path_1 cancel out

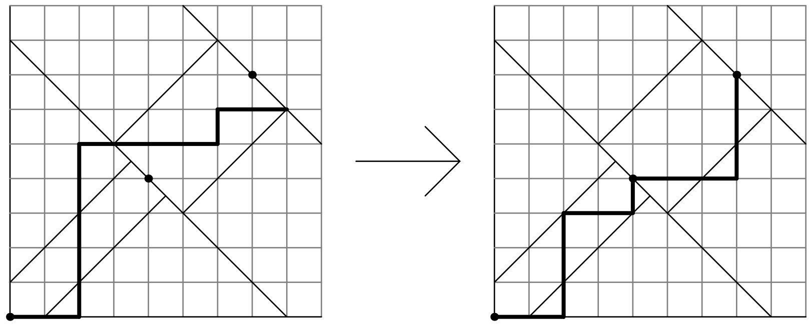

FIGURE 2. A mapping from Path_{-1} to Path_0 for $m = 4, s = 3$.

exactly with the paths in Path_0 that intersect both lines ℓ_2 and ℓ_3. Moreover, any path in Path_0 does not intersect both ℓ_3 and ℓ_4 since $s \leq 3$. This guarantees that the paths in Path_0 that Path_{-1} and Path_1 cancel do not overlap. Therefore, $S(m, m+s)$ counts the number of paths in Path_0 that do not intersect both lines ℓ_1 and ℓ_4 or both ℓ_2 and ℓ_3.

□

3.2. **Enumeration of the Paths in Theorem 3.1.** From (2), we can find explicit expressions for the super Catalan numbers when s is small. In this subsection, we count the paths remaining under the injection defined in Theorem 3.1 and show how they match with these explicit expressions.

For $s = 0$, we have that

$$S(m, m) = \frac{(2m)!(2m!)}{m!m!(m+m)!} = \binom{2m}{m},$$

where the central binomial coefficient counts the paths from $(0, 0)$ to (m, m). These are exactly the paths specified in Theorem 3.1 for $s = 0$.

For $s = 1$, we have that

$$S(m, m+1) = \frac{(2m)!(2m+2)!}{m!(m+1)!(2m+1)!} = 2\binom{2m}{m},$$

which counts the number of paths from $(0, 0)$ to $(m+1, m+1)$ going through (m, m). Since no paths from (m, m) to $(m+1, m+1)$ intersect ℓ_3 or ℓ_4, these are also exactly the paths specified in Theorem 3.1.

For $s = 2$,

$$S(m, m+2) = \frac{(2m)!(2m+4)!}{m!(m+2)!(2m+2)!} = 2(2m+3)C_m.$$

By Theorem 3.1, the paths remaining in Path_0 under the injection either intersect only one of ℓ_1 and ℓ_2 or intersect both ℓ_1 and ℓ_2. We first count those that only intersect ℓ_1. There are C_m ways to choose the first $2m$ steps since this segment of the path must stay below the line $y = x$. Then, there are 5 possible paths from (m, m) to $(m+2, m+2)$ that do not intersect ℓ_4. Hence, we have totally $5C_m$ paths from $(0, 0)$ to (m, m) to $(m+2, m+2)$ that intersect only ℓ_1. By symmetry, we have $5C_m$ paths that only intersect ℓ_2.

We now count the paths that intersect both ℓ_1 and ℓ_2. Consider the segment from $(0,0)$ to (m,m). We notice that to get the number of paths that intersect both ℓ_1 and ℓ_2, we subtract the number of paths that intersect only ℓ_1 or ℓ_2 from the total number of paths from $(0,0)$ to (m,m). We have

$$\binom{2m}{m} - 2C_m = \binom{2m}{m} - \frac{2}{m+1}\binom{2m}{m} = (m-1)C_m$$

such paths. Since there are 4 paths from (m,m) to $(m+2, m+2)$ that do not intersect ℓ_3 or ℓ_4, we have $4(m-1)C_m$ paths that intersect both ℓ_1 and ℓ_2. Therefore, the total number of paths is

$$2 \cdot 5C_m + 4(m-1)C_m = 2(2m+3)C_m,$$

as desired.

Finally, for $s = 3$, we have that

$$S(m, m+3) = \frac{(2m)!(2m+6)!}{m!(m+3)!(2m+3)!} = 4(2m+5)C_m.$$

Applying a similar method as in the $s = 2$ case, we can count $2 \cdot 14C_m$ paths that only cross ℓ_1 or ℓ_2 and $8(m-1)C_m$ paths that cross both ℓ_1 and ℓ_2. It is clear that

$$2 \cdot 14C_m + 8(m-1)C_m = 4(2m+5)C_m.$$

3.3. **A Note on the q-analog.** Warnaar and Zudilin [6] proved that the q-super Catalan numbers

$$[S(n,m)] = \frac{[2n]![2m]!}{[n]![m]![n+m]!}$$

are polynomials with positive coefficients, where the q-integer

$$[n] = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}.$$

Using our lattice paths interpretation, it is not hard to see why this is the case for $s = n - m \leq 3$.

For $s = 0$, $S(m,m) = \binom{2m}{m}$ has the q-analog

$$[S(m,m)] = \begin{bmatrix} 2m \\ m \end{bmatrix}.$$

It is well-known that $\begin{bmatrix} 2m \\ m \end{bmatrix}$ counts the number of unit squares in an $m \times m$ box above the lattice paths that remain under our injection in Theorem 3.1.

For $s = 1$, $S(m, m+1) = 2\binom{2m}{m}$ has the q-analog

$$[S(m, m+1)] = (1 + q^{m+1})\begin{bmatrix} 2m \\ m \end{bmatrix}.$$

For $s = 2$,

$$S(m, m+2) = 2(5C_m + 2(m-1)C_m) = 2(2m+3)C_m$$

has the q-analog

$$\begin{aligned}[S(m,m+2)] &= \frac{[2m]![2m+4]!}{[m]![m+2]![2m+2]!} \\ &= \frac{[2m+4]}{[m+2]}[2m+3][C_m] \\ &= (1+q^{m+2})([2m-2]+q^{2m-2}[5])[C_m] \\ &= (1+q^{m+2})((1+q^{m-1})[m-1]+q^{2m-2}[5])[C_m],\end{aligned}$$

where the q-analog of C_m,

$$[C_m] = \frac{1}{[m+1]}\begin{bmatrix}2m \\ m\end{bmatrix},$$

counts the *major index* of the length $2m$ words in the alphabet $-1, 1$, and for any given Catalan path, up and right steps correspond to 1 and -1, respectively (see exercise 6.34 in [5]).

Similarly, for $s = 3$,

$$S(m, m+3) = 2 \cdot 2(2m+5)C_m$$

has the q-analog

$$\begin{aligned}[S(m,m+3)] &= \frac{[2m]![2m+6]!}{[m]![m+3]![2m+3]!} \\ &= \frac{[2m+6]}{[m+3]}\frac{[2m+4]}{[m+2]}[2m+5][C_m] \\ &= (1+q^{m+3})(1+q^{m+2})[2m+5][C_m] \\ &= (1+q^{m+3})(1+q^{m+2})([2m-2]+q^{2m-2}[7])[C_m] \\ &= (1+q^{m+3})(1+q^{m+2})((1+q^{m-1})[m-1]+q^{2m-2}[7])[C_m].\end{aligned}$$

4. An Attempt at $s = 4$

In general, the methods used for finding a combinatorial interpretation for the super Catalan numbers $S(m, m+s)$ for $s \leq 3$ in Section 3 do not generalize nicely to higher s. In this section, we examine the case of $s = 4$.

We first define the following lines in addition to $\ell_1, \ell_2, \ell_3, \ell_4$ as defined in Section 3. These lines are pictured in Figure 3.

ℓ_5 connects $(m-1/2, m+1/2)$ and $(m+s-1/2, m+s+1/2)$,
ℓ_6 connects (m, m) and $(m+s, m+s)$, and
ℓ_7 connects $(m+1/2, m-1/2)$ and $(m+s+1/2, m+s-1/2)$.

Theorem 4.1. *For $s = 4$, $S(m, m+s)$ counts the number of paths from $(0, 0)$ to (m, m) to $(m+s, m+s)$ that do not touch both lines ℓ_1 and ℓ_4 or ℓ_2 and ℓ_3 such that if they have an intersection of ℓ_1 before an intersection of ℓ_2, they do not also remain between lines ℓ_5 and ℓ_6 or between ℓ_6 and ℓ_7.*

Proof. For $s = 4$, we notice that (4) gives us five terms corresponding to $k, -2 \leq k \leq 2$. Interpreting these terms with lattice paths, we again have that for any $i, -2 \leq i \leq 2$, the $k = i$ term counts the number of paths in $\mathrm{Path_i}$ where $\mathrm{Path_i}$ is defined as the set of lattice paths from $(0, 0)$ to $(m+i, m-i)$ to $(m+s-i, m+s+i)$.

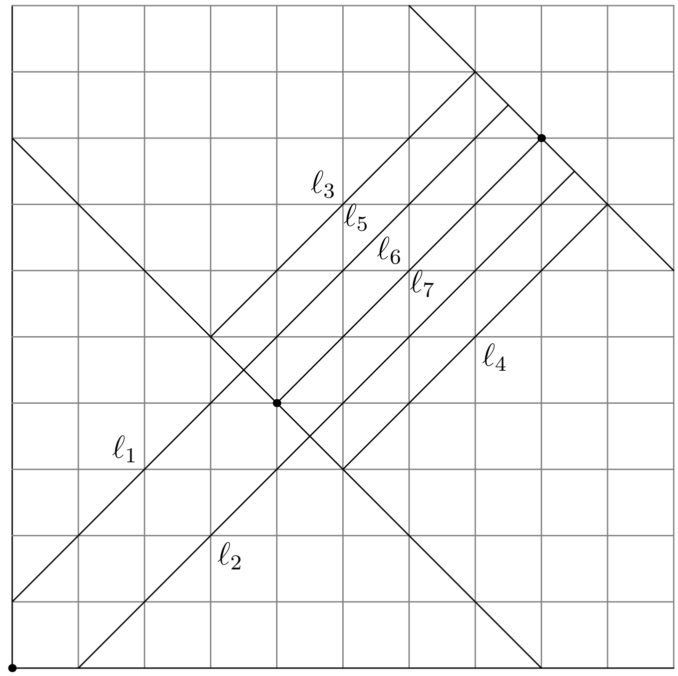

FIGURE 3. ℓ_1 through ℓ_7 for $m = 4, s = 4$.

We again map paths in Path_1 and Path_{-1} to Path_0 using the mapping defined in the proof of Theorem 3.1. Under this mapping however, we have double cancellation of paths that intersect all four lines ℓ_1, ℓ_2, ℓ_3, and ℓ_4. Therefore,

$$|\text{Path}_0| - |\text{Path}_1| - |\text{Path}_{-1}| = |\{\text{Paths that do not intersect both } \ell_1, \ell_4 \text{ or } \ell_2, \ell_3\}| \\ - |\{\text{Paths that intersect } \ell_1, \ell_2, \ell_3, \text{ and } \ell_4\}|.$$

We may also map the terms in Path_2 and Path_{-2} into the set Path_0. We first define ℓ_8 to be the line segment from $(0, 3)$ to $(m - 3/2, m + 3/2)$ and ℓ_9 to be the line segment from $(3, 0)$ to $(m + 3/2, m - 3/2)$. Then all paths in Path_{-2} must intersect ℓ_8. We find the last such intersection and reflect the tail end of these first $2m$ steps over line ℓ_8. Now we have a segment that intersects ℓ_8 and ends at $(m - 1, m + 1)$. This segment must also intersect ℓ_1 before its last intersection of ℓ_8. We find the last such intersection of ℓ_1 and reflect the tail end of the segment over ℓ_1. This then gives us a segment from $(0, 0)$ to (m, m) that has an intersection of ℓ_2 somewhere before an intersection of ℓ_1.

We note that this is also an invertible operation. Also, we transform the last $2s$ steps of the path to the segment from (m, m) to $(m + s, m + s)$ that intersects first ℓ_3 and then ℓ_4, see Figure 4 for an example.

By symmetry, we can map the paths in Path_2 to the paths in Path_0 that have an intersection of ℓ_1 before an intersection of ℓ_2 and also hit lines ℓ_4 and ℓ_3 in that order. Hence, the paths in Path_{-2} and Path_2 are mapped to exactly the paths in Path_0 that hit all four lines $\ell_1, \ell_2, \ell_3, \ell_4$, but at some point hit ℓ_1 before hitting ℓ_2.

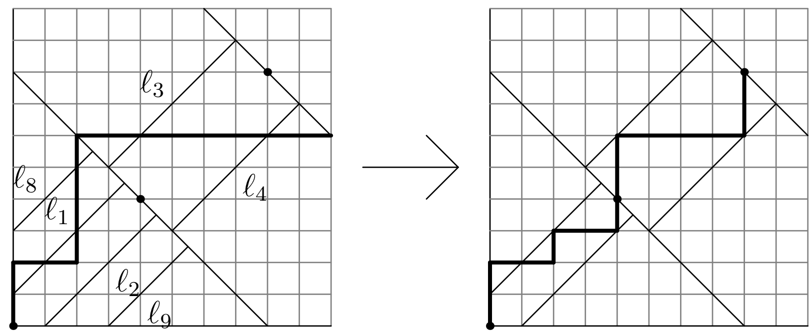

FIGURE 4. A transformation from Path_{-2} to Path_0 for $m = 4, s = 4$.

Adding this to our count for $|\text{Path}_0| - |\text{Path}_1| - |\text{Path}_{-1}|$, we have

$$S(m, m+4) = |\{\text{Paths that do not intersect both } \ell_1, \ell_4 \text{ or } \ell_2, \ell_3\}| \\ - |\{\text{Paths that intersect } \ell_1, \ell_2, \ell_3, \text{ and } \ell_4, \text{ but never } \ell_1 \text{ before } \ell_2\}|.$$

But if we define the sets S_1 and S_2 where

$$S_1 = \{\text{Paths that intersect } \ell_1, \ell_2, \ell_3, \text{ and } \ell_4, \text{ but not } \ell_1 \text{ before } \ell_2\}$$

and

$$S_2 = \{\text{Paths that intersect } \ell_1 \text{ and } \ell_2, \text{ not } \ell_1 \text{ before } \ell_2 \\ \text{that stay between } \ell_5 \text{ and } \ell_6 \text{ or } \ell_6 \text{ and } \ell_7\},$$

we can find a bijection between the two sets as follows. For paths in S_1, we map the last $2s$ steps of paths that intersect first ℓ_4 and then ℓ_3 to those that remain between ℓ_5 and ℓ_6, and the last $2s$ steps of paths that intersect first ℓ_3 and then ℓ_4 to those that remain between ℓ_6 and ℓ_7.

Hence, we can find that $S(m, m+4)$ counts the number of paths from $(0,0)$ to (m,m) to $(m+s, m+s)$ that do not touch both lines ℓ_1 and ℓ_4 or ℓ_2 and ℓ_3 such that if they have an intersection of ℓ_1 before an intersection of ℓ_2, they do not also remain between lines ℓ_5 and ℓ_6 or between ℓ_6 and ℓ_7.

$\square$

Remark 4.2. This is not a very satisfying interpretation of $S(m, m+4)$ because of the many conditions that it imposes on paths that are counted. The examination of the case of $s = 4$, however, reveals some of the difficulties of using this method for $s \geq 4$, namely that we have to deal with double cancellation as well as the addition of more paths. For these reasons, it is unlikely that the methods used in the proof of Theorem 3.1 will generalize to higher s.

5. Connection to Annular Non-crossing Partitions

In this section, we point out a connection between the numbers

$$P(p,q) = \frac{q}{2(p+q)} \binom{2p}{p} \binom{2q}{q}$$

(see Section 7 of [2]) and annular non-crossing partitions.

Definition 5.1. The set of annular non-crossing partitions of type B, denoted by $NC^{(B)}(p,q)$, consist of set partitions of

$$B(p,q) = \{1, 2, \cdots, p+q\} \cup \{-1, -2, \cdots, -(p+q)\}$$

such that

(1) If A is a block of the partition, then $-A$ is also a block of the partition.
(2) We draw two circles: one with $2p$ points labeled $1, 2, \cdots, p, -1, -2, \cdots, -p$ oriented clockwise on the outside, and the other with $2q$ points labeled $p+1, p+2, \cdots, p+q, -(p+1), -(p+2), \cdots, -(p+q)$ oriented counterclockwise on the inside.
(3) We draw a non-self-intersecting clockwise closed contour (following the order of $B(p,q)$) for each block of the partition such that the region enclosed stays in the annulus.
(4) Regions enclosed by different contours are mutually disjoint. In other words, different blocks of the partition do not cross each other.
(5) If a block A satisfies $A = -A$, then A must include points on both the inside and outside circles. We call such a block a *zeroblock* and notice that an annular non-crossing partition has at most one zeroblock.

Example 5.2. We refer the readers to [4, Fig. 1] for an example.

We also need the following definition in order to state the connection between annular non-crossing partitions and the super Catalan numbers.

Definition 5.3. An annular non-crossing partition is said to have *connectivity* c if it has c pairs of non-zero blocks with points on both the inside and outside circles. Such blocks are called *connected blocks*.

Theorem 5.4. *The number of annular non-crossing partitions with connectivity greater than or equal to* 1 *is*

$$|NC^{(B)}(p,q;c \geq 1)| = \frac{pq}{p+q}\binom{2p}{p}\binom{2q}{q}.$$

Remark 5.5. For the proof of the Theorem, we refer the reader to [4, Theorem 4.5].

With this result, we notice then that

$$P(p,q) = \frac{q}{2(p+q)}\binom{2p}{p}\binom{2q}{q} = \frac{|NC^{(B)}(p,q;c \geq 1)|}{2p}. \tag{5}$$

Gessel [2] gave a combinatorial interpretation for $P(p,q)$ as the number of lattice paths from $(0,0)$ to $(p+q, p+q-1)$ with unit up and right steps such that the path never touches the points $(q,q), (q+1,q+1), \cdots$ on the diagonal. It would also be interesting to find a combinatorial interpretation of the connection between the number $P(p,q)$ and the number of annual non-crossing partitions with connectivity greater than or equal to 1 as stated in (5).

Moreover, the super Catalan numbers can be obtained by choosing a subclass of the annular non-crossing partitions and put a signed weight using connectivity c, although it is not clear how we pick the subclass.

6. Other Integer Factorial Ratios

The super Catalan numbers (2) are a natural extension of the Catalan numbers (1), which are in one variable, to an integer factorial ratio in two variables. In this section, we extend the idea of the super Catalan numbers to other integer factorial ratios in three variables. We then find a general family of factorial ratios that are integral.

Two natural extensions of the super Catalan numbers to three variables are defined in (6) and (7). We first give algebraic proofs of the integrality of these factorial ratios.

Proposition 6.1. *For $a, b, c \geq 0$,*

$$\frac{(3a)!(3b)!(3c)!}{a!b!c!(a+b)!(a+c)!(c+b)!} \tag{6}$$

is an integer.

Proof. To show that (6) is an integer, it suffices to show that the order (i.e. multiplicity) of any prime p in the expression is non-negative. To do this, we first recall a well-known fact that the order (i.e. multiplicity) of a prime p in $a!$ is

$$\operatorname{ord}_p a! = \left\lfloor \frac{a}{p} \right\rfloor + \left\lfloor \frac{a}{p^2} \right\rfloor + \left\lfloor \frac{a}{p^3} \right\rfloor + \cdots .$$

Letting $l = \frac{a}{p^k}, m = \frac{b}{p^k}, n = \frac{c}{p^k}$, we see that to show

$$\frac{(3a)!(3b)!(3c)!}{a!b!c!(a+b)!(a+c)!(c+b)!}$$

is an integer, it suffices to prove that

$$\lfloor 3l \rfloor + \lfloor 3m \rfloor + \lfloor 3n \rfloor - \lfloor l \rfloor - \lfloor m \rfloor - \lfloor n \rfloor - \lfloor l+m \rfloor - \lfloor l+n \rfloor - \lfloor n+m \rfloor \geq 0.$$

Further letting $l_0 = l - \lfloor l \rfloor, m_0 = m - \lfloor m \rfloor, n_0 = n - \lfloor n \rfloor$, this inequality reduces to

$$\lfloor 3l_0 \rfloor + \lfloor 3m_0 \rfloor + \lfloor 3n_0 \rfloor - \lfloor l_0 + m_0 \rfloor - \lfloor l_0 + n_0 \rfloor - \lfloor n_0 + m_0 \rfloor \geq 0,$$

which is not hard to verify by dividing into 4 cases. □

Proposition 6.2. *For $a, b, c \geq 0$,*

$$\frac{(4a)!(4b)!(4c)!}{a!b!c!(a+b)!(a+c)!(c+b)!(a+b+c)!} \tag{7}$$

is an integer.[1]

Proof. The proof follows using a similar method as in the proof of Proposition 6.1. □

Moreover, Warnaar and Zudilin [6] observed that the above integrality of factorial ratios has interesting q-counterpart when we let

$$[n] = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}.$$

The integrality implies that the q-analogs of the factorial ratios are polynomials with integer coefficients. We can also examine the positivity of the coefficients of these q-analogs. They are positive for small a, b, and c, which leads us to the following conjectures:

[1]The authors would like to thank Rohit Agrawal for suggesting this factorial ratio.

Conjecture 6.3. *For* $a, b, c \in \mathbb{Z}_{\geq 0}$*,*

$$\frac{[3a]![3b]![3c]!}{[a]![b]![c]![a+b]![a+c]![c+b]!} \in \mathbb{N}[q].$$

Conjecture 6.4. *For* $a, b, c \in \mathbb{Z}_{\geq 0}$*,*

$$\frac{[4a]![4b]![4c]!}{[a]![b]![c]![a+b]![a+c]![c+b]![a+b+c]!} \in \mathbb{N}[q].$$

6.1. **Generalized Integer Factorial Ratios.** We may also generalize (7) to n variables $x_1, \dots, x_n$ as follows. We let $X = \{1, \dots, n\}$ and $P = \{S \subseteq X : S \neq \emptyset\}$. Then, we can consider whether the factorial ratio

$$\frac{\prod_{i=1}^{n} (2^{n-1} x_i)!}{\prod_{S \in P} \left(\sum_{i \in S} x_i \right)!} \tag{8}$$

is integral. We can also interpret (8) as the factorial ratio on a Boolean algebra with the denominator the product of factorial of each atom in the Boolean algebra poset and the numerator the product of the factorial for the total number of each atom in the poset. We can confirm the integrality of such factorial ratios for $n \leq 6$ by computer experimentation.

We can also consider the integrality of such ratios for generalized posets. To do so, we let $a_{i,k} \in \mathbb{Z}_{\geq 0}, 1 \leq i \leq l$ and

$$A_i(x_1, x_2, \dots, x_r) = \sum_{k=1}^{r} a_{i,k} x_k,$$

where for each i, at least one $a_{i,k} \neq 0$. Then, we can look at factorial ratios of the form

$$\frac{\prod_{j=1}^{r} \left(\left(\sum_{i=1}^{l} a_{i,k} \right) x_k \right)!}{\prod_{i=1}^{l} A_i(x_1, x_2, \dots, x_r)!}. \tag{9}$$

We note that not all such ratios are integral. For example, the factorial ratio [2]

$$\frac{(8a)!(6b)!(6c)!}{a!b!c!(a+b)!^3(a+c)!^3(b+c)!(a+b+c)!}$$

is not an integer for $a = 6, b = 1, c = 1$. Thus, an interesting question to consider is if there are sufficient conditions on the $a_{i,k}$ for the integrality of (9).

6.1.1. *An Attempt a Generalization.* In this section, we look at the family of integer factorial ratios where $\sum_{i=1}^{l} a_{i,k} = m$ for all k, some fixed m. We attempt to find a sufficient condition for the integrality of such ratios.

We let $a_{i,k} \in \mathbb{Z}_{\geq 0}, 1 \leq i \leq l$ and

$$A_i(x_1, x_2, \dots, x_r) = \sum_{k=1}^{r} a_{i,k} x_k,$$

[2]We would like to thank Prof. Dennis Stanton for showing us this example.

where for each i, at least one $a_{i,k} \neq 0$ and $\sum_{i=1}^{l} a_{i,k} = n$ for all k. Then, we look at the factorial ratio

$$\frac{(nx_1)!(nx_2)!\cdots(nx_r)!}{\prod_{i=1}^{l} A_i(x_1, x_2, \ldots, x_r)!} \tag{10}$$

We first define $y_k = x_k - \lfloor x_k \rfloor$ and notice that by similar reasoning as in the proof of Proposition 6.1, to show the integrality of (10), it suffices to show that

$$\sum_{k=1}^{r} \lfloor ny_k \rfloor - \sum_{i=1}^{l} \left\lfloor \sum_{k=1}^{r} a_{i,k} y_k \right\rfloor \geq 0$$

for all $\{y_1, \ldots, y_r\}$ where $0 \leq y_k < 1$ for all $1 \leq k \leq r$.

But we have that

$$\begin{aligned} \left\lfloor \sum_{k=1}^{r} a_{i,k} y_k \right\rfloor &\leq \sum_{k=1}^{r} a_{i,k} \left(\frac{(ny_k - \lfloor ny_k \rfloor) + \lfloor ny_k \rfloor}{n} \right) \\ &\leq \frac{b_i + \sum_{k=1}^{r} a_{i,k} \lfloor ny_k \rfloor}{n}, \end{aligned}$$

if $b_i = \sum_{k=1}^{r} a_{i,k} - 1$ where the last step follows since $(ny_k - \lfloor ny_k \rfloor) < 1$ for all y_k and $\lfloor \sum_{k=1}^{r} a_{i,k} y_k \rfloor$ must be an integer. Thus, we have that

$$\begin{aligned} \sum_{i=1}^{l} \left\lfloor \sum_{k=1}^{r} a_{i,k} y_k \right\rfloor &\leq \sum_{i=1}^{l} \left(\frac{b_i + \sum_{k=1}^{r} a_{i,k} \lfloor ny_k \rfloor}{n} \right) \\ &= \frac{\sum_{i=1}^{l} b_i + \sum_{k=1}^{r} \left(\sum_{i=1}^{l} a_{i,k} \right) \lfloor ny_k \rfloor}{n}. \end{aligned}$$

Therefore, if $\sum_{i=1}^{l} b_i < n$, we would have that $\lfloor \sum_{k=1}^{r} a_{i,k} y_k \rfloor < 1 + \sum_{k=1}^{r} \lfloor ny_k \rfloor$ from which it follows that since the sum of floor functions is integral, we have that

$$\sum_{k=1}^{r} \lfloor ny_k \rfloor - \sum_{i=1}^{l} \left\lfloor \sum_{k=1}^{r} a_{i,k} y_k \right\rfloor \geq 0.$$

Hence, this method gives us that a sufficient condition for the integrality of (10) is that $\sum_{i=1}^{l} b_i < n$.

Remark 6.5. We notice that this says that $\sum_{i=1}^{l} (\sum_{k=1}^{r} a_{i,k} - 1) = nr - l < n$, which is not strong enough of a condition to find a non-trivial family of integer factorial ratios. This suggests that only looking at the sum of $a_{i,k}$ terms is not enough to find sufficient conditions for integrality and that we would need to look at the interactions between $a_{i,k}$ terms to find a non-trivial condition.

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