ON THE LATTICE STRUCTURE OF SHARD INTERSECTION ORDERS.

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Abstract. Garver and McConville recently introduced two lattices associated to a tree: the lattice of biclosed sets and the oriented flip graph. They showed that both of these lattices are congruence-uniform lattices, and thus both admit a lattice-theoretic shard intersection order. We show that the shard intersection order of the oriented flip graph is graded and rank-symmetric. We also construct a CU-labeling of the lattice of biclosed sets and use this to understand the lattice structure of its shard intersection order.

1. Introduction

The study of triangulated surfaces is important to understanding the combinatorics of and representation theory related to cluster algebras. However, not as much is understood about the connections between polygonal subdivisions of surfaces (partial triangulations) and similar combinatorial and representation-theoretic objects. In [3], the authors study this question, and obtain an isomorphism between a lattice of polygonal subdivisions, subdivisions of surfaces (partial triangulations) and similar combinatorial and representation-theoretic objects.

2. Preliminaries

2.1. Lattices. Let $L$ be a finite, graded lattice. For $x, y \in L$, if $x \prec y$ and no $z \in L$ satisfies $x \prec z \prec y$, we write $x \prec y$. Let \( Cov(L) := \{(x, y) \in L^2 \mid x \prec y\}\) be the set of covering relations of $L$. A map $\lambda : Cov(L) \to \mathbb{Q}$, where $(Q, \leq_Q)$ is some poset is called an edge labeling. Given two maximal chains $C = c_1 < \cdots < c_n$ and $C' = c'_1 < \cdots < c'_n$ in $L$, we say $C$ is lexicographically smaller than $C'$ if $(\lambda(c_1), c_2), \ldots, (\lambda(c_{n-1}), c_n)$ lexicographically precedes $(\lambda(c'_1), c'_2), \ldots, (\lambda(c'_{n-1}), c'_n))$.

We call a labeling $\lambda : Cov(L) \to \mathbb{Q}$ an EL-labeling of $L$ if for every interval $[x, y]$ of $L$,

1. there is a unique increasing maximal chain $C$ in $[x, y]$, and
2. $C$ is lexicographically smaller than any other maximal chain $C'$ in $[x, y]$.

If $L$ admits an EL-labeling, it is said to be EL-shellable.

We need the concepts of join and meet irreducibility to discuss another important type of labeling. We say that an element $j \in L$ is join irreducible if $j \neq 0$ and whenever $j = x \lor y$, either $j = x$ or $j = y$ holds. Meet-irreducible elements are defined dually. We denote the subset of join-irreducible (resp. meet-irreducible) elements by JI$(L)$ (resp. MI$(L)$). For $j$ (resp. $m$) in JI$(L)$ (resp. MI$(L)$), we let $j_\ast$ (resp. $m_\ast$) denote the unique element of $L$ covered by (resp. covers) $j$ (resp. $m$). With this in hand, we arrive at the notion of a CN-labeling, which plays a prominent role in this paper.

Definition 2.1. A labeling $\lambda : Cov(L) \to \mathbb{Q}$ is a CN-labeling if $L$ and its dual $L^\ast$ satisfy the following: For elements $x, y, z \in L$ with $(z, x), (z, y) \in Cov(L)$ and maximal chains $C_1$ and $C_2$ in $[z, x \lor y]$ with $x \in C_1$ and $y \in C_2$,

(CN1) the elements $x' \in C_1, y' \in C_2$ such that $(x', x \lor y), (y', x \lor y) \in Cov(L)$ satisfy $\lambda(x', x \lor y) = \lambda(z, y), \lambda(y', x \lor y) = \lambda(z, x)$;

(CN2) if $(u, v) \in Cov(C_1)$ with $z \prec u, v \prec x \lor y$, then $\lambda(z, x), \lambda(z, y) \prec_Q \lambda(u, v)$;
(CN3) the labels on Cov(C1) are pairwise distinct.

We say that \( \lambda \) is a **CU-labeling** if, in addition, it satisfies

(CU1) \( \lambda(j_*, j) \neq \lambda(j'_*, j') \) for \( j, j' \in IJ(L), j \neq j' \), and

(CU2) \( \lambda(m, m^*) \neq \lambda(m, m') \) for \( m, m' \in MI(L), m \neq m' \).

If \( L \) admits a CU-labeling, it is said to be **congruence-uniform**, see [3].

Given a lattice \( L \) with a CU-labeling, one can define a new poset, \( \Psi(L) \), called the **shard intersection order** of \( L \). Reading introduced this concept in [4].

**Definition 2.2.** Let \( L \) be a congruence-uniform lattice with CU-labeling \( \lambda : \text{Cov}(L) \rightarrow P \). Let \( x \in L \) and let \( y_1, \ldots, y_k \) be the elements of \( L \) satisfying \( (y_i, x) \in \text{Cov}(L) \). We denote the set \( \{ \lambda(y_i, x) \} \) by \( \lambda(x) \). Define the **shard intersection order** of \( L \), denoted \( \Psi(L) \), to be the collection of sets of the form

\[
\psi(x) := \{ \lambda(w, z) \mid \bigwedge_i y_i \leq w < z, (w, z) \in \text{Cov}(L) \}
\]

partially ordered by inclusion.

2.2. **Tree preliminaries.** A **tree** is a finite connected acyclic graph. Any tree may be embedded in a disk \( D^2 \) in such a way that a vertex is on the boundary if and only if it is a leaf. We will always assume that any tree comes equipped with such an embedding. We will refer to non-leaf vertices of a tree as **interior vertices**, and denote the set of interior vertices of a tree \( T \) by \( V^o(T) \). We assume that any interior vertex of a tree has degree at least 3. Given trees \( T, T' \) embedded in \( D^2 \), we consider \( T \) and \( T' \) to be **equivalent** if there is an isotopy between the spaces \( D^2 \backslash T \) and \( D^2 \backslash T' \).

Let \( T \) be a tree embedded in \( D^2 \). The embedding of \( T \) in \( D^2 \) determines a collection of 2-dimensional regions in \( D^2 \) that we will refer to as **faces**. A **corner** of a tree is a pair \((v, F)\) consisting of an interior vertex \( v \) and a 2-dimensional face \( F \) containing \( v \). The embedding that accompanies \( T \) endows each interior vertex with a cyclic ordering. Given two corners \((u, F), (u, G)\), we say that \((u, G)\) is **immediately clockwise** (resp. **immediately counterclockwise**) from \((u, F)\) if \( F \cap G \neq \emptyset \) and \( G \) is clockwise (resp. counterclockwise) from \( F \) according to the cyclic ordering at \( u \).

An **acyclic path** supported by a tree \( T \) is a sequence \((v_0, \ldots, v_t)\) of vertices of \( T \) such that \( v_i \) and \( v_j \) are adjacent if and only if \( |i - j| = 1 \). We typically identify acyclic paths with their underlying vertex sets; that is, we do not distinguish between paths of the form \((v_0, \ldots, v_t)\) and \((v_1, \ldots, v_0)\).

An **arc** \( p = (v_0, \ldots, v_t) \) is an acyclic path whose endpoints are leaves and for all \( i \), the edges \((v_{i-1}, v_i)\) and \((v_i, v_{i+1})\) are adjacent to a common face—another way of saying this is that \( p \) “turns sharply” at every vertex it contains. We say \( p \) **contains a corner** \((v, F)\) if \( v = v_i \) for some \( i = 0, 1, \ldots, t \) and \( F \) is the face that is incident to both \((v_{i-1}, v_i)\) and \((v_i, v_{i+1})\). Since an arc \( p \) divides \( D^2 \) into two components, it determines two disjoint subsets of the set of faces of \( T \) that we will call **regions**.

A **segment** is an acyclic path consisting of at least two vertices and with the same incidence condition that is required of arcs, but whose endpoints are not leaves. Since trees have unique geodesics between any two vertices, if the endpoints of a segment or arc are \( v, w \), we may denote the path by \([v, w]\). We say that two segments \( s \) and \( t \) are **composable** if they agree only at one endpoint and their concatenation at this endpoint is also a segment. We denote their composition by \( s \circ t \).

2.3. **Noncrossing tree partitions.** We now introduce a partially ordered set defined on a tree, the **noncrossing tree partitions**, which is one of this paper’s central objects of study. For the proof of all the results in this subsection, see [3].

Given a tree \( T \), a **red admissible curve** \( \gamma : [0, 1] \rightarrow D^2 \) for a segment \( s = [v_0, v_t] \) is a simple curve where

- its endpoints are \( v_0 \) and \( v_t \),
- \( \gamma \) may only intersect edges of \( T \) of the form \((v_{i-1}, v_i)\), where \( i \in [t] \), and
- \( \gamma \) must leave \( v_0 \) (resp. \( v_t \)) on the face immediately clockwise from \( v_0 \) (resp. \( v_t \)), orienting \( T \) from \( v_0 \) to \( v_t \).

Two segments are **noncrossing** if they admit red admissible curves that do not intersect. A **green admissible** curve is the same as a red admissible curve, replacing the word “clockwise” with “counterclockwise.”

Given a collection of vertices \( B \subseteq V^o(T) \), we say that \( B \) is **segment-connected** if for \( v, w \in B \) there exists a sequence of segments \( s_1 = [v_1 = v, v_2], s_2 = [v_2, v_3], \ldots, s_{n-1} = [v_{n-1}, v_n = w] \) such that \( s_1 \circ \cdots \circ s_{n-1} = [v, w] \), with \( v_i \in B \), and \([v_1, v_{i+1}] \cap (B \backslash \{v_i, v_{i+1}\}) = \emptyset \).

It was shown in [3] that given two such vertices \( u \) and \( v \) in a segment-connected set \( B \subseteq V^o(T) \), the sequence of segments described in the above definition is unique. The union of all segments appearing in the
Let Theorem 2.5. We have already defined what it means for two segments to cross; there is a similar notion for arcs.

We say that two arcs \( p = (v_0, \ldots, v_k), q = (w_0, \ldots, w_k) \) are crossing along a segment \( s = (u_0, \ldots, u_r) \) if

i) each vertex of \( s \) appears in both \( p \) and \( q \)

ii) if \( R_p \) and \( R_q \) are regions defined by \( p \) and \( q \), respectively, then \( R_p \cap R_q \neq \emptyset \).

We say they are noncrossing otherwise. The noncrossing complex \( \Delta^{NC}(T) \) is defined to be the abstract simplicial complex whose simplices are pairwise noncrossing collections of arcs supported by a tree \( T \).

If \( p \) is an arc whose vertices all lie on a common face, then \( p \) is non-crossing with every arc supported by \( T \). We call such an arc a boundary arc. The reduced noncrossing complex \( \tilde{\Delta}^{NC}(T) \) is the abstract simplicial complex whose faces are the faces of \( \Delta^{NC}(T) \) that contain no boundary arcs.

We now introduce a partial ordering on arcs that contain a particular corner of \( T \). Let \( F \) be a face of \( \Delta^{NC}(T) \) and let \( (v, F) \) be a corner that is contained in at least one arc of \( F \). The arcs of \( F \) that contain \( (v, F) \) are partially ordered in the following way: \( p \leq_{(v, F)} q \) if and only if the region defined by \( p \) containing \( F \) is contained in that of \( q \).

If \( F \) is a face of \( \Delta^{NC}(T) \) and \( (v, F) \) is a corner contained in at least one arc of \( F \), then the partially ordered set \( \{ p \in F : p \text{ contains } (v, F) \}, \leq_{(v, F)} \) is a linearly ordered set. In particular, it has a unique maximal element, which we will denote by \( p(u, F) \). We say that an arc \( p \) of \( F \) is marked at \( (v, F) \) if \( p = p(v, F) \). The following theorem discussing marked corners is crucial to defining the oriented flip graph.

Theorem 2.5. Let \( F \) be a face of \( \Delta^{NC}(T) \), let \( p \in F \), and let \( \text{Reg}_1, \text{Reg}_2 \) denote the regions defined by \( p \).

1) The arc \( p \) is marked at a corner of \( T \).
2) If \( p \) is not a boundary arc, then \( p \) is marked at a corner in \( \text{Reg}_1 \) and at a corner in \( \text{Reg}_2 \).
3) Assume that \( p \) is marked at two distinct corners \( (v, F), (w, G) \in \text{Cor}(T) \) and that \( F \) and \( G \) belong to the same region defined by \( p \). Then there exists an arc \( p' \neq F \) that contains \( (v, F) \) and \( (w, G') \) such that \( G' \neq G \) and where \( F \cup \{ p' \} \in \Delta^{NC}(T) \).
4) If \( F \) is a facet and \( p \in F \) is not a boundary arc, then there exists a unique arc \( q \neq F \) such that \( (F \setminus \{ p \} \cup \{ q \}) \) is a facet. Moreover, if \( p \) is marked at two distinct corners \( (v, F), (u, G) \in \text{Cor}(T) \), then \( [v, u] \) is the unique longest segment along which \( p \) and \( q \) cross.

A corollary of this result is that the simplicial complex \( \tilde{\Delta}^{NC}(T) \) is pure (i.e. every facet has the same cardinality).

We refer to the operation \( \mathcal{F} \mapsto (\mathcal{F} \setminus \{ p \}) \cup \{ q \} \) sending facet \( \mathcal{F} \) of \( \tilde{\Delta}^{NC}(T) \) to a new facet of \( \tilde{\Delta}^{NC}(T) \) as a flip of \( \mathcal{F} \) at \( p \), and denote it by \( \mu_p \). We define the flip graph of \( T \), denoted \( \text{FG}(T) \), to be the graph whose vertices are facets of \( \tilde{\Delta}^{NC}(T) \) and such that two vertices are connected by an edge if and only if the corresponding facets can be obtained from each other by a single flip.

Given a facet \( \mathcal{F} \) and an arc \( p \in \mathcal{F} \) marked at \( (u, F) \) and \( (v, G) \), we call \( p \) red (resp. green) if \( F \) and \( G \) are the immediately clockwise (resp. counterclockwise) faces of their corresponding vertices, orienting from \( u \) to \( v \); it is shown in [3] that for a facet \( \mathcal{F} \), every arc in \( \mathcal{F} \) is either green or red.

Definition 2.6. Let \( \mathcal{F}_1, \mathcal{F}_2 \in \text{FG}(T) \) and assume that \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are connected by an edge in \( \text{FG}(T) \). Let \( \mathcal{F}_2 = \mu_p \mathcal{F}_1 \) and let \( q \) denote the unique arc produced by flipping \( p \) in \( \mathcal{F}_1 \). If \( p = p(u, F) = p(v, G) \) and \( q = p(u, F') = p(v, G') \), we orient the edge connecting \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) so that \( \mathcal{F}_1 \rightarrow \mathcal{F}_2 \) if and only if the corner \( (u, F') \) (resp. \( (v, G') \)) is immediately clockwise from the corner \( (u, F) \) (resp. \( (v, G) \)) about vertex \( u \) (resp. \( v \)). We refer to the resulting directed graph as the oriented flip graph of \( T \) and denote it by \( \text{FOG}(T) \).

The following result concerns the structure of \( \text{FOG}(T) \), and allows to define the shard intersection of \( \text{FOG}(T) \).
Theorem 2.7. The poset $\overline{FG}(T)$ is a congruence-uniform lattice.

With this in hand, we can consider $\Psi(\overline{FG}(T))$, and this is where one witnesses the connection between $\text{NCP}(T)$ and $\overline{FG}(T)$.

Theorem 2.8. The poset $\Psi(\overline{FG}(T))$ is a lattice, and $\text{NCP}(T) \cong \Psi(\overline{FG}(T))$.

There exists a bijection $\rho: \overline{FG}(T) \rightarrow \text{NCP}(T)$ given as follows. For a facet $F \in \overline{FG}(T)$, each red arc in $F$ defines a segment whose endpoints are the arc’s two marked corners. Let $S$ denote this set of segments. Recall that for a segment-connected block in $\text{NCP}(T)$, there exists a unique set of segments realizing this segment-connectedness. The authors showed in [3] that $S$ defines an element of $\text{NCP}(T)$ in this manner, and that $\rho$ is in fact a bijection.

Accordingly, the isomorphism $\varphi$ from $\text{NCP}(T)$ to $\Psi(\overline{FG}(T))$ is given by mapping $B \in \text{NCP}(T)$ under $\rho^{-1}$ and then letting $\varphi(B) = \psi(\rho^{-1}(B))$.

2.5. Biclosed collections of segments. All of the results in this subsection are proved in [3]. Let $\text{Seg}(T)$ be the set of segments supported by a tree $T$. For $X \subseteq \text{Seg}(T)$, we say $X$ is closed if for segments $s, t \in \text{Seg}(T)$, if $s, t \in X$ and $s \circ t \in \text{Seg}(T)$ then $s \circ t \in X$. If $X$ is any subset of $\text{Seg}(T)$, its closure $\overline{X}$ is the smallest closed set containing $X$. Say $X$ is biclosed if $X$ and $\text{Seg}(T) \setminus X$ are both closed. We let $\text{Bic}(T)$ denote the poset of biclosed subsets of $\text{Seg}(T)$, ordered by inclusion.

Theorem 2.9. The poset $\text{Bic}(T)$ is a lattice.

We now discuss an important injective map $\phi$ from $\overline{FG}(T)$ to $\text{Bic}(T)$ that will be used later in the report.

For an arc $p = (v_0, \ldots, v_l)$ oriented from $v_0$ to $v_l$, let $C_p$ be the set of segments $(v_i, \ldots, v_j)$, $0 < i < j < l$ such that

- $p$ turns right at $v_i$, and
- $p$ turns left at $v_j$.

Given $F \in \overline{FG}(T)$, we define $\phi(F)$ to be $\bigcup_{p \in F} C_p$.

Theorem 2.10. The map $\phi: \overline{FG}(T) \rightarrow \text{Bic}(T)$ is an injective lattice map.

Corollary 2.11. The lattice $\overline{FG}(T)$ is isomorphic to its image under $\phi$, and is thus a sublattice of $\text{Bic}(T)$.

3. Lattice properties of $\text{NCP}(T)$.

Throughout this section, we will let $T$ denote a tree as defined in the previous section. It is known that $\text{NCP}(T)$ is a lattice; in this section we will prove several results about the structure of $\text{NCP}(T)$, as well as state future directions for exploration. We begin with results that discuss ways to decompose $\text{NCP}(T)$ for a general tree.

3.1. Decomposing $\text{NCP}(T)$.

Definition 3.1. Assume that $T$ has an interior vertex $w$ that is incident to at least two leaf vertices—we call such a vertex a wall vertex. Observe that there is a unique acyclic path of $T$ connecting the leaves in these edges, and that this path divides $T$ into two regions, $R_1$ and $R_2$. Given a wall vertex $w$, we can define two new trees, $T_1$ and $T_2$, where $T_i$ is given by removing all vertices in $R_j$ ($j \neq i$) and all edges between these vertices, except for all vertices in $R_j$ incident to $w$, which become boundary vertices of $T_i$ (keeping also the edges that realize this incidence). It is clear from the definition that $T_i$ is a tree with all interior vertices of degree at least three.

Lemma 3.2. Let $w$ be a wall vertex of $T$. Then $\text{NCP}(T) \cong \text{NCP}(T_1) \times \text{NCP}(T_2)$.

Proof. We first define a bijection $\varphi$ from $\text{NCP}(T)$ to $\text{NCP}(T_1) \times \text{NCP}(T_2)$. Let $B \in \text{NCP}(T)$. Note that if $u_1 \in R_1 \setminus \{w\}$ and $u_2 \in R_2 \setminus \{w\}$ are in the same block $B$ of $B$, then $B$ contains $w$. We define $\varphi$ as follows: for any block $B \in B$ that does not contain $w$, we let it be a block of $B_i$, $\in \text{NCP}(T_i)$, where $B$ is contained in $R_i$. For the block $B'$ of $B$ that contains $w$, we create two new blocks, $B_1$ and $B_2$, where $B_i := B'(\cap (R_i \setminus \{w\}))$, and then let $B_i$ be a block of $B_i$. The pair $(B_1, B_2)$ is in $\text{NCP}(T_1) \times \text{NCP}(T_2)$, and $\varphi$ is a bijection.

To see that $\varphi$ is order-preserving, suppose that $B \leq B'$ in $\text{NCP}(T)$. Thus, every block of $B$ is contained in a block of $B'$. Let $(B_1, B_2)$ be a pair of blocks in $\varphi(B)$. If $w \in B_i$, then there exists $B' \in B'$ such that $w \in B'$ and $B_i \leq B'$, where $B_i$ is seen as a collection of vertices in $R_i \subseteq V^o(T)$. It follows that $B_i \leq B'_i$. A similar argument
holds when \( w \notin B_i \). Hence, if \( \varphi(B') = (B'_1, B'_2) \), then there exist \( B'_1 \in B'_1, B'_2 \in B'_2 \) such that \( B_1 \subseteq B'_1 \) and \( B_2 \subseteq B'_2 \), so \( \varphi(B) \leq \varphi(B') \). The proof that \( \varphi^{-1} \) is order-preserving is similar.

Next, we discuss a result concerning decomposing lower intervals of \( \text{NCP}(T) \) into the direct product of non-crossing tree partitions of smaller trees. As will be seen later in the report, such a decomposition has many consequences and applications.

**Definition 3.3.** The following construction is from [3]. Let \( B \) be a segmented-connected subset of \( V^o(T) \), and let \( S = \text{Seg}_r(B) \). We define the **contracted tree** \( T_B \) such that

- \( B \) is the set of interior vertices of \( T_B \),
- \( S \) is the set of interior edges of \( T_B \), and
- for edges \( e \) with one endpoint \( u \) in \( B \) and the other endpoint not between two vertices of \( B \), there is an edge from \( u \) to the boundary in the direction of \( e \).

It is clear that \( T_B \) defines a tree with respect to any interior vertex of degree at least three.

If \( B \in \text{NCP}(T) \) and \( B \) is one of its blocks, then we can further partition \( B \) into sub-blocks, \( \{B'_1, \ldots, B'_m\} \). If each sub-block of such a partition of \( B \) is segmented-connected and for any segments \( s \in \text{Seg}_r(B'_i), t \in \text{Seg}_r(B'_j) \), \( s \) and \( t \) have non-crossing admissible curves, we say this partition is an element of the set that we call \( \text{NCP}(B) \), and define a partial-ordering on \( \text{NCP}(B) \) by refinement. It is clear that \( \text{NCP}(B) \) is isomorphic to \( \text{NCP}(T_B) \), since removing vertices in \( V^o(T) \setminus B \) does not affect the crossing of admissible curves between elements of \( B \).

**Theorem 3.4.** For \( B = \{B_1, \ldots, B_m\} \in \text{NCP}(T), \{0, B\} \cong \prod_{i=1}^m \text{NCP}(B_i) \).

**Proof.** In view of the above discussion, it suffices to show that \( \{0, B\} \cong \prod_{i=1}^m \text{NCP}(B_i) \). For any \( B' = \{B'_1, \ldots, B'_n\} \in \{0, B\} \), it is clear that \( B' \) defines a collection of elements in \( \text{NCP}(B_i) \), where for some \( B_i \in B \), we let \( B_i = \{B'_i | B'_j \subseteq B_i\} \). This map defines an order-preserving injection from \( \{0, B\} \) to \( \prod_{i=1}^m \text{NCP}(B_i) \). It remains to show that it is surjective. To do this, we will define an injection from \( \prod_{i=1}^m \text{NCP}(B_i) \) to \( \{0, B\} \).

An element of \( \prod_{i=1}^m \text{NCP}(B_i) \) naturally defines a collection of segment-connected blocks of interior vertices, \( B' = \{B'_1, \ldots, B'_n\} \). Thus, to complete the proof it suffices to show that any two segments \( s, t \in \text{Seg}_r(B') \) are non-crossing. Let \( s \in \text{Seg}_r(B'_i) \) and \( t \in \text{Seg}_r(B'_j), i \neq j \), where \( B'_i \subseteq B_k \in B, B'_j \subseteq B_r \in B, k \neq r \); we want to show that \( s \) and \( t \) have non-crossing red-admissible curves. Observe that \( s \) is a composition of segments in \( \text{Seg}_r(B_k) \), \( \{s_i\}_{i=1}^n \), and \( t \) is a composition of segments in \( \text{Seg}_r(B_r), \{t_j\}_{j=1}^m \). Moreover, since \( B \in \text{NCP}(T) \), given \( \{s_i\} \), there exists a corresponding sequence of red-admissible curves, and none of these curves cross any of the curves in the analogous sequence for \( \{t_j\} \). Thus, the composition of all the curves associated to \( \{s_i\} \) and the composition of all the curves associated to \( \{t_j\} \) define two new red admissible curves associated to \( s \) and \( t \) that do not cross, completing the proof.

This result, and the isomorphism constructed in its proof, implies that we can reduce every interval in \( \text{NCP}(T) \) to a product of upper intervals, as the following result shows.

**Corollary 3.5.** Every interval in \( \text{NCP}(T) \) is isomorphic to a product of upper intervals of noncrossing tree partitions.

**Proof.** Let \( B = \{B_1, \ldots, B_n\} \in \text{NCP}(T) \) and let \( B' = \{B'_1, \ldots, B'_m\} \in \text{NCP}(T) \), and \( B \leq B' \). From the above Theorem, we know that \( \{0, B'\} \cong \prod_{i=1}^m \text{NCP}(T_{B'_i}) \). Moreover, we know that under this isomorphism, \( B \) corresponds to \( (x_1, \ldots, x_m) \), where \( x_i = \{B'_1, \ldots, B'_i\} \) is the collection of blocks in \( B \) that are contained in \( B'_1, \ldots, B'_i \) in \( B' \). Moreover, since \( B' \) corresponds to \( \{1_{\text{NCP}(T_{B'_1})}, \ldots, 1_{\text{NCP}(T_{B'_m})}\} \), we can conclude that \( [B, B'] \cong \prod_{i=1}^m [x_i, 1_{\text{NCP}(T_{B'_i})}] \).

We conclude this subsection with a result concerning decomposing upper intervals in certain cases.

**Definition 3.6.** Let \( B \in \text{NCP}(T) \), and \( B \in B \) be a block. If there exist two vertices \( u, v \in B \) that are connected by an edge, we call the segment \( [u, v] \) a simple segment. Given such a segment \( s = [u, v] \), we can define a new tree, called the **reduced tree with respect to** \( s \), given by identifying \( u \) with \( v \), denoted \( T_s \). The element \( B \) maps naturally to an element of \( \text{NCP}(T_s) \), denoted \( B_{T_s} \).
Lemma 3.7. Let $B \in \text{NCP}(T)$, and $B \in B$ be a block that contains a simple segment $s$. Then $[B, 1_{\text{NCP}(T)}] \cong [B_{T_i}, 1_{\text{NCP}(T)}]$.

Proof. The result follows from the fact that for any $B' \subseteq [B, 1_{\text{NCP}(T)}]$, $u$ and $v$ are in the same block. Thus, sending an element of $[B, 1_{\text{NCP}(T)}]$ to its natural image in $[B_{T_i}, 1_{\text{NCP}(T)}]$ is an order-preserving injection. Moreover, given an element $B^* \subseteq [B_{T_i}, 1_{\text{NCP}(T)}]$, we can define an element $B^* \subseteq [B, 1_{\text{NCP}(T)}]$ by expanding the vertex at which $u$ and $v$ are identified according to the way $u$ and $v$ appear in $T$, and then keeping all blocks the same except for the block containing $u \sim v$, which we replace with $u$ and $v$. Such a construction clearly creates a partition of $\nu(T)$ into segment-connected blocks. To see that this partition is noncrossing, note first that if neither $s$ nor $t$ in $\text{Seg}_o(B^*)$ is equal to $[u, v]$, then we can extend the admissible curves corresponding $s$ and $t$ in $\text{Seg}_o(B^*)$ to noncrossing curves for the images of $s$ and $t$ in $T$. Now if $s = [u, v]$, then in order for $t$ to cross $s$, we know that $t$ must be either pass through $v$ to reach $u$ or vice versa; assume the former, and denote the subsegment of $t$ approaching $u$ from this direction by $[u, v]$. If $t$ turns away from $u$ at $v$, then it does not cross $s$, so we can assume it turns toward $u$ at $v$. Note that in $T$, the two sharp turns at $u \sim v$, from $w$ to $v$, are either the two sharp turns at $v$ (with the same orientation in $T$) if the turn toward $u$ is not one of these turns in $T$, or the one sharp turn at $v$ away from $u$, and the first turn that we encounter by walking from $v$ back to $v$, starting along $[u, v]$. In order for $t$ to cross $s$, $t$ must not make either of these turns, which is a contradiction, since then its corresponding segment in $T$ does not turn sharply at $u \sim v$.

3.2. Results concerning the oriented flip graph. As discussed in the first section, $\text{NCP}(T)$ is naturally isomorphic to the shard intersection order of the lattice called the oriented flip graph, $\overline{\nu}(T)$. Thus, in order to prove results about $\text{NCP}(T)$, it is helpful to investigate properties of $\overline{\nu}(T)$. In this subsection, we will prove several results concerning $\overline{\nu}(T)$, and then tie all of these results in with our results from the last subsection in order to discuss a proof for why $\text{NCP}(T)$ is graded. Our first result concerns decomposing certain subsets of $\overline{\nu}(T)$.

Definition 3.8. Let $p$ be an arc of $T$. Let $S_p = \{F \in \overline{\nu}(T) \mid p \in F\}$, where $S_p$ inherits the partial order defined on $\overline{\nu}(T)$. Let $R_1$ and $R_2$ be the regions defined by $p$. Let $V^\alpha(p)$ denote the set of interior vertices contained in $p$. Note that for every element in $S_p$, every corner that $p$ contains in $R_i$ cannot be a marked corner of an arc in $R_i \setminus V^\alpha(p)$, since the region containing any face in $R_i$ cut out by $p$ contains any such region cut out by an arc in $R_i \setminus V^\alpha(p)$ by construction. Note further that for such a corner $(v, F)$, there is an edge from $v$ to two unique vertices in $R_i$, both contained in $p$, since if this were not the case, $p$ would not contain $(v, F)$.

For the remainder of this section, we work with a fixed tree $T$, and a fixed arc $p$ of $T$.

Similar to our definition of the contracted tree earlier, we can define two new trees, $T_1$ and $T_2$. We define $T_i$ by the following procedure:

1. Delete all vertices in $R_i \setminus p$ and edges between these vertices. This gives a new tree, $T'_i$; denote the image of $p$ in $T'_i$ by $p'$.
2. Note that if $p'$ contains a corner $(v, F)$, then $F$ is either the image of a face in $R_i$, or is the unique face in $T'_i$ that is incident only to vertices in $p'$. If the former, delete $v$ and any edges incident to it. As discussed in Definition 3.8, such a $v$ is incident to two unique vertices in $R_i$. After deleting $v$, connect these two vertices by a new edge. We obtain a new tree.
3. Repeat the second step with the resulting tree until no corners in the image of $R_i$ remain. The resulting tree is $T_i$.

It is clear that $T_i$ is a tree with every interior vertex of degree at least three. We now provide an example of the construction of $T_1$ and $T_2$.

Example 3.9. Figure 1 shows the above procedure. We start with our tree $T$ on the far left, with arc $p$ outlined in green. Each new picture represents a step in the procedure, where here $i = 1$. Figure 2 shows the result of applying the procedure to $T$ when $i = 2$.
The following results show that the poset $S_p$ has a very interesting structure.

**Theorem 3.10.** Letting $S_p$ be the same poset defined above, $S_p \cong \overline{F\tilde{G}}(T_1) \times \overline{F\tilde{G}}(T_2)$.

**Proof.** There is a natural bijection $\varphi$ from $S_p$ to $\overline{F\tilde{G}}(T_1) \times \overline{F\tilde{G}}(T_2)$. For $\mathcal{F}' \in S_p$ and $i \in \{1, 2\}$, we can remove all arcs in $\mathcal{F}'$ containing any vertices in $R_i \setminus p$ and then remove vertices and contract edges according to the construction of $T_i$. We claim that the remaining collection of arcs, $\mathcal{F}'_i$, is in $\overline{F\tilde{G}}(T_i)$. To see that $\mathcal{F}'_i$ is a facet of $\overline{\Delta}^{NC}(T_i)$, note that by construction the total rank of $T_1$ and $T_2$ is the rank of $T$, and since we split any $\mathcal{F}'$ into $\mathcal{F}'_1$ and $\mathcal{F}'_2$ such that the total number of arcs in both of these facets is the number of arcs in $\mathcal{F}$ minus one ($p$ becomes a boundary arc in both facets), we cannot add any noncrossing arc to $\mathcal{F}'_i$. This is because every member of $V^o(T)$ has a unique image in exactly one $T_i$, and so the total number of arcs in any facet of $\overline{\Delta}^{NC}(T_1)$ and any facet of $\overline{\Delta}^{NC}(T_2)$ sums to $|V^o(T)| - 2 = |\mathcal{F}| - 1$.

We construct $\mathcal{F}'_i$ for $i = 1$ and $i = 2$, thus producing a pair $(\mathcal{F}'_1, \mathcal{F}'_2) \in \overline{F\tilde{G}}(T_1) \times \overline{F\tilde{G}}(T_2)$, so that $\varphi$ is well-defined. It is clear that $\varphi$ is injective, and surjectivity follows from the fact that a pair of facets, one in $\overline{F\tilde{G}}(T_1)$ and the other in $\overline{F\tilde{G}}(T_2)$, determine an element of $\overline{F\tilde{G}}(T)$ by taking the union of the image in $T$ of each arc in $\mathcal{F}_i$ for $i = 1, 2$ and then adding in $p$—this operation is clearly injective, since if $\mathcal{F}_1$ contains an arc that $\mathcal{F}'_1 \in \overline{F\tilde{G}}(T_1)$ does not, any element of $\overline{F\tilde{G}}(T)$ corresponding to $\mathcal{F}'_1$ will also lack that arc.

We will first show that $\varphi^{-1}$ is order-preserving; it suffices to show that $\varphi^{-1}$ preserves the markings of every arc in $\mathcal{F}'_i$, since if every marking of $\mathcal{F}'_i$ is preserved under $\varphi^{-1}$, if a facet $\mathcal{F}'_i \in \overline{F\tilde{G}}(T_i)$ is obtained from $\mathcal{F}'_i$ by a flip at arc $t$, then $\mathcal{F}'_i$ can be obtained from $\mathcal{F}'$ by a flip at the arc in $\overline{F\tilde{G}}(T)$ corresponding to $t$. Observe that the image of any arc $t$ in some $\mathcal{F}'_i$ under $\varphi^{-1}$ can never be marked at a corner that $p$ contains in $R_i$, since the region cut out by $p$ containing the corresponding face is always larger. Thus, the markings of the image of $t$ depend only on the other arcs in $\mathcal{F}'_i$, and $\varphi^{-1}$ does not alter these markings, so the result follows.

The proof that $\varphi$ is order-preserving is more difficult.

The bijection $\varphi$ naturally induces a surjection $\varphi_1 : \overline{F\tilde{G}}(T) \to \overline{F\tilde{G}}(T_1)$. Suppose by contradiction that we have some $\mathcal{F} \subset \overline{F\tilde{G}}(T)$ with $\mathcal{F}_1 := \varphi_1(\mathcal{F}) \not\leq \overline{F\tilde{G}}(T_1)$ and $\mathcal{F}'_1 := \varphi_1(\mathcal{F}')$. Recall the injective lattice map $\phi$ from $\overline{F\tilde{G}}(T)$ to $\text{Bic}(T)$. We have that $\phi(\mathcal{F}) \not\leq \phi(\mathcal{F}_1)$ in $\text{Bic}(T_1)$ and $\phi(\mathcal{F}) \not\leq \phi(\mathcal{F}')$ in $\text{Bic}(T)$. Since in this proof we will frequently be switching between $T$ and $T_1$, if $s$ is a segment of $T_1$, we write it as $s_1$, and its image in $T$ under the
process of obtaining $T$ from $T_1$ by $s$, similarly with arcs.

Since $\phi(F_1) \notin \phi(F'_1)$, there exists $s^1 \in \phi(F_1) \setminus \phi(F'_1)$. The first claim is that $s^1$ must overlap $p^1$ along a segment $s^1_2$ in $T_1$. If $s^1$ does not overlap $p^1$ at all, then $s$ does not overlap $p$ at all, and thus when we pass from $F$ to $F'$, we cannot obtain $s$, since no segment of $C_p$ or $C_q$ for $q \in R_2$ can be composed with other segments to form $s$. Suppose that $s^1$ just overlaps with $p^1$ at a vertex $v^1$. Since $s^1 \notin \phi(F_1)$, there is no composition of segments in $C_q^1$ for $q^1 \in F'_1$ that can form $s^1$. Note that $v$ is an interior vertex in $T$, and hence $s$ only overlaps with $p$ at $v$ in $T$. It follows that no segment in $C_p$ or in $C_q$ for $q \in R_2$ can be used in a composition of $s$, since all segments in $s$ are not subsegments of $p$. Thus, $s^1$ overlaps $p^1$ along a segment $s^1_2 = [v_1, v_2]$ contained in $p^1$. Note that $s_2$ is a segment contained in $p$. We can assume that $s^1 \in C_{q_1}$ for $q^1 \in F'_1$, since if we show that all such segments are contained in some $C_{r_1}$ for $r^1 \in F'_1$, then any composition of such segments is as well.

Since $s^1 \notin \phi(F'_1)$, there must be some $t = [v_1, v_2] \in C_p$ or $C_q$ for $q \in R_2$ that is part of a decomposition of $s$, specifically of $s_{p_2}$ in $\phi(F')$. Let $q$ be any such arc containing $t$. Note that since $t \in C_q$, orienting $q$ from $v_1$ to $v_2$ (without loss of generality) must turn right at $v_1$ and left at $v_2$. We claim that one of $v_1$ or $v_2$ is a corner of $p$ on the $R_1$ side. If $q = p$, then since $p$ must turn a different direction at $v_1$ than at $v_2$, one of these turns implies that $v_1$ is a corner of a segment $(v_1, F)$ of $p$ in $R_1$. If $q \in R_2$ and by contradiction neither $v_1$ nor $v_2$ is a corner of $q$ in $R_1$, then both are parts of corners of $q$ in $R_2$. However, this implies that $q$ turns at the same direction at $v_1$ and $v_2$, which is a contradiction—thus, the claim is true. Now since one endpoint of $t$ is $v_1$ such that $(v_1, F)$ is a corner of $p$ in $R_1$ and $v_1^1$ is not an interior vertex of $T_1$, the segment $s_p$ cannot have $v_1$ as an endpoint, since both of its endpoints are vertices of $T$ that are also interior vertices of $T_1$. Thus, we must have some other segment $t' = [v_1, v_2]$ in some $C_{q_i}$, where now $q'$ is any arc in $F'$. Note that since $t'$ only agrees with $t$ at $v_1$, $q'$ must turn a different direction than $q$ at $v_1$, since if it turns the same direction, the turn at $v_1$ is always either an initial left turn, or a terminal right turn, neither of which allows $t'$ to be in $C_{q_i}$. It follows that $q' \in R_2$, since $(v_1, F')$ is a corner in $R_1$, and the corner that $q'$ contains at $v_1$ is a corner in $R_2$. Hence, the other corner of $q'$, $(v_3, F')$, cannot be in $R_2$, and is thus a corner of $p$ in $R_1$. But now $v_3$ cannot be an endpoint of $s_p$, since $v_3$ is not an interior vertex in $T_1$. It is clear that this process will continue indefinitely, a contradiction.

Corollary 3.11. The poset $S_p$ a lattice.

Corollary 3.12. Let $p$ be an arc in some tree $T$. If there exists a facet $F \in \overline{FG}(T)$ such that $p$ is green in $F$ and all other arcs are red, then $F$ is the unique facet of $\overline{\Delta}^{NC}(T)$ with this property.

Proof. Let $F$ be such a facet. As described in the proof of Theorem 3.10, $F$ defines facets $F_1 \in \overline{FG}(T_1)$ and $F_2 \in \overline{FG}(T_2)$. Moreover, the proof of Theorem 3.10 also implies that $F_i = \overline{\Delta}^{NC}(T_i)$, since the color of every arc is preserved. If there exists another $F_i \in \overline{FG}(T)$ containing $p$ as a green arc and the rest red, then $F_i' \neq F_i$, for some $i \in \{1, 2\}$. This implies that $\overline{FG}(T_i)$ has two distinct maximal elements, which contradicts the fact that $\overline{FG}(T_i)$ is a lattice. Thus, $F$ is unique.

The following result is more general, and addresses how arcs can turn from red to green in a facet of $\overline{FG}(T)$, and is important in light of the result that will follow it.

Proposition 3.13. Let $F \in \overline{FG}(T)$. If $p = [u_1, u_2]$ is a green arc in $F$, then $p$ cannot change color due to a sequence of flips that do not involve $p$.

Proof. Suppose by contradiction that this can occur. Then there must be a facet $F''$ in which $p$ is green, and it must be separated by a single flip from facet $F'$, in which $p$ is red. We can assume without loss of generality that $F'' = F$. Suppose that $p$ is marked at $(v_1, F_1)$ and $(v_2, F_2)$. We will use the Figure 3 (a) throughout the proof as a reference:

In order to turn $p$ red, we need to change at least one marking of $p$. It is easy to see that our flip cannot change both markings. In order to do this, we $p$ must no longer be marked at these corners, which it will be unless a new arc in $F'$ inherits these markings. Thus, the arc that we obtain after the flip to reach $F''$ is marked at both $(v_1, F_1)$ and $(v_2, F_2)$, making it green, a contradiction, since such an arc is always red.

Now suppose that $p$ can be made red by changing one of its markings. Assume without loss of generality that we change the marking $(v_1, F_1)$. In order to do this, the new arc $t' = [w_1, w_2]$ that we obtain in $F''$ after flipping
an arc $t = [u_3, u_4]$ in $F$ must be marked at $(v_1, F_1)$. Since $t'$ is red in $F'$, the fact that it is marked at $(v_1, F_1)$ implies that the other marking of $t'$ must lie on $[w_1, v_1]$, where $w_1$ is the endpoint of $t'$ whose path to $v_2$ passes through $v_1$; see Figure 3 (b). Call this other marking of $t'$ $(v_3, F_4)$. Note that $t$ is marked at $(v_3, F_5)$ and $(v_1, F_3)$, where $F_3$ is the face paired with $F_4$ during the flip.

The key observation now is that in order for $p$ to become red, its new marking in $F'$, $(v_4, F_6)$, must lie on $[v_2, u_2]$; see Figure 3 (c). However, $p$ is not marked at $(v_4, F_6)$, although it contains the corner. Since the only change from $F$ to $F'$ is the position of $t$, it follows that no arc except $t$ can be marked at $(v_4, F_6)$, since if some other arc $r$ were marked there, $p$ would not be marked there in $F'$, since $r$ does not change from $F$ to $F'$. Thus, $t$ is marked at $(v_4, F_6)$, which contradicts the fact that one of its markings, $(v_3, F_5)$, must lie on $[w_1, v_1]$, which is disjoint from $[v_2, u_2]$. It follows that $p$ cannot be turned red with a flip.

We now come to a result that follows from the above theorem and Theorem 3.10 that discusses when an arc cannot be contained in certain facets of $\overline{FG}(T)$.

**Theorem 3.14.** Let $F \in \overline{FG}(T)$ and $p$ a green arc in $F$. Let $F'$ be the facet obtained by flipping $p$ in $F$. Then $[F', \hat{1}] \cap S_p = \emptyset$.

**Proof.** By Theorem 3.10 if $F' < F''$ for some $F'' \in S$ where $p$ is red, then $F''$ can be reached from $F$ by a sequence of flips not involving $p$. This contradicts the result of Proposition 3.13, since $p$ is green in $F$ and red in $F''$. 

**Corollary 3.15.** For a fixed arc $p$, $S_p$ is a sublattice of $\overline{FG}(T)$.

**Proof.** It suffices to show that $S_p = [0_{S_p}, \hat{1}_{S_p}]$, where this is an interval in $\overline{FG}(T)$. First, it is clear that $S_p$ is contained in this interval. Suppose by contradiction that we have some facet $F$ that is larger than $0_{S_p}$ and smaller than $\hat{1}_{S_p}$ but is not in $S_p$; i.e., $F$ does not contain $p$. Since $F > 0_{S_p}$, we can reach $F$ from $0_{S_p}$ via a sequence of flips, and since $p \in 0_{S_p}$, this sequence of flips must involve a flip at $p$. However, we can also reach $\hat{1}_{S_p}$ from $F$ via a sequence of flips, which is a contradiction, since we have already flipped $p$ by the time we reach $F$ from $0_{S_p}$, contradicting Theorem 3.14, as $\hat{1}_{S_p} \in S_p$. 

The result of this theorem can be used alongside the results of the past two subsections to prove that the lattice $\text{NCP}(T)$ is graded, as follows.

**Theorem 3.16.** The lattice $\text{NCP}(T)$ is graded of rank $\#V^w(T) - 1$. 

}\]
Proof. Let $B, B' \in \text{NCP}(T)$, and $n = \#V^o(T)$. We will show that if we can merge $m$ blocks of $B$, where $3 \leq m \leq n$, to obtain $B'$, then we can merge $m - 1$ of these blocks to obtain an element $B'' \in \text{NCP}(T)$. Let $B'$ be the union of $m$ blocks of $B$. By Theorem 3.4, we can assume without loss of generality that $m = n$, since if this equality does not hold, we can consider $B$ as an element $\text{NCP}(T_B)$.

For a facet $F \in \overline{FG}(T)$, define $I_F := \bigwedge_{i=1}^{m} \{F_i, \overline{S}_p \} \in \overline{\text{Cov}(F_G(T))}$. Let $F \in \overline{FG}(T)$ be the unique facet corresponding to $B$. By Theorem 2.8, this facet satisfies $\lambda_1(F) = \text{Seg}(B)$ and $\lambda_2(F) = \overline{\text{Seg}(B)}$. Recall that $\psi(F)$ is the set of all labels appearing in $I_F$. We will construct a facet $F' \in \overline{FG}(T)$ where $I_F \subset I_{F'}$ and $\#\lambda_1(F') = n - 2$.

By assumption, $F$ contains at least two green arcs, $p$ and $q$. By Corollary 3.15, $S_p$ is a sublattice of $\overline{FG}(T)$, and hence $\overline{\text{Cov}(F_G(T))} \cap S_p = \bigwedge_{i=1}^{m} \{F_i, \overline{S}_p \} \in \overline{\text{Cov}(F_G(T))}$, from now on all meets and joins involving $S_p$ are taken in $\overline{FG}(T)$ unless otherwise specified. We claim that $S_p$ is our desired facet. By Theorem 3.14, $p$ is green in $\bigwedge_{i=1}^{m} \{F_i, \overline{S}_p \}$, and $p \in S_p$. This implies that $\bigwedge_{i=1}^{m} \{F_i, \overline{S}_p \}$ is the unique facet in $\overline{FG}(T)$ with $p$ green and all other arcs red, since if $\bigwedge_{i=1}^{m} \{F_i, \overline{S}_p \}$ contains any other green arcs, it would be strictly less than another element of $S$. Hence, we know that $\#\lambda_1(S_p) = n - 2$, as desired.

Observe now that every element that $\bigwedge_{i=1}^{m} \{F_i, \overline{S}_p \}$ covers in $\overline{FG}(T)$ is contained in $S_p$. Moreover, $\bigwedge_{i=1}^{m} \{F_i, \overline{S}_p \} \in \overline{\text{Cov}(F_G(T))}$, since

\[
\{F_i, \overline{S}_p \} \in \overline{\text{Cov}(F_G(T))} = \varphi^{-1}(S_p), S_T := \{(F_i, \overline{F_G(T_p)}), F'_i \in \overline{F_G(T)} \mid (F_i, \overline{F_G(T)}) \in \overline{\text{Cov}(F_G(T))}\}
\]

and hence $\bigwedge_{i=1}^{m} \{F_i, \overline{S}_p \} \in \overline{\text{Cov}(F_G(T))}$, $S_T := \{(F_i, \overline{F_G(T_p)}), F'_i \in \overline{F_G(T)} \mid (F_i, \overline{F_G(T)}) \in \overline{\text{Cov}(F_G(T))}\}$, follows from the fact that $\overline{F_G(T)}$ is a CU-lattice.

Hence, we have that $S_p = \bigwedge_{i=1}^{m} \{F_i, \overline{S}_p \} = I_{\bigwedge_{i=1}^{m} \{F_i, \overline{S}_p \}}$. To finish the proof, it suffices to show that $I_F \subset S_p$. Since $p$ is green in $F$, all of the facets that $F$ covers in $\overline{FG}(T)$ are in $S_p$, and therefore their meet is in $S_p$. Thus, if any member of $I_F$ is not in $S_p$, then $p$ must be flipped in some facet strictly between the minimal element of $I_F$ and $F$, which by Theorem 3.14 cannot occur, since $p \in F$.

Now if $B \subseteq B'$ in $\text{NCP}(T)$, we must combine two blocks of $B$ to obtain $B'$, or else by the above result there is at least one element strictly between $B$ and $B'$. The theorem follows.

\[\Box\]

3.3. Further lattice properties. In this subsection, we show that, unlike the classical noncrossing partition lattice $\text{NC}(n)$, in general the lattice $\text{NCP}(T)$ is not self-dual. We investigate a bijection defined on $\text{NCP}(T)$ in [3] called the Krener Complement and show that $\text{NCP}(T)$ is rank-symmetric.

Definition 3.17. For a tree $T$, a red-green tree $T$ is a collection of pairwise noncrossing colored segments such that every pair of vertices in $V^o(T)$ is connected by a sequence of curves in $T$. The segments in $T$ are allowed to be red or green.

Theorem 3.18. Let $B$ be a noncrossing tree partition. There exists a unique red-green tree $T$ whose set of red segments is $\text{Seg}(B)$.

See [3] for the proof of this result.

Definition 3.19. Let $B$ be a noncrossing tree partition. Let $T$ be the tree corresponding to $B$ as discussed in the above theorem. Garver and McConville show in [3] that the green segments in $T$ define a noncrossing tree partition of $T$. We call this partition the Krener Complement of $B$, and denote it by $\text{Kr}(B)$.

Theorem 3.20. [3] The map $\text{Kr} : \text{NCP}(T) \rightarrow \text{NCP}(T)$ is a bijection.
The following result discusses what we can deduce about $\text{Kr}(B)$ given what we know about $B$.

**Theorem 3.21.** If $B \in \text{NCP}(T)$ has $m$ blocks, then $\text{Kr}(B)$ has $\#V^\circ(T) - m + 1$ blocks.

**Proof.** In order to obtain a red-green tree, there must be a path consisting of green segments from every block of $B$ to every other block. For a block $B \in B$, if a vertex in $B$ is connected by a segment in $\text{Seg}(\text{Kr}(B))$ to another block of $B$, then we call this vertex a connecting vertex. Otherwise, we call it an isolated vertex. Observe that every isolated vertex in a block of $B$ corresponds to a singleton block in $\text{Kr}(B)$.

We claim that a maximal sequence of green segments $[w_1, w_2], [w_2, w_3], \ldots, [w_{n-1}, w_n]$, where $w_i$ is a connecting vertex, defines a block of $\text{Kr}(B)$, and that the two types of blocks of $\text{Kr}(B)$ that we have described make up all of its blocks. It is clear that the above sequence is at least contained in a block of $\text{Kr}(B)$. To see that in fact it is the entire block, note that if this block contained any more elements, they would be other connecting vertices, contradicting the maximality of our construction. The fact that the two types of blocks previously described encompass all of the blocks of $\text{Kr}(B)$ is clear. Note that in the above sequence of green segments, for $i \neq j$, $w_i$ and $w_j$ cannot be in the same block of $B$, or else $T$ contains a cycle. Thus, it suffices to show that the number of isolated vertices and maximal sequences of connecting vertices, call this sum $S(B)$, is always $\#V^\circ(T) - m + 1$.

By the above remarks, the blocks of $B$ form the vertices of a tree, where each edge in this tree is given by a segment of a block in $\text{Kr}(B)$. Hence, there are $m - 1$ such segments. Note that the upper bound for the number of blocks of $\text{Kr}(B)$ is $\#V^\circ(T)$. We claim that every one of the green segments in $T$ between blocks of $B$ reduces the numbers of blocks in $\text{Kr}(B)$ by 1. If we add a green segment $s$ between two vertices $u$ and $v$, we have three cases:

1. $u$ and $v$ are connecting vertices prior to adding $s$. In this case, the number of blocks in $\text{Kr}(B)$ decreases by 1, since we combine two separate blocks of sequences of green segments;
2. (without loss of generality) $v$ is a connecting vertex prior to adding $s$, but $u$ is not. In this case, we lose 1 block of $\text{Kr}(B)$, since $u$ is no longer an isolated vertex;
3. neither $u$ nor $v$ is a connecting vertex prior to adding $s$. In this case, we lose two blocks of $\text{Kr}(B)$, since both $u$ and $v$ are no longer isolated vertices, but gain a block of $\text{Kr}(B)$, since we now have the beginning of a sequence of green segments.

The result follows.

In light of this result, we see that the noncrossing tree partition lattice has the following additional structure:

**Corollary 3.22.** The noncrossing tree partition lattice is rank-symmetric.

The name “Kreweras Complement” comes from a similar bijection defined on the NC($n$) lattice, which has the important property of being order-reversing. In fact, it is modified in [5] to prove that the NC($n$) lattice is self-dual. We now show that in general there does not exist an order-reversing bijection on the noncrossing tree partition lattice. In particular, unlike this classical bijection, the Kreweras Complement is not order-reversing.

**Theorem 3.23.** In general, there does not exist an ordering-reversing bijection on the noncrossing partition lattice.

**Notation 3.24.** When we write a noncrossing tree partition $B$, we only write the blocks of $B$ that are not singletons.

**Proof.** We employ the following tree as a counterexample:
Suppose by contradiction that there exists some order-reversing bijection on \( \text{NCP}(T) \). Consider \( B = \{1, 2, 5\} \{3, 4\} \in \text{NCP}(T) \). Note that \( B \) covers four elements of \( \text{NCP}(T) \): \( \{1, 2, 5\}, \{1, 5\} \{3, 4\}, \{2, 5\} \{3, 4\}, \{1, 2\} \{3, 4\} \). Moreover, \( B \) is covered by three elements: \( \{1, 2, 5, 6\} \{3, 4\}, \{1, 2, 3, 4, 5\}, \{1, 2, 5\} \{3, 4, 6\} \). In order for an order-reversing bijection to exist, \( \text{NCP}(T) \) must contain an element that covers three elements and is covered by four elements. We claim there does not exist such a partition. First, it is trivial that no element consisting of five blocks can satisfy this property—these are the atoms. Similarly, no elements consisting of two blocks can satisfy this property. Moreover, this element cannot contain three blocks either: for a partition three blocks, there are a maximum of three elements covering it.

We now explain why no four-block element of \( \text{NCP}(T) \) can satisfy the desired properties. Note that we have two types of four-block elements, either two pairs of two vertices and the rest singletons, or a triple of vertices and the rest singletons. Our element cannot be of the first type, as such partitions cover exactly two elements. By symmetry, it suffices to check that none of \( \{1, 2, 3\}, \{1, 2, 5\}, \{1, 2, 6\}, \{1, 2, 4\}, \{2, 3, 5\}, \{1, 3, 4\} \), and \( \{1, 3, 6\} \) are covered by four elements, which is a straightforward computation.

**Corollary 3.25.** In general, the noncrossing tree partition lattice is not self-dual.

The fact that the noncrossing tree partition lattice is not self-dual marks a significant difference from the classical noncrossing partition lattice. However, we conjecture that the following structural similarity does hold:

**Conjecture 3.26.** The noncrossing tree partition lattice is rank-unimodal.

## 4. Shellability

In [1], the author shows that the lattice \( \text{NC}(n) \) is EL-shellable. In this short section, we discuss the way in which a similar labeling system to that employed in [1] fails in general for the noncrossing tree partition lattice.

**Definition 4.1.** For a tree \( T \) with \( \#V^o(T) = n \), assume that we label the interior vertices with \( \{1, \ldots, n\} \), with each number appearing as only one label. We can define a labeling of \( \text{NCP}(T) \), where if \( (B, B') \in \text{Cov}(\text{NCP}(T)) \), then \( \lambda(B, B') = \max(\min B_1, \min B_2) \), where \( B_1 \) and \( B_2 \) are the blocks of \( B \) that are merged to obtain \( B' \). We call this labeling an **ordered vertex labeling**. Note that for \( \text{NC}(n) \), this same function \( \lambda \) is an EL-labeling.

**Proposition 4.2.** In general, there does not exist an ordered vertex labeling of the noncrossing tree partition lattice that is an EL-labeling.

**Proof.** We employ the following tree as a counterexample:

The three circled subgraphs in our picture are called the **outer triples** of \( T \). In a given outer triple, the lone vertex with three edges to other interior vertices of \( T \) is called the **center vertex** of the triple, and the other two vertices in the triple are called the **edge vertices** of the triple. We call the very center vertex the **base vertex** of the tree, denoted by \( b \). Note that the center vertex of one outer triple must receive a label that is greater than
or equal to 3; denote this vertex by \(v_1\), its triple by \(G\), and the edge vertices of \(G\) by \(v_2\) and \(v_3\).

More generally, let \([B,B']\) be an interval in \(NCP(T)\). Note that any maximal chain in this interval is also a maximal chain in the lattice of set partitions of \([n]\), \(\Pi(n)\), since every covering relation in \(NCP(T)\) is given by combining two blocks. It is known that \(\lambda\) gives an EL-labeling for \(\Pi(n)\), and so there exists a unique increasing lexicographically-minimal maximal chain in \([B,B']\) \(\subseteq \Pi(n)\), and this must be the same chain in \(NCP(T)\). Moreover, this maximal chain is given by combining two blocks that yield the smallest possible covering label under \(\lambda\) at every step. Denote the smallest possible covering relation label in \([B,B']\) involving \(B\) by \(\pi(B,B')\). Hence, if we can show that in every case, there is such an interval where the smallest possible covering label under \(\lambda\) can never be achieved by an element covering \(B\) in \(NCP(T)\), then we are done.

Suppose by contradiction that some ordered vertex labeling of \(NCP(T)\) is an EL-labeling. Denote the label of a vertex \(v \in V^q(T)\) by \(\ell(v)\). Note that one edge vertex in \(G\) must have a label greater than that of \(v_1\); if both of their labels are smaller, then \(B = \{v_2,b\}\{v_3\}\) and \(B' = \{v_1,v_2,v_3,b\}\) are both in \(NCP(T)\) with \(B < B'\). We have that \(\pi(B,B')\) is achieved by merging \(\{v_2,b\}\) and \(\{v_3\}\). However, the resulting set partition is not in \(NCP(T)\), so that we cannot have an increasing chain from \(B\) to \(B'\).

Suppose first that \(\ell(v_1) > 3\). Observe that two of \(\{1,2,3\}\) must be labels of vertices outside of \(G\); call the two vertices with these labels \(w_1\) and \(w_2\) (there may be three of such vertices—in that case just pick the ones with labels 1 and 2). Let \(p_1\) be the path in \(T\) from \(w_1\) to \(v_2\), and \(p_2\) the path in \(T\) from \(w_2\) to \(v_3\) (including the endpoints). Then \(B = \{v_2,w_1\}\{v_3,w_2\}\) and \(B' = \{V^q(p_1),V^q(p_2)\}\); we remark that the roles of \(w_1\) and \(w_2\) may have to be interchanged in order to ensure that \(B \in NCP(T)\) — the important thing to note is that no matter where \(w_1\) and \(w_2\) are located in \(T\), (at least) one of \(\{v_2,w_1\}\{v_3,w_2\}\) and \(\{v_2,w_2\}\{v_3,w_1\}\) is in \(NCP(T)\), and we assume without loss of generality that the configuration given above is in \(NCP(T)\). Thus, we have that \(B\) and \(B'\) are in \(NCP(T)\), with \(B < B'\). Moreover, \(\pi(B,B')\) is given by merging \(\{v_2,w_1\}\) and \(\{v_3,w_2\}\) in \(B\), which gives a set partition that is not in \(NCP(T)\).

Hence, we can assume that \(\ell(v_1) = 3\). If \(\ell(v_2), \ell(v_3) > 3\), then the identical argument used in the paragraph above can be used to show that there exist \(B,B' \in NCP(T)\) with \(B < B'\) and no increasing maximal chain in \([B,B']\) \(\subseteq NCP(T)\). Therefore, assume further that \(\ell(v_2) \in \{1,2\}\). However, this means that a center vertex of one of the two other outer triples has a label strictly greater than 3, so that again the same argument from the above paragraph carries over to provide the same contradiction. It follows that our ordered vertex labeling cannot be an EL-labeling.

\[\square\]

5. A CU-LABELING OF Bic\((T)\)

In [3], the authors prove that \(Bic(T)\) is congruence-uniform. Thus it admits a CU-labeling. In this section, we will explicitly construct such a labeling.

**Definition 5.1.** A lattice is **congruence-uniform** if it can be constructed from the element lattice by a finite sequence of interval doublings. Alternatively, the set of its covering relations permits a CU-labeling.

**Definition 5.2.** For a tree \(T\), segments \(s_1, s_2\) are composable if \(s_1 \circ s_2 \in Seg(T)\). A subset \(B \subseteq T\) is closed if for \(s_1, s_2 \in B, s_1 \circ s_2 \in B\). \(B\) is biclosed if both \(B\) and its complement, \(B^C\), are closed.

**Definition 5.3.** We say a segment \(s\) is a **split** of segment \(t\) if \(s\) is contained in \(t\), and \(s\) and \(t\) share exactly one endpoint. A **break** of a segment \([a,c]\) is a pair of splits \(([a,b], [b,c])\), of \([a,c]\) for some vertex \(b\) of segment \([a,c]\) lying between \(a\) and \(c\). We say that \(b\) is the **faultline** of the break \(([a,b], [b,c])\).

**Definition 5.4.** Define the poset \(P\) whose elements are of the form \(s_{f_1,f_2,...,f_m}\), where \(s\) is a segment of \(T\) with \(m\) breaks and each \(f_i\) is a split of \(s\) with no two \(f_i\)'s from the same break. The ordering is given by \([a,c]_{s_1,s_2,...,s_k} \geq [d,e]_{m_1,m_2,...,m_l}\) if \([a,c]\) contains \([d,e]\).

**Theorem 5.5.** \(Bic(T)\) has a CU-labeling, \(\widetilde{\lambda} : Cov(Bic(T)) \rightarrow (Seg(T) \times 2^{Seg(T)})\), given by: \(\widetilde{\lambda}(S \leq (S \cup \{[a,c]\})) = [a,c]_{s_1,s_2,...,s_k}\) where \(s_1, s_2, \cdots, s_k\) are the splits of \([a,c]\) which are contained in \(S\).
Example 5.6. Consider the tree $T = P_4$, consisting of a path of 4 vertices numbered 1,2,3,4 in order. Consider the biclosed sets $B = \{[1,2], [1,3]\}$ and $C = \{[1,2], [1,3], [1,4]\}$.

Then $\lambda(B < C) = [1,4]_{[[1,2],[1,3]]}$.

Proof. of Theorem 5.5

We will verify axioms (CN1)-(CN3), as well as (CU1) and (CU2) [3]. For convenience, we will let $\lambda : Cov(Bic(T)) \to Seg(T)$ denote the first coordinate function of $\lambda$. Note that $\lambda(B < B \cup \{s\}) = s$.

(CN1): Given $C$ and $D$ which both cover $B$ where $C = B \cup \{a,c\}$ and $D = B \cup \{d,e\}$. The biclosed set $C \cap D \supset \{[a,c], [d,e]\}$. Say $C \cap D$ covers $C' \supset C$ and $D' \supset D$. If $C' \supset \{[d,e]\}$, then $C' \supset C \supset B$ so $C' \supset D$ meaning that $C \cap D$ is not the least upper bound of $C$ and $D$, a contradiction. So $C'$ does not contain $[d,e]$ and similarly, $D'$ does not contain $[a,c]$. Thus $\lambda(C' \cap C \cap D) = [d,e]$ with some subscript, while $\lambda(D' \cap C \cap D)$ is in $[a,c]$ with some subscript.

The label $\lambda(B < C) = [a,c]_{s_1,s_2,\ldots,s_k}$ and $\lambda(B < D) = [d,e]_{m_1,m_2,\ldots,m_l}$. Thus $C$ contains $s_1,s_2,\ldots,s_k$, so $C'$ does as well. The set $D$ contains $m_1,m_2,\ldots,m_l$, so $D'$ does as well. Thus the subscripts for $\lambda(C' \cap C \cap D)$ are $s_1,s_2,\ldots,s_k$ and the subscripts for $\lambda(D' \cap C \cap D)$ are $m_1,m_2,\ldots,m_l$. So $\lambda(C' \cap C \cap D) = \lambda(B < D)$ while $\lambda(D' \cap C \cap D) = \lambda(B < C)$.

(CN2): Let $C$ and $D$ cover $B$ and both $C' > C$ and $D' > D$ be covered by $C \cap D$. Labels of the form $\lambda(G < H)$ where $C \leq G, H \leq C'$ or $D \leq G, H \leq D'$ consist of elements of $C \cap D$ which are not elements of $B$, with some subscript. $C \cap D$ is the closure of $C \cup D$. It consists of four types of segments:

1. segments in $B$
2. a composition of $[a,c]$ with segments in $B$, which is necessarily in $C$.
3. a composition of $[d,e]$ with segments in $B$, which is necessarily in $D$.
4. a composition of both $[a,c], [d,e]$ and perhaps some segments in $B$.

A segment of type 1 is in $B$ and thus cannot be $\lambda(G < H)$. A segment of type 2 cannot be used as a label between $C$ and $C'$ since it is contained in $C$. It cannot be $\lambda(G < H)$ where $D \leq G < H \leq D'$ because it is a segment including $[a,c]$ and thus not included in $D'$. A segment of type 3 cannot be used as a label between $D$ and $D'$ since it is contained in $D$. It cannot be $\lambda(G < H)$ where $C \leq G < H \leq C'$ because it is a segment including $[d,e]$ and thus not included in $C'$. Hence any label $\lambda(G < H)$ is a segment of type 4) with some subscript.

A segment of type 4) contains both $[a,c]$ and $[d,e]$ so $\lambda(G < H) \supset [a,c] \cup [d,e] \supset \lambda(B < C), \lambda(B < D)$.

(CN3): Within a maximal chain of the interval $[x,y]$, the label given to the covering relation $S \subset C \subset T$ is the element of $T$ not contained in $S$, (possibly) along with some subscripts. For $S_1 \subset T_1$ and $S_2 \subset T_2$ where $S_1 \subset S_2$, we have $T_1 \supset S_2 \supset S_2$ already contains the element of $T$ not contained in $S$ and thus such an element cannot be involved in the label of $S_2 \subset T_2$. So all labels in a maximal chain are unique.

(CU2): Consider elements $M$ and $M'$ of Bic(T) which are uniquely covered by $M^*$ and $M'^*$ respectively. Assume for the sake of contradiction that $\lambda(M \leq M^*) = \lambda(M' \leq M'^*)$. Thus $M^* = M \cup \{[a,c]\}$ and $M'^* = M' \cup \{[a,c]\}$ for some segment $[a,c]$. The sets $M$ and $M'$ contain the same split $[a,b]$ or $[b,c]$ for each break $([a,b],[b,c])$ of $[a,c]$.

Also $M \cap M' \geq M^*, M'^*$. Thus the closure of $M \cup M'$ contains $[a,c]$. So there exist some composable set of segments $[a,b_1], [b_1,b_2], [b_2,b_3], \ldots, [b_{n-1},b_n], [b_n,c]$ in $M \cap M'$. Without loss of generality, $[a,b_1] \in M$, $[b_1,b_2] \in M'$, $[b_2,b_3] \in M$, etc. Since $[a,b_1] \in M$, then $[a,b_1] \notin M'$. $[b_1,b_2] \in M'$ as well as $[a,b_2] \in M'$ via composition. Thus $[a,b_2] \in M$. $[b_2,b_3] \in M'$, so $[a,b_3] \notin M$ via composition. Thus $[a,b_3] \notin M'$. Continuing in this manner, we see that $[a,c] \notin M$, which is a contradiction. Thus $M \leq M^*$ and $M' \leq M'^*$ have different labels.

(CU1): Consider elements $J$ and $J'$ of Bic(T) which uniquely cover $J_a$ and $J'_b$ respectively. Thus $J_a, J'_b \supset J \cap J'$. Assume for the sake of contradiction that $\lambda(J_a \leq J) = \lambda(J'_a \leq J')$. Say that this label is $[a,c]$ with some subscripts. Thus $J$ and $J'$ both contain $[a,c]$, so $[a,c] \in J \cap J'$. $J_a$ and $J'_b$ do not contain $[a,c]$ so $J \cap J'$ does not either. The set $J \cap J'$ is the complement of $J^C \cap J'^C$ so $J^C \cap J'^C$ contains $[a,c]$. So $[a,c]$ is in the closure of the union of the biclosed sets $J^C$ and $J'^C$. For each break of $[a,c]$, $J_a$ and $J'_b$ and thus $J$ and $J'$ contain the same split. Thus $J^C$ and $J'^C$ contain the same split. We then apply argument used for (CU2). □

6. Shard Intersection Poset

Throughout this section, we let $\lambda : Cov(Bic(T)) \rightarrow (Seg(T) \times 2^{\text{Seg}(T)})$ denote the CU-labeling constructed in Section 5 and $\lambda : Cov(Bic(T)) \rightarrow Seg(T)$ denote the first coordinate function of $\lambda$. 14
Definition 6.1. For an element $B$ of the poset of biclosed sets, define $\psi(B)$ as $\{\lambda(u, v) \mid R \leq u < v \leq B\}$ where $R$ is the meet of the biclosed sets covered by $B$.

Definition 6.2. $\Psi(T)$ is the poset whose elements are $\psi(B)$ for biclosed sets $B$ of $T$, partially ordered by inclusion.

Conjecture 6.3. $\Psi(T)$ is a lattice.

Lemma 6.4. The shard intersection order of $Bic(T)$ has a unique maximal element.

Proof. The top element, $X$ of $Bic(T)$ is the set of all segments of $T$. It covers every set consisting of all segments of $T$ except $[a, b]$ where $a$ and $b$ are adjacent in $T$: Such a set is necessarily closed because it contains any segment $[c, d]$ that is formed from composing two segments of $T$. Its complement is necessarily closed since it contains only one segment. It is covered by $X$ because it contains all but one segment of $T$.

Thus the meet of the elements covered by $X$ cannot contain any segments $[a, b]$ where $a, b$ are adjacent in $T$. Thus the complement of the meet contains all segments $[a, b]$ of $T$ where $a, b$ are adjacent. By composition, the complement of the meet must contain all segments in $T$ so the meet contains no segments. Thus $\psi(X)$ consists of the set of all labels.

Given a biclosed set $B$, which covers $B_1, B_2, \ldots, B_k$, with $\lambda(B_i, B) = \lambda_i$ for $i = 1, 2, \ldots, k$, we let $S = \{s_1, s_2, \ldots, s_k\}$ where $s_i$ is the segment part of $\lambda_i$, given by $\lambda(B_i, B)$. The set of compositions of segments in $S$ is denoted by $S$.

Definition 6.5. Let $s \in S$ be a segment expressed as $s_{i_1} \circ s_{i_2} \circ \ldots \circ s_{i_w}$ with each $s_{i_j} \in S$. Let $t$ be a split of $s$ such that there exists a break $(t, t')$ or $(t', t)$ of $s$. We say that $t$ is a faultline split if the faultline of such a break occurs at the endpoint of some $s_{i_j}$. Otherwise, we say that $t$ is a non-faultline split.

Proposition 6.6. Given a biclosed set $B$, which covers $B_1, B_2, \ldots, B_k$, with $\lambda(B_i, B) = s_{i\delta}$, for $i = 1, 2, \ldots, k$, we can describe $\psi(B)$ as consisting of all labels of the following form:

$$(s_{i_1} \circ s_{i_2} \circ \ldots \circ s_{i_w})\delta$$

where $[a, c] = s_{i_1} \circ s_{i_2} \circ \ldots \circ s_{i_w} \in S$ and $\delta$ is a list of segments containing one split per break of $s$ with any of the $2^{w-1}$ possible choices for the faultline splits and with the non-faultline splits determined as follows: for $b$, a vertex within $[x, y] = s_{i_j}$ for some $1 \leq j \leq w$, where $x$ is the endpoint of $s_{i_j}$ closer to $a$, then $\delta$ contains $[a, b]$ if $\delta_{i_j}$ contains $[x, b]$ and $\delta$ contains $[b, c]$ if $\delta_{i_j}$ contains $[b, y]$.

Example 6.7. Consider the tree $T = P_4$ given by a path of 4 vertices numbered 1,2,3,4 in order. Consider the biclosed set $B = \{(1,2), (2,3), (3,1), (1,4)\}$. In $Bic(T)$, $B$ covers $B_1 = \{(1,2), (1,3), (1,4)\}$ and $B_2 = \{(1,2), (2,3), (1,3)\}$. The meet of the elements covered by $B$ is $B_1 \wedge B_2 = \{(1,2), (1,3)\}$.

Thus, the set $\psi(B)$ is given by $\{(2,3), (1,4),(1,2),(1,3)\}$.

Proof. of Proposition 6.6

Lemma 6.8. For a biclosed set $B$ and any $(B_1, B)$ and $(B_2, B)$ in Cov(Bic(T)), the segment $\lambda(B_1, B)$ is not a split of $\lambda(B_2, B)$.

Proof. Suppose for the sake of contradiction that there exist some biclosed sets $B_1, B_2$ such that $\lambda(B_1, B)$ is a split of $\lambda(B_2, B)$. Then $B_1 = B - \{[a, b]\}$ and $B_2 = B - \{[a, c]\}$ where $[a, b]$ is a split of $[a, c]$.

Consider $L = B - \{[a, b], [a, c]\}$. Since $L \subset B_1, B_2$, a composition of composable elements in $L$ lies in $B_1$ and $B_2$. Thus it lies in $B_1 \cap B_2 = L$ so $L$ is closed. A composition of two elements in $L^C$ is a composition of two elements in $B_1^C \cap B_2^C$ and perhaps $[a, c]$ and $[a, b]$. A composition of two elements in $B_1^C$ is in $B_1^C$ and thus in $L^C$. A composition of two elements in $B_2^C$ is in $B_2^C$ and thus in $L^C$. The only other possible composition of elements would be $[a, b]$ and $[a, c]$ but those are not composable, so $L^C$ is closed. Thus $L$ is biclosed.

Consider the segment $[b, c]$. If it is not contained in $B$, then $B_1$ contains $[a, c]$ without containing $[a, b]$ or $[b, c]$, meaning it is not biclosed. If $[b, c]$ is contained in $B$, then it is contained in $B_2$, which then contains $[a, b]$ and $[b, c]$ without containing $[a, c]$. But $B_1$ and $B_2$ are biclosed, so this is a contradiction.

Lemma 6.9. Let $B_1, \ldots, B_k \in Bic(T)$ denote the biclosed sets covered by $B$ and define $s_i = \lambda(B_i, B)$ for $i = 1, \ldots, k$. Then

$$\bigwedge_{i=1}^{k} B_i = B - \{s_1, \ldots, s_k\}$$
Proof.

\[ \bigwedge_{i=1}^{k} B_i = \bigwedge \{ B - \{ s \} \mid s \in S \} \]
\[ = (\bigvee \{ B - \{ s \} \mid s \in S \})^C \]
\[ = (\bigvee \{ B^C \cup \{ s \} \mid s \in S \})^C \]

The set \( \bigvee \{ B^C \cup \{ s \} \mid s \in S \} \) contains \( B^C \) and \( \overline{S} \). Any composition of two composite elements of \( B^C \) is in \( B^C \) and any composition of two composite elements of \( \{s_1, \ldots, s_k\} \) is in \( \{s_1, \ldots, s_k\} \). The composition of \( n \in B^C \) and \( s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_m} \) can be rewritten as \( (n \circ s_{i_1}) \circ (s_{i_2} \circ \cdots \circ s_{i_m}) \), but \( B^C \cup \{ s_i \} \) is biclosed, so \( n_1 = n \circ s_{i_1} \in B^C \). Continuing in this manner, utilizing that each \( B^C \cup \{ s_i \} \) is biclosed, we see that the composition of composite elements from \( B^C \) and from \( \{s_1, \ldots, s_k\} \) is itself in \( B^C \) and thus in \( B^C \cup \{s_1, \ldots, s_k\} \). So simply, \( \bigwedge_{i=1}^{k} B_i = (B^C \cup \{s_1, \ldots, s_k\})^C = B - \{s_1, \ldots, s_k\} \).

\[ \square \]

Lemma 6.10. If \( s = \lambda(B_i, B) \) for some \( B_i \) covered by the biclosed set \( B \), then \( R \cup \{s\} \) is biclosed.

Proof. A composition of two composite elements in \( R \cup \{s\} \) is either a composition of two elements in \( R \) which is therefore in \( R \) or the composition of \( s \) with an element, \( r \) in \( R \). A segment \( s \circ r \) is in \( B \) so it is either in \( R \) or in \( \overline{S} \). If \( s \circ r \) is in \( \overline{S} \) then we can write it as \( s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_m} \), where \( s_{i_k} \) shares an endpoint with \( r \) and \( s_{i_m} \) shares an endpoint with \( s \). However \( s \) is not a split of \( s_{i_m} \) or vice versa, so \( s = s_{i_m} \). If \( m = 1 \), this would mean \( r \circ s = s \), so \( m > 1 \). Thus \( r = s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_{m-1}} \), a contradiction since \( r \in R = B - \overline{S} \). Thus \( s \circ r \in R \cup \{s\} \). So \( R \{s\} \) is closed. A composition of two composite elements in \( (R \{s\})^C = R^C - \{s\} \) is a composition of elements in \( R^C \) so the only way it would not be in \( (R \{s\})^C \) if it equals \( s \). Suppose \( n_1 \circ n_2 = s \) for \( n_1, n_2 \in R^C \). If \( n_1, n_2 \in R^C - \overline{S} \), then \( n_1 \circ n_2 \in R - \overline{S} \), a contradiction, so either \( n_1 \) or \( n_2 \) is in \( \overline{S} \). Without loss of generality, say \( n_1 = s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_{m-1}} \), where \( s_{i_k} \) shares an endpoint with \( s \). This means \( s_{i_1} \) is a split of \( s \), contradicting 6.8. So \( (R \cup \{s\})^C \) is closed, and thus \( R \cup \{s\} \) is biclosed.

Consider the set of segments in \( \overline{S} \). We can consider a subset, \( A \) of \( \overline{S} \) to be biclosed if for every two composite segments \( a, b \in A \), we have \( a \circ b \in A \) and for every break of a segment \( s \in A \) where the splits of the break are in \( \overline{S} \), at least one of those splits is in \( A \).

Lemma 6.11. The biclosed subsets in the interval \([R, B]\) are precisely those of the form \( R \cup A \) where \( A \) is a biclosed subset of \( \overline{S} \).

Proof. Let \( A \) be a subset of \( \overline{S} \). \( R \subset R \cup A \) and \( A \subset \overline{S} \subset (B) \), so \( R \cup A \subset B \). Thus, if \( R \cup A \) is biclosed, it is in the interval \([R, B]\). (If \( A \) is not a subset of \( \overline{S} \), then \( R \cup A \) would either contain some element not in \( B \) or would be equivalent to \( R \cup A' \) for some \( A' \in \overline{S} \))

If \( A \) is a biclosed subset of \( \overline{S} \), consider the composition of two composite segments in \( R \cup A \). If they are both in \( R \), their composition is in \( R \) and if they are both in \( A \), their composition is in \( A \). If one is \( R \) and the other is some \( s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_m} \in A \), then their composition is \( r \circ s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_m} \). However \( r \circ s_{i_k} \) is in the biclosed set \( R \cup \{s_i\} \), but it is not equal to \( s_{i_k} \), so it must be some other \( r_1 \in R \). In general for \( r_j \in R \), \( r_j \circ s_{i_{j+1}} \in R \cup \{s_{i_{j+1}}\} \) but is not equal to \( s_{i_{j+1}} \), so \( r_j \circ s_{i_{j+1}} \in R \cup A \) is closed.

For two composite elements of \( (R \cup A)^C = R^C - A \), their composition is in \( R^C \). \( (R \cup A)^C \) is closed unless the composition of some \( n_1, n_2 \in R^C - A \) is in \( A \). If \( n_1, n_2 \in R^C - \overline{S} \), their composition is in \( R^C - \overline{S} \), so without loss of generality, \( n_1 \in \overline{S} \). Given \( n_1 \circ n_2 = (s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_m}) \circ n_2 = s_{j_1} \circ s_{j_2} \circ \cdots \circ s_{j_d} = a \in A \), where \( s_{i_k} \) and \( s_{j_k} \) share an endpoint, we have that one of \( s_{i_k} \) and \( s_{j_k} \) is a split of the other unless \( s_{i_k} = s_{j_k} \). Thus \( (s_{i_1} \circ \cdots \circ s_{i_m}) \circ n_2 = s_{j_2} \circ \cdots \circ s_{j_d} \) where \( s_{i_k} \) and \( s_{j_k} \) share and endpoint, so one is a split of the other unless they are the same. Thus \( (s_{i_1} \circ \cdots \circ s_{i_m}) \circ n_2 = s_{j_2} \circ \cdots \circ s_{j_d} \). Continuing in this manner, we see that \( n_2 = s_{j_{m+1}} \circ \cdots \circ s_{j_d} \).

This means that both \( n_1, n_2 \in \overline{S} \). Since \( A \) is a biclosed subset of \( \overline{S} \) and \( a \in A \), either \( n_1 \) or \( n_2 \) must be in \( A \).
So the composition of \( n_1, n_2 \in R^C - A \) cannot be in \( A \), and we get that \( R^C - A \) is closed, and hence \( R \cup A \) is biclosed.

If \( A \) is not a biclosed subset of \( \overline{S} \), it either contains (i) some \( a, b \) without \( a \circ b \) or (ii) there exist some \( a, b \in \overline{S} \) for which \( a \circ b \notin A \), but \( a \circ b \in A \). In case (i), \( R \cup A \) contains \( a, b \) but neither \( R \) nor \( A \) contains \( a \circ b \), so \( R \cup A \) is not closed. In case (ii), \( R \cup A \) contains \( a \circ b \) but since \( a, b \notin R, A \), it does not contains \( a \) or \( b \). Thus \( (R \cup A)^C \) is not closed. So \( R \cup A \) is not biclosed in either case.

\[ \square \]

For a segment \( t = [e, f] = s_{i_1} \circ s_{i_2} \circ \ldots \circ s_{i_m} \) in \( \overline{S} = B - R \), it has \( m - 1 \) breaks. To show that \( t \) with any of the \( 2^{m-1} \) choices of one split per break occurs as a label in \([R, B]\) consider the following:

We will build a set \( D \) with the property that \( D \) is a biclosed subset of \( \overline{S} \) which contains an arbitrarily chosen split for each faultline break of \( t \).

First, choose one split for each of the \( m - 1 \) breaks of \( t \) and add them to \( D \). For each split \( p \) included in \( D \):

1. Consider a faultline break of \( p \) into \( a, b \) where a shares an endpoint with \( t \). If \( a \) was one of the original splits chosen to be in \( D \), do nothing. If \( a \notin D \), add \( b \) to \( D \).

2. Now add compositions of segments in \( D \) to \( D \).

3. Repeat this process for the next break of \( p \).

We never add something that is a split of \( t \), except perhaps in the phase where we add compositions of segments already in \( D \). Since none of the original splits \( p \in D \) are composable, such a composition will include at least one \( b = [x, y] \) (with \( x \) closer to \( e \) than \( y \)) that was added later. This would require the presence of either \([e, x]\) or \([y, f]\) in \( D \) in order for a split of \( t \) to result from a composition. Initially, there is no break of \( t \) for which both splits are contained in \( D \). By induction, this is true before we add \( b \) to \( D \). In the case that \([x, y]\) was added to \( D \) as a split of \( p = [e, y] \): If \([e, x]\) was already in \( D \), then it was one of the original elements of \( D \) and we would already have the split \( a = [e, x] \) of the split \( p = [e, y] \) and not have added \([x, y]\) to \( D \) per the rules outlined earlier. If \([y, f]\) was already in \( D \), then \([e, y]\) was not, a contradiction.

In the case that \([x, y]\) was added to \( D \) as a split of \( p = [x, f] \): If \([e, x]\) was already in \( D \), then \([x, f]\) could not have been, a contradiction. If \([y, f]\) was already in \( D \) then the split \( a = [y, f] \) of \([x, f]\) would already been in \( D \) and we would not have added \([x, y]\) as per the rules outlined earlier.

The last step of this process involves taking compositions of all the elements already in \( D \) and adding them to \( D \) so the resulting \( D \) is closed. \( D \) will be biclosed if for every element, \( q = [x, y] \) of \( D \) and every break of \( q \), some split corresponding to that break is also in \( D \).

By virtue of how the \( b \)'s were added to \( D \) any split, \( p \), of \( t \) that was used as a seed for \( D \) has a split in \( D \) corresponding to each of its breaks.

Consider any other \([x, y]\in D\) where \( x \) is closer to \( e \) than \( y \). Since this does not have \( e \) or \( f \) as an endpoint, it is the composition of some set of \( b \)'s added during phase (1), including some \([g, y]\). Assume for the sake of contradiction that there is some vertex \( c \in [x, y] \) where neither \([e, c]\) nor \([c, y]\) is in \( D \). Either \([e, c]\) or \([c, f]\) was one of the splits used to seed \( D \). Without loss of generality, say \([e, f]\) is in \( D \). Since the split \([c, y]\) of \([c, f]\) is not in \( D \), we must have \([y, f] \in D \). However, this means \([y, f] \) was one of the original splits used to seed \( D \). Since \([g, y], [y, f] \in D \), then \([g, f] \in D \). Thus \([g, f] \) was one of the splits used to seed \( D \). But since the split \([y, f]\) of \([g, f]\) was already in \( D \), the split \([g, y]\) would not have been added in phase (1) except perhaps as a split of \([e, y]\). However, the split \([y, f] \in D \) so \([e, y] \notin D \) and thus there is no way \([g, y]\) could have been added, a contradiction.

So \( D \) is a biclosed subset of \( \overline{S} \). The only way it could include \( t \) would be if \( t \) had been added as a composition of two splits corresponding to the same break of \( t \). But that situation never occurs so the biclosed set \( R \cup D \) does not include \( t \), but \( D \) does. Some covering relation on a chain from \( R \cup D \) to \( B \) will have a label \( t_\mu \) where \( \mu \) includes the splits we originally chose as seeds for \( D \).

For label \( t_\mu \in \Psi(B) \) and each non-faultline break, \([e, h], [h, f]\) of \( t \), either \([e, h]\) or \([f, h]\) is in \( \delta \). \( h \) is a vertex inside some segment \( s_i = [x, y] \), where \( x \) is the endpoint of \( s_i \) closer to \( e \). If \( \lambda_i \) contains \([x, h]\), then \( B \) (and \( R \)) contains \([x, h]\). \( R \) contains either \([e, h]\) or \([h, f]\). If \( R \) contains \([h, f]\), then \( R \) contains \([x, f]\) since \( R \) is biclosed. This contradicts \([x, f] \in \overline{S} \), so \( R \) contains \([e, h]\). Thus, \( \delta \) contains \([e, h]\).

Similarly, if \( \lambda_i \) contains \([h, y]\), then \( B \) (and \( R \)) contains \([h, y]\). If \( R \) contains \([e, h]\), then \( R \) contains \([e, y]\) in \( \overline{S} \), a contradiction. So \( R \), and thus \( \delta \), contains \([h, f]\). \( \square \)

For biclosed \( B \) and \( B' \) of \( Seg(T) \), we can consider the structure of \( \psi(B) \cap \psi(B') \). Suppose \( \lambda(B_i, B) = s_i \) while \( \lambda(B'_i, B') = t_j \). Then any label in \( \psi(B) \cap \psi(B') \) consists of a segment \( s \) which can be expressed as a composition of both \( s_i \)'s and \( t_j \)'s along with some subscripts. If this composition of \( s_i \)'s and the corresponding composition of \( t_j \)'s share \( m \) faultlines, then we can consider this label as corresponding to both a composition
of $m + 1 \ s'$s and a combination of $m + 1 \ t'$s which do not share any faultlines. We call labels in $\psi(B) \cap \psi(B')$ corresponding to compositions of $s_i's$ and $t_i's$ that share no faultlines pseudominimal. An element of $\psi(B) \cap \psi(B')$ is of the form $s_j$ where $s$ is a composition of pseudominimal elements of $\psi(B) \cap \psi(B')$ and $\delta$ consists of one split per break of $s$. There are independent choices for which faultline splits are included in $\delta$ but which non-faultline splits are included are predetermined by the splits in the subscripts of the pseudominimal elements, in much the same way the non-faultline splits of an element of $\psi(B)$ depend on the subscripts of the labels on the covering relations $(B', B)$.

As a result, if the pseudominimal elements of $\psi(B) \cap \psi(B')$ form a canonical join representation, then we can take $B''$ equal to their join to obtain $\psi(B'') = \psi(B) \cap \psi(B')$, which would prove Conjecture 6.3.

**Remark 6.12.** $\Psi(T)$ is not always graded. For example, $\Psi(S_4)$ and $\Psi(S_6)$ are not graded.

**Proof.** Label the degree 4 vertex of $S_4$ as 0 and label the others clockwise as 1,2,3,4. For convenience, let $a = [0,1], b = [0,2], c = [0,3], d = [0,4], e = [1,2], f = [2,3], g = [3,4], h = [1,4]$.

We demonstrate that $\Psi(S_4)$ is not graded by producing two maximal chains of differing lengths:

$$\psi(\{\}) = \{s, \} < \psi(\{b, e\}) = e_{[b]} < \psi(\{b, e, f\}) = e_{[b]} - f_{[b]} < \psi(\{b, d, e, f, g, h\}) = e_{[b]} - f_{[b]} - g_{[d]} - h_{[d]} < \psi(\{a, b, c, d, e, f, g, h\}) = a_{[b]} - b_{[b]} - c_{[d]} - d_{[d]} - e_{[a]} - f_{[b]} - g_{[c]} - h_{[d]} - h_{[a]}$$

and

$$\psi(\{\}) = \{s, \} < \psi(\{b, e\}) = e_{[b]} < \psi(\{b, e, f\}) = e_{[b]} < \psi(\{a, b, c, e, f\}) = a_{[b]} - b_{[b]} - c_{[c]} - e_{[a]} - f_{[b]} - f_{[c]} - g_{[c]} - h_{[d]} - h_{[a]}$$

The results developed in the Section 3 can be used to make calculations concerning the noncrossing tree partition lattice. In particular, there are nice methods to calculate the numbers of maximal chains for the noncrossing tree partition lattices corresponding to one-parameter families of trees. For a lattice $L$, we let $\text{mc}(L)$ denote the number of maximal chains in $L$. We focus on a family of trees called stars.

**Definition 7.1.** Define the star $S_k$ to be a tree with $k + 1$ interior vertices, one of which is adjacent to the other $k$, all of which have degree 3.

**Proposition 7.2.** The number of maximal chains, $\text{mc}(\text{NCP}(S_k))$ is $\frac{F_k F_{k+1}}{2}$ where $F_k$ is the $k$th Fibonacci number with $F_0 = 1, F_1 = 1$.

**Proof.** Let $\{a_i\}_{i=1}^m$ be the set of coatoms of $S_k$.

First note that $\text{mc}(\text{NCP}(S_k)) = \sum_{i=1}^m \text{mc}(\{0, a_i\})$. Also, each $a_i$ consists of two blocks. There are two possibilities: either there is a block of size 1 (not the vertex of degree $k$) and a block of size $k$ or there is a block of size $k - 1$ involving the vertex of degree $k$ along with a block of size 2.

For $k \geq 3$, there are $k$ of each type. So if a cocomordin $a_i$ consists of blocks $A$ and $B$, we have $\text{mc}(\{0, a_i\}) = \text{mc}(\text{NCP}(A) \times \text{NCP}(B))$.

Thus, $\text{mc}(\text{NCP}(S_k)) = k(\text{mc}(\text{NCP}(S_1) \times \text{NCP}(S_{k-2})) + \text{mc}(\text{NCP}(S_0) \times \text{NCP}(S_{k-1}))) = k(\text{mc}(\text{NCP}(S_1) \times \text{NCP}(S_{k-2})) + \text{mc}(\text{NCP}(S_{k-1})))$.

The minimal element in $\text{NCP}(S_{k-2})$ has $k - 1$ blocks and the maximal element has 1, so a maximal chain has $k - 2$ covering relations. Thus there are $(k - 2) + 1 = k - 1$ ways to insert the single covering relation of $\text{NCP}(S_1)$ into a maximal chain of $\text{NCP}(S_{k-2})$ to form a maximal chain of $\text{NCP}(S_1) \times \text{NCP}(S_{k-2})$. So $\text{mc}(\text{NCP}(S_k)) = k((k - 1) \ast \text{mc}(\text{NCP}(S_{k-2})) + \text{mc}(\text{NCP}(S_{k-1})))$.

Note that $\text{mc}(\text{NCP}(S_1)) = 1$ and $\text{mc}(\text{NCP}(S_2)) = 3$. These don’t meet the specifications for $T$ but it is still possible to count their maximal chains. Now for $l \geq 3$, assume we have $\text{mc}(\text{NCP}(S_l)) = \frac{k! F_{k+1} - 2}{2}$ for $1 \leq k < l$.

Then $\text{mc}(\text{NCP}(S_l)) = l(\l - 1) \ast \text{mc}(\text{NCP}(S_{l-2})) + \text{mc}(\text{NCP}(S_{l-1})) = l\l - 1\l(l - 2)! \frac{F_{l+1}}{2} + \l(l - 1)! \frac{F_{l+1}}{2} = l\l! \frac{F_{l+1}}{2}$ and the induction is complete.

**Definition 7.3.** Two vertices of degree 3 in $S_k$, which are each adjacent to 2 leaves, are called outer neighbors if there is a segment with those endpoints. Such a segment would necessarily consist of two edges.
Proposition 7.4. \(|NCP(S_3)| = 14, |NCP(S_4)| = 34. For \(k \geq 5\), \(|NCP(S_k)| = 2|NCP(S_{k-1})| + |NCP(S_{k-2})|\). These are known as the Pell–Lucas numbers.

Proof. One can verify \(|NCP(S_3)| = 14, |NCP(S_4)| = 34. For \(n \geq 5\), consider some interior vertex, \(A\) of \(S_k\) which has degree 3. \(A\) is either (i) in a block of size 1, (ii) in a block of size 2 along with some other interior vertex of degree 3, or (iii) in the same block as the vertex of degree \(k\).

Let \(Sub_k\) denote the number of noncrossing partitions of \(S_k\) such that a given pair of vertices of degree 3 (in particular vertices of degree 3 adjacent to two leaves) form a block of size 2.

Case (i): The remaining vertices other than \(A\) form a \(S_{k-1}\) but the two vertices which had been outer neighbors of \(A\) cannot be in the same block of size 2. Thus there are \(|NCP(S_{k-1})| - Sub_{k-1}\) possibilities.

Case (ii): There are two choices for which outer neighbor, \(B\), of \(A\) is included in the same block as \(A\). The remaining vertices form a \(S_{k-2}\) where the vertices which had been outer neighbors of \(A\) and \(B\) respectively cannot constitute a block of size 2. Thus there are \(2 \times (|NCP(S_{k-2})| - Sub_{k-2})\) possibilities.

Case (iii): We can contract the edge between \(A\) and the vertex of degree \(k\) in order to form a \(S_{k-1}\). Choosing a noncrossing tree partition of the \(S_{k-1}\) which does include a block of size 2 constituted by what were the outer neighbors of \(A\) and then uncontracting the edge we contracted is equivalent to choosing a noncrossing tree partition of \(S_k\) in which \(A\) is in the same block as the vertex of degree \(k\). Thus there are \(|NCP(S_{k-1})| - Sub_{k-1}\) possibilities.

In total:

\[
|NCP(S_k)| = |NCP(S_{k-1})| - Sub_{k-1} + 2 \times (|NCP(S_{k-2})| - Sub_{k-2}) + |NCP(S_{k-1})| - Sub_{k-1} \\
= 2|NCP(S_{k-1})| + 2|NCP(S_{k-2})| - 2(Sub_{k-1} + Sub_{k-2})
\]

Note that \(Sub_3 = 2, Sub_4 = 5,\) and \(Sub_5 = 12\). We claim that for \(k \geq 3\): \(|NCP(S_k)| = 2Sub_{k+1} + 2Sub_k\). We will prove this by induction.

Base cases: \(|NCP(S_3)| = 14 = 2 \times 5 + 2 \times 2 = 2Sub_4 + 2Sub_3\) and \(|NCP(S_4)| = 34 = 2 \times 12 + 2 \times 5 = 2Sub_5 + 2Sub_4\)

For some \(n \geq 3\), we have \(NCP(S_k) = 2Sub_{k+1} + 2Sub_k\) for all \(k \leq n\). Now consider

\[
|NCP(S_{n+1})| = 2|NCP(S_n)| + 2|NCP(S_{n-1})| - 2(Sub_n + Sub_{n-1})
\]

\[
= 2(2Sub_{n+1} + 2Sub_n) + 2(2Sub_n + 2Sub_{n-1}) - 2(Sub_n + Sub_{n-1})
\]

\[
= 4Sub_{n+1} + 6Sub_n + 2Sub_{n-1}
\]

Now consider \(Sub_{k+1}\). It is the number of noncrossing partitions of \(S_{k+1}\) which include a given block of size 2 constituted by 2 outer neighbors. This is equivalent to

\[
|NCP(S_{k+1})| = 2Sub_k + Sub_{k-1}
\]

Thus \(|NCP(S_{n+1})| = 2Sub_{n+2} + 2Sub_{n+1}\), completing the induction. So

\[
|NCP(S_k)| = 2|NCP(S_{k-1})| + 2|NCP(S_{k-2})| - 2(Sub_{k-1} + Sub_{k-2})
\]

\[
= 2|NCP(S_{k-1})| + |NCP(S_{k-2})|
\]

\(\square\)

In the case of a star, it is also possible to explicitly compute \(|Bic(T)|\).

Proposition 7.5. For \(k \geq 3\), \(|Bic(S_k)| = 3^k + (-1)^k\)

Proof. Each member of \(Bic(S_k)\) is analogous to choosing some vertices and edges of an \(k\)-gon such that if two adjacent edges are chosen, the vertex between them must be chosen and if two adjacent edges are both not chosen, then the vertex between them cannot be chosen either.

The number of vertices for which one neighboring edge is chosen and the other is not is necessarily even. For each choice of such vertices, there are two ways to pick which of the sets of intervening edges are chosen and which are not. There are then two ways to pick whether each of these vertices is chosen.

Thus

\[
|Bic(S_k)| = \sum_{i=0}^{i=k} \binom{k}{i} \times 2^{i+1}
\]
Note that
\[ 3^k = (2 + 1)^k = \sum_{i=0}^{k} \binom{k}{i} \cdot 2^i \]
and
\[ 1 = (2 - 1)^k = \sum_{i=0}^{k} (-1)^{k-i} \cdot \binom{k}{i} \cdot 2^i \]
Adding these expressions together gives us:
\[ 3^k + 1 = 2 \cdot \sum_{\substack{i=0 \atop i \text{ even}}}^{k} \binom{k}{i} \cdot 2^i \]
for even \( k \) and
\[ 3^k - 1 = 2 \cdot \sum_{\substack{i=0 \atop i \text{ even}}}^{k} \binom{k}{i} \cdot 2^i \]
for odd \( k \).
So in either case, \( |\text{Bic}(S_k)| = 3^k + (-1)^k \)

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