RIBBON LATTICES AND RIBBON FUNCTION IDENTITIES

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Abstract. We construct a family of generalized lattice models depending on a positive integer n whose partition functions are equal to the n-ribbon functions introduced by Lascoux, Leclerc and Thibon. Using the ribbon lattice model, we provide combinatorial proofs of ribbon function identities relying on the lattice structure and demonstrate the exact solvability of the model for 1,2, and 3 ribbons.

1. Introduction

The study of symmetric functions is of great importance, and has given rise to many interesting discoveries in combinatorics, algebraic geometry, and representation theory in recent years. There has also been a strong interest in q-analogues and q,t-analogues of symmetric functions. For example, Lascoux, Leclerc, and Thibon's ribbon functions [LLT97] are q-analogs of the well known Schur functions, and have led to the discovery of many interesting results in symmetric function theory. In particular, many important families of symmetric functions $\{F_\lambda(x_1, x_2, \dots) : \lambda \in S\}$ over a field K with index set S satisfy three important properties:

(1) They are generating functions for a set of nice tableaux

$$F_\lambda(x_1, x_2, \dots) = \sum_T s(T), \tag{1}$$

where the sum is taken over all tableaux of shape λ. The composition $\mathrm{wt}(T)$ is the *weight* of T and $s(T)$ is some other statistic of T taking values in K.

(2) Along with a related dual family $\{G_\lambda(x_1, x_2, \dots) : \lambda \in S\}$ of symmetric functions, they satisfy a Cauchy identity

$$\sum_{\lambda \in S} F_\lambda(x_1, x_2, \dots) G_\lambda(y_1, y_2, \dots) = \prod_{i,j=1}^{\infty} \sum_{k=0}^{\infty} b_k x_i^k x_j^k, \tag{2}$$

where b_k is a set of parameters in K.

(3) They satisfy a Pieri identity

$$h_k(x_1, x_2, \dots) F_\lambda(x_1, x_2, \dots) = \sum_{\mu \rightharpoonup_k \lambda} b_{\lambda,\mu} F_\mu(x_1, x_2, \dots), \tag{3}$$

where k is a positive integer, $\{h_1, h_2, \dots\}$ is a set of symmetric polynomials over K, and $b_{\lambda,\mu} \in K$ are coefficients for each λ, μ satisfying some condition $\mu \rightharpoonup_k \lambda$.

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In most cases K is either $\mathbb{Q}$ or $\mathbb{Q}(q)$. The simplest family satistfying these three conditions is the Schur functions, where the indexing set S is given by the set of partitions. For the first property the tableaux are the semistandard Young tableaux, $s(T)$ is just 1, and $\mathrm{wt}(T)$ is the usual weight of a tableaux. In the Cauchy identity, the dual family is also equal to the Schur functions, and all of the coefficients b_i are equal to 1. Finally for the Pieri rule, the h_k are the homogeneous symmetric functions, the condition $\mu \rightharpoonup_k \lambda$ is that λ/μ is a horizontal strip of size k, and all the coefficients $b_{\lambda,\mu}$ are equal to 1. The ribbon functions are another example of a family of symmetric functions satisfying these three properties. In [Lam05a], it was shown that certain representations of Heisenberg algebras give rise to families of symmetric functions satisfying these three properties including the Schur and ribbon functions. This relation between representation theory of Heisenberg algebras and families of symmetric functions is known as the combinatorial boson fermion correspondence.

In the case of Schur functions, the Heisenberg algebra representation can be viewed directly as a lattice model [BBF09] and these three identities can be proven combinatorially. These lattice models are intimately related with a multitude of combinatorial objects including semistandard Young tableaux, non-intersecting lattice paths, and free fermionic models [ZJ09]. For ribbon functions along with many other families, however, it was not thought that there was a corresponding lattice model related to the family, and all the identities known about the ribbon functions arose from algebraic methods using representation theory of infinite dimensional lie algebras or the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$ [Lam05b].

In our work, we construct a lattice model, called a ribbon lattice, generalizing the five vertex model in [BBF09] that naturally gives rise to the ribbon functions. Using the properties of the ribbon lattice, we are able to give proofs of ribbon function identities. Finally, we demonstrate that the lattice model is exactly solvable or integrable, meaning that it satisfies an appropriate Yang-Baxter equation [BBF09] for 1, 2, and 3-ribbons by a direct computation. From the perspective of symmetric function theory, all the Yang Baxter equation implies is that the ribbon functions are symmetric, however is of extreme importance from the perspective of lattice models and statistical mechanics, for it is an indispensable tool for computing the partition function of a fixed set of boundary conditions.

It would be interesting to investigate if there are other families of symmetric functions that arise naturally from lattice models. In particular, seeing as the properties of the lattices give natural relations to tableaux and can be used to prove dual Cauchy and Pieri identities for Schur functions, it would be particularly interesting to see if families of symmetric functions arising from Lam's combinatorial boson fermion correspondence can be written as partition functions of generalized lattices, and more generally if the combinatorial boson fermion correspondence has a nice interpretation as generalized lattice models.

2. Tableaux and Symmetric Functions

The families of symmetric functions we are interested in arise as generating functions of nice families of tableaux with shape specified by a fixed partition, which we now review. A partition λ is any decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of nonnegative integers. We denote the *length* of λ by $l(\lambda) = r$, and the *weight* of λ by $|\lambda| = \lambda$ is $\lambda_1 + \lambda_2 + \ldots + \lambda_r$. Furthermore, we do not distinguish a partition from its Young diagram. If $\mu \subset \lambda$ for two partitions λ, μ, then λ/μ is said to be a *skew shape* with size $|\lambda/\mu| = |\lambda| - |\mu|$. We will occasionally abuse notation and identify partitions that differ only by a sequence of zeros.

The ring of symmetric functions over $\mathbb{Q}$ contains a distinguished basis $\{s_\lambda\}$ indexed by partitions known as the *Schur functions*. These are given by the generating functions of semistandard Young tableaux

$$s_\lambda(X) = \sum_{T \in \mathrm{SSYT}(\lambda)} x^{\mathrm{wt}(T)}, \tag{4}$$

where $\mathrm{SSYT}(\lambda)$ denotes the set of all semistandard Young tableaux of shape λ, and we will write $X = (x_1, x_2, \ldots)$ for simplicity. Similarly, one can define the *skew Schur functions* $s_{\lambda/\mu}$ as the generating functions of semistandard Young tableaux of skew shape λ/μ. For the purposes of this paper, it will be most useful to think of a semistandard Young tableaux of skew shape λ/μ as a sequence of partitions $\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^l = \lambda$ such that for each i the skew shape λ_i/λ_{i-1} contains at most one box in each column (Figure 1). Such a skew shape is called a *horizontal strip*. The weight of some $T \in \mathrm{SSYT}(\lambda/\mu)$ is then given by $\mathrm{wt}(T) = (|\lambda^1/\lambda^0|, |\lambda^2/\lambda^1|, \ldots, |\lambda^{l-1}/\lambda^l|)$.

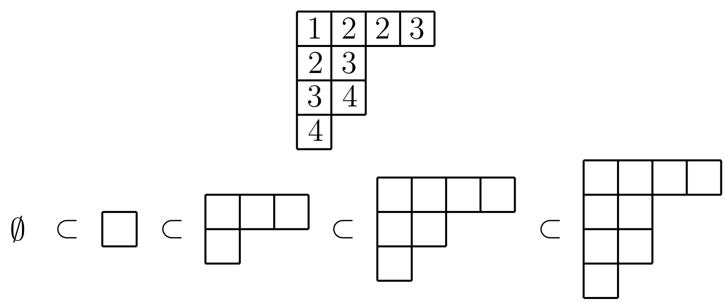


FIGURE 1. Semistandard Young Tableaux of shape (4,2,2,1) and the corresponding sequence of Young diagrams.

We now describe the fundamental objects of this paper: ribbon tableaux and ribbon functions. We follow the notation in [Lam05b]. Let n be a fixed positive integer. An *n-ribbon* is a skew shape r containing n boxes that is connected and contains no 2 by 2 squares. The *spin* of a ribbon r is defined to be the height of r minus one, and is denoted $\mathrm{spin}(r)$. We define a *semistandard n-ribbon tableaux* of skew shape λ/μ to be a sequence $\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^l = \lambda$ of partitions such that for each i the skew shape λ_i/λ_{i-1} is a *horizontal n-ribbon strip*: A tiling of a skew shape λ/μ by n-ribbons such that toprightmost square of each ribbon touches the northern boundary of λ/μ. We define the *weight* of a semistandard n-ribbon tableaux T to be $\mathrm{wt}(T) = (|\lambda^1/\lambda^0|, |\lambda^2/\lambda^1|, \ldots, |\lambda^{l-1}/\lambda^l|)$ and we define the *spin* of T to be the sum of the spins of the n-ribbons tiling T. We denote the set of all semistandard n-ribbon tableaux by $\mathrm{SSRT}_n(\lambda)$. Note that is not always possible to tile a given partition λ with n-ribbons, and we will restrict our attention to partitions or skew shapes that are tileable by n-ribbons.

We can now define one of the central objects of this paper, first introduced by Lascoux, Leclerc, and Thibon in [LLT97].

Definition 1. Let $n \geq 1$ be fixed and λ/μ a skew skape tileable by n-ribbons. Then the *n-ribbon function of shape λ/μ* is defined as the generating function

$$\mathcal{G}^{(n)}_{\lambda/\mu}(X;q) = \sum_{T \in \mathrm{SSRT}_n(\lambda/\mu)} q^{\mathrm{spin}(T)} x^{\mathrm{wt}(T)}. \tag{5}$$

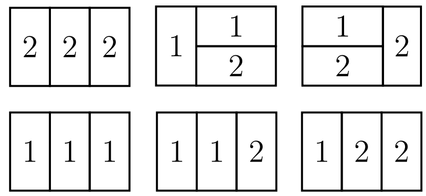


FIGURE 2. Ribbon Tableaux corresponding to $\mathcal{G}^{(2)}_{(3,3)}$.

For example, suppose that $\lambda = (3,3)$ and $n = 2$. It is not difficult to see that the only possible 2-ribbon tableaux are those depicted in Figure 2. Now we can directly compute that

$$\mathcal{G}^{(2)}_{(3,3)}(x_1, x_2; q) = q^3({x_1}^3 + {x_1}^2 x_2 + x_1 {x_2}^2 + {x_2}^3) + q({x_1}^2 x_2 + x_1 {x_2}^2).$$

We will often abuse notation and write $\mathcal{G}^{(n)}_{\lambda/\mu}(X;q) = \mathcal{G}_{\lambda/\mu}(X;q)$ when n is understood. The ribbon functions are q-analogues of the Schur functions in the sense that $\mathcal{G}_{\lambda/\mu}(X;1)$ is equal to a product of n Schur functions. Having defined all the relevant tableaux and symmetric functions, we now describe the lattice models of interest.

3. Lattice Models

Exactly solvable two dimensional Ising models are of great interest in statistical mechanics, as they provide useful information about the behavior of phase transitions. The six vertex model or ice-type model from statistical mechanics is a lattice model defined on a rectangular grid. An *admissible state* for the lattice is a labeling of each edge with arrows so that at each vertex, there are two arrows pointing inward and two pointing outward. We denote a vertex by the two directions from which arrows are pointing inward, for example southwest (SW) and so on. To each vertex v, we assign a weight $\mathrm{wt}(v)$ taking values in $\mathbb{Q}(x_1, x_2, \ldots)$, usually depending on which row the vertex lies in. For a given admissible state S, the weight of the lattice is defined to be the product of the weights at each of the vertices, that is

$$\mathrm{wt}(S) = \prod_{v \in S} \mathrm{wt}(v). \tag{6}$$

Given a fixed set of boundary conditions B on a rectangular grid, we define the *partition function* of a square grid to be the sum of the weights of all admissible states:

$$\mathcal{P}_B(X) = \sum_{\substack{\text{admissible states } S \text{ with} \\ \text{boundary conditions } B}} \mathrm{wt}(S), \tag{7}$$

We are interested in lattice models whose partition functions give interesting families of symmetric polynomials. In this section, we describe a five vertex lattice model whose partition functions are the Schur functions, followed by a more general lattice type model that gives rise to the ribbon functions. We can then deduce interesting properties of these symmetric functions from the properties of the lattice models.

3.1. **A Five Vertex Model and Schur Functions.**

Given a fixed skew shape λ/μ, where λ has length r, we convert λ from a weakly decreasing sequence to a strictly decreasing sequence by adding the vector $\rho = (r-1, r-2, \ldots, 1, 0)$, and do the same to μ, viewing μ as a vector with r components by adding zeroes to the end if necessary. Now consider a square grid with r rows and and $\lambda_1 - \mu_1 + r$ columns, labeled

0 through $\lambda_1 - \mu_1 + r - 1$. In the top row, place an upward arrow in the columns labeled by the entries of $\mu + \rho$, and downward arrows for all other vertices. Similarly place upward arrows in the bottom row at every column labeled by the entries of $\lambda + \rho$. Finally for the left and right boundaries of the square lattice, put rightward pointing arrows at every edge. We call these boundary conditions *Schur boundary conditions of shape* λ/μ. We then define the *partition function* $\mathcal{P}_{\lambda/\mu}$ of skew shape λ/μ to be the partition function with respect to Schur boundary conditions of shape λ/μ. When $\mu = \emptyset$, we often omit μ, and just write $\mathcal{P}_\lambda$ in place of $\mathcal{P}_{\lambda/\mu}$.

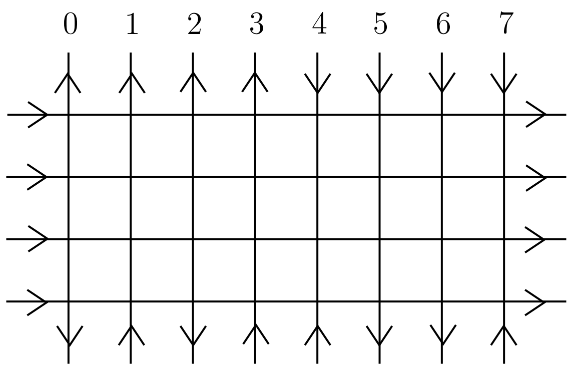


FIGURE 3. Schur boundary conditions for the six vertex model of shape λ/μ with $\lambda = (4, 2, 2, 1)$ and $\mu = (0, 0, 0, 0)$, so that $\lambda + \rho = (7, 4, 3, 1)$ and $\mu + \rho = (3, 2, 1, 0)$

Now consider the weights picture in Figure 4, noting that the weight of a vertex depends on the row it lies in. Since the SE vertex has a weight of zero, any admissible state containing a SE vertex will not contribute to the partition function. Therefore we can view this lattice as a five vertex model. Therefore, going forward, we will not consider any state with weight 0 as an admissible state. A similar five vertex model appears in [Lam05a] whose partition functions are equal to $x^\rho s_\lambda(x_1 x_2, \ldots)$. The advantage using the set of weights in Figure 4 is that the resulting admissible states can also be interpreted as non intersecting lattice paths (NILPs) on a square grid. Furthermore, these vertex weights have a nice combinatorial interpretation that is key to understanding the connection between the five vertex model and the Schur functions:

Proposition 3.1. *For any admissible state* S*,* $wt(S) = \prod_{i=1}^{r} x_i^{a_i}$*, where* a_i *is the number of leftward pointing arrows in row* i*.*

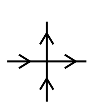
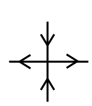
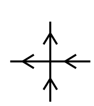
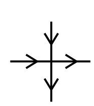
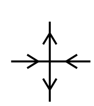
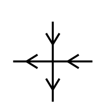
Label	SW	NS	SE	NW	EW	NE
Vertex						
Weight	1	1	0	1	x_i	x_i

FIGURE 4. The weights of vertices lying in row i for the five vertex model labeled by cardinal directions.

This observation motivates the following result.

Theorem 3.2. *There is a weight preserving bijection between admissible lattice states with Schur boundary conditions of skew shape λ/μ and semistandard Young tableaux of shape λ/μ. More precisely, there is a bijective mapping Φ from the set $SSYT(\lambda/\mu)$ to the set of admissible lattice states with Schur boundary conditions of shape λ/μ such that $wt(\Phi(T)) = x^{wt(T)}$ for $T \in SSYT(\lambda/\mu)$.*

Proof. Consider some semistandard Young tableaux $T = (\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^l = \lambda)$. By adding zeros if necessary, assume each λ_i has length $l(\lambda) = r$. Now define $\gamma_i = \lambda_i + \rho$, where $\rho = (r-1, r-2, \ldots, 0)$.

1	2	2	3
2	3		
3	4		
4			

FIGURE 5. Semistandard Young tableaux of the partition $(4, 2, 2, 1)$.

Given a grid with Schur boundary conditions of shape λ/μ with $\lambda_1 - \mu_1 + r$ columns and r rows, assign the interior vertical arrows according to the following rule: for a fixed i in $2, 3, \ldots, r$ the vertical arrows directly above row i and in column j are pointing upward if j is an entry of γ_{i-1}, and are pointing downwards otherwise (recall that we are labeling the columns starting at 0 and ending at $\lambda_1 - \mu_1 + r - 1$).

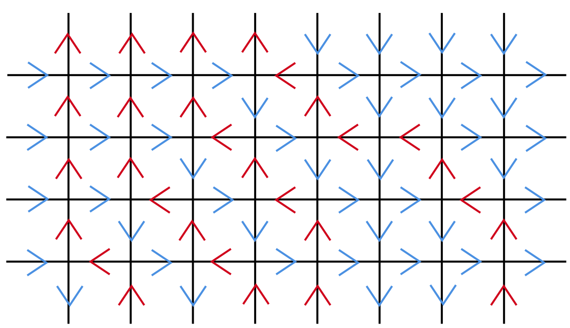


FIGURE 6. Lattice model associated to the semistandard Young tableaux in Figure 5.

Since the no admissible state contains a SE vertex, it is easy to see that the assignment of vertical arrows will determine a unique assignment of horizontal arrows corresponding naturally to a NILP. Now in view of Proposition **??**, the weight preserving bijection follows from the weight bijection between semistandard Young tableaux and NILP's described in [Sta99b] Section 7.16.

□

We obtain the following as an immediate corollary, which explains the naming of the Schur boundary conditions.

Corollary 3.3. *For any skew shape λ/μ, the partition function $\mathcal{P}_{\lambda/\mu}(x_1, \ldots, x_r)$ is equal to the skew Schur function $s_{\lambda/\mu}(x_1, \ldots, x_r)$.*

In view of the previous corollary, many nice identities of Schur functions can be proved using properties of lattice models. For example, consider some admissible lattice state S with Schur boundary conditions of shape λ. Below the r^{th} row, there are vertical arrows positioned according to the entries of $\lambda + \rho$. Directly above the r^{th} row, there are vertical arrows positioned according to $\gamma + \rho$, where λ/γ is a horizontal strip, which we will denote by $\gamma \prec \lambda$. For a fixed $\gamma \prec \lambda$, exactly $|\lambda/\gamma|$ vertices in the r^{th} row will have weight x_r, and all the remaining vertices in that row will have weight 1. It readily follows that

Proposition 3.4. *If λ/μ is a skew shape, then*

$$s_{\lambda/\mu}(x_1, \ldots, x_r) = \sum_{\gamma \prec \lambda} x_r^{|\lambda/\gamma|} s_{\gamma/\mu}(x_1, \ldots, x_{r-1}). \tag{8}$$

$$\tag{9}$$

Splitting the lattice diagram at any row instead of the bottom row gives rise to more general branching identities:

Theorem 3.5. *Let λ/μ be a skew shape of length r and $\gamma \subset \lambda$ a fixed partition such that λ/γ has length k. Then*

$$s_{\lambda/\mu}(x_1, \ldots, x_r) = \sum_{T \in SSYT(\lambda/\gamma, k)} x_{-(k+1)}^{wt(T)} s_{\gamma/\mu}(x_1, \ldots, x_k), \tag{10}$$

$$s_{\lambda/\mu}(x_1, \ldots, x_r) = \sum_{T \in SSYT(\gamma/\mu, k)} x^{wt(T)} s_{\lambda/\gamma}(x_{r-k+1}, \ldots, x_r), \tag{11}$$

where we write $x_{-(k+1)}^{wt(T)} = x_{k+1}^{a_1} \cdots x_r^{a_k}$ whenever $wt(T) = (a_1, \ldots, a_k)$, and $SSYT(\lambda/\gamma, k)$ is the set of semistandard Young tableaux of skew shape λ/γ with labels in $[k]$.

Overall, the lattice model structure allows us to prove interesting identities about Schur functions by computing a partition function in two different ways. In fact, we can generalize the five vertex model in order to give combinatorial proofs of many ribbon function identities.

3.2. Ribbon Lattices.

Throughout this section, fix some $n \geq 1$. We now describe an extension of the 5 vertex model that is connected to n ribbon functions, which we call the *n-ribbon lattice model.* In each row, instead of a single line, we will have n horizontal lines crossing one vertical line at each vertex. For example, Figure 7 shows an example of a single row inside a 3-ribbon lattice, where each vertex has one vertical line passing through and three tangled lines passing through.

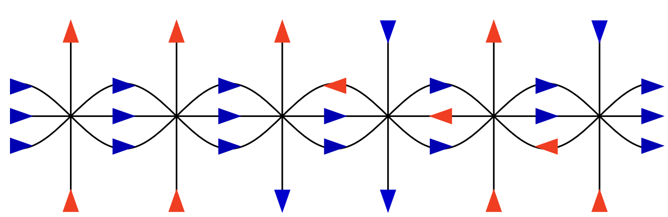


FIGURE 7. One row of a 3-Ribbon Lattice

We describe a vertex v by two n-tuples $v_L = (v_L(1), \cdots, v_R(n))$ and $v_R = (v_R(1), \cdots, v_R(n))$ with entries labeled either left or right, and two labels v_U and v_D given by up or down. The entries $v_L(1), v_L(2)$ to $v_L(n)$ encode the arrows at the edges to the left hand side of a vertex, from bottom to top; and v_R encodes the arrows right to a vertex in the same way. We think of these labels as arrows pointing either in or out of the vertex v, and will loosely call the entries of v_U, v_D, v_L, and v_R arrows. There are a few conditions that are necessary for a vertex v to be admissible. First of all, the number of arrows pointing inward must equal the number of vertices pointing outward. Furthermore, we require that $v_R(i) = v_L(i+1)$ for $i = 1, 2, \ldots, n-1$. It is important to note that $v_R(n)$ need not be equal to $v_L(1)$.We can summarize this condition pictorially by drawing lines between $v_R(i)$ and $v_L(i+1)$ for $i = 1, 2, \ldots, n-1$ in addition to a line between $v_R(1)$ and $v_L(n)$ (Figure 8).

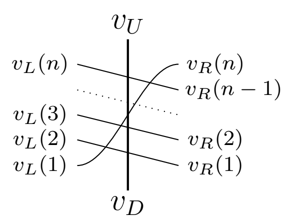


FIGURE 8. A general n-ribbon vertex

We call the edge from $v_L(1)$ to $v_R(n)$ to be the *twisted edge*, and every other horizontal edge to be *straight*. We can summarize the second condition for an admissible vertex concisely by requiring that arrows do not change along any straight edge.

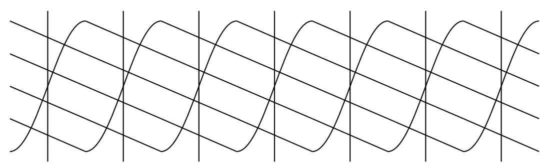


FIGURE 9. One row of a 5 ribbon lattice with arrows omitted.

Formally, we define a *n-Ribbon Lattice* to be a rectangular grid of admissible vertices $v[i, j]$ where $1 \leq i \leq a$ and $1 \leq j \leq b$ for fixed positive integers a, b such that the left vertices of one vertex v agree with the right vertices of the vertex w located to the left of v, and so on. Notice that when $n = 1$ this is just the 5 vertex model described earlier. In general, there are 2^{n-1} choices for the vertices $v_R(1), v_R(2), \ldots, v_R(n-1)$, and these uniquely determine the vertices $v_L(2), v_R(3), \ldots, v_L(n)$. Then we just need to choose values for $v_U, v_D, v_L(1)$, and $v_R(n)$, and it is easy to see that there are exactly 5 choices for an admissible vertex, so there are a total of $5 \cdot 2^{n-1}$ admissible. Some of these, however, will have weight zero and will not contribute to the partition functions of interest.

Unlike the 5-vertex model, the vertices of ribbon lattices have weights in $\mathbb{Q}(q, x_1, x_2, \ldots)$. First off, any vertex that is not admissible will be given a weight of 0. Furthermore, any vertex v such that $(v_R(1), v_L(n), v_U, v_D) = (\text{left}, \text{left}, \text{up}, \text{up})$ is also given weight 0. Given any other admissible vertex v in row i, we define its weight by

$$\text{wt}(v) = x_i^{\varepsilon(v)} \cdot q^{\sigma(v)}, \tag{12}$$

where

$$\varepsilon(v) = \begin{cases} 1 & \text{if } v_R(n) = \text{left} \\ 0 & \text{otherwise;} \end{cases} \tag{13}$$

$$\sigma(v) = \begin{cases} L(v) & \text{if } v_U = \text{up}, \, v_D = \text{up} \\ L(v) & \text{if } v_U = \text{down}, \, v_D = \text{up} \\ L(v) - 1 & \text{if } v_U = \text{up}, \, v_D = \text{down} \\ L(v) - 1 & \text{if } v_L(1) = \text{left}, \, v_R(n) = \text{left} \end{cases} \tag{14}$$

where $L(v)$ is the number of left arrows in v_R. The spin weight has a nice pictorial interpretation pictured in Figure 10: the exponent $\sigma(v)$ of q is exactly the number of left arrows in each blue circled area.

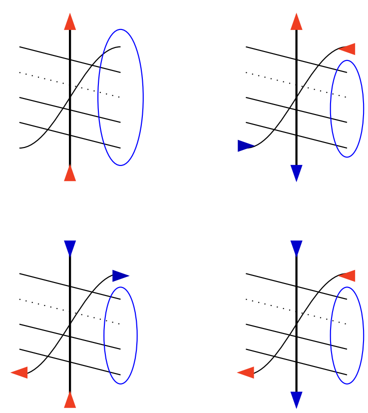


FIGURE 10. The spin of a vertex.

The ribbon lattices have the graphical interpretation that when moving from one row to the row above it, the up arrows are allowed to travel to the left in steps of size divisible by n, where the weight picks up a factor of x_i each time an up arrow moves to the left for n steps and a factor of q each time one up arrow crosses over another arrow.

The boundary conditions of the ribbon lattices are very similar to the Schur boundary conditions for the 5-vertex model. For a skew shape λ/μ, the top and lower boundaries of the rectangular grid are defined exactly as before by adding ρ to λ and μ and using the shifted entries to determine where to place the up arrows. Along the left and right boundary, we force that all the outer arrows point right, that is $v[1, i]_L$ and $v[a, i]_R$ consist of only right arrows for all i. We call these boundary conditions *n-ribbon boundary conditions*. We can now define the *n-ribbon partition functions* $\mathcal{R}^{(n)}_{\lambda/\mu}(X; q)$ to be the partition of the n-ribbon lattice with the above boundary conditions. We will show that these partition functions are exactly the ribbon functions $\mathcal{G}^{(n)}_{\lambda/\mu}(X, q)$.

Lemma 3.6. *Consider a ribbon lattice boundary $\mathcal{B}$ with a single row with upper and lower boundary conditions determined by the skew shape λ/μ. If λ/μ is a horizontal n-ribbon strip, then there is a unique admissible state in the n-ribbon with this set of boundary conditions. If λ/μ is not a horizontal n-ribbon strip, then there are no admissible states with boundary $\mathcal{B}$.*

Proof. The condition $v_R(i) = v_L(i+1)$ implies that any admissible state can be broken up into n different admissible states of the 1-ribbon lattice where each vertex only interacts with vertices a multiple of n away from them. Theorem 3.2 then implies there is at most one admissible lattice state for λ/μ.

We create a map from a set of n-ribbons which form a horizontal n-ribbon strip in λ/μ to an admissible state with boundary $\mathcal{B}$. Our map will "peel" off ribbons in our horizontal n-ribbon strip by creating a larger lattice which we will merge into a single row in our lattice. We start at the rightmost ribbon r and create an admissible state for $\lambda/(\lambda \setminus r_1)$. Consider the furthest right vertex v in our lattice with an up arrow below v and a down arrow above v. We then may peel off r in our lattice by adding left arrows at n vertices by adding $v_L(1)$ and using the rule $v_R(i) = v_L(i+1)$ to add a left arrow out of to $n-1$ of v's leftward neighbors. Filling out the lattice with all other arrows as right arrows gives an admissible lattice state for $\lambda/(\lambda \setminus r_1)$. Now we recursively apply the above process to $(\lambda \setminus r_1)/(\lambda \setminus r_1 \cup r_2)$ where r_2 is the rightmost ribbon in $(\lambda \setminus r_1)$. When we finish this process we will have a set of admissible lattices $(\lambda \setminus \bigcup_{i=1}^{k} r_i)/(\lambda \setminus \bigcup_{i=1}^{k+1} r_i)$.

Now we construct the admissible lattice state for λ/μ by taking the union of the sets of left arrows in each of the above lattice states. There are 4 things to check to make sure this is an admissible lattice state: (1) For every vertex v, $v_R(i) = v_L(i+1)$, (2) every vertex has an equal number of in and out arrows, (3) the boundary conditions are the correct boundary conditions, and (4) no vertex has an up arrow above it and a left arrow at $v_L(1)$.

Since each ribbon has an upper right corner which borders the top of the horizontal n-ribbon strip, we know that $\lambda \setminus \cup_{i=1}^{k} r_k$ must agree with μ everywhere to the right of and including the rightmost vertex of r_k. If follows that if $v_L(1)$ points left that there is no up arrow above v proving (4). For (2), $v_R(i) = v_L(i+1)$ and the above statement shows that all sets of left arrows in $(\lambda \setminus \cup_{i=1}^{k} r_i)/(\lambda \setminus \cup_{i=1}^{k+1} r_i)$ are distinct from those in $(\lambda \setminus \cup_{i=1}^{s} r_i)/(\lambda \setminus \cup_{i=1}^{s+1} r_i)$ for $s \neq k$, this then follows from each of $(\lambda \setminus \cup_{i=1}^{k} r_i)/(\lambda \setminus \cup_{i=1}^{k+1} r_i)$ being an admissible lattice state. (1) follows from all the sets of left arrows we are unioning including $v_R(i) = v_L(i+1)$. (3) follows in a similar manner as (1) for the left and right boundaries. The up and down boundaries follow from our construction. We can reverse the above process starting from the leftmost side of an admissible state and construct ribbons in λ/μ whose union forms a horizontal n-ribbon. This implies the second statement in our proposition. □

Theorem 3.7. *There is a weight preserving bijection between admissible n-ribbon lattice states with n-ribbon boundary conditions of shape λ/μ and semistandard n-ribbon tableaux of shape λ/μ.*

Proof. First view a ribbon tableaux as a sequence of partitions $\mu = \lambda^1 \subset \lambda^1 \subset \cdots \subset \lambda^r = \lambda$. We assign the vertical arrows in the same manner as in Theorem 3.2. By lemma 3.6, it follows that this map is a bijection.

That the map is weight preserving follows naturally from the path interpretation of the ribbon lattices. In a fixed row i, the weight is given by a power of q times the $x_i^{a_i}$ where a_i is the number of times an arrow moves n steps to the left. This corresponds exactly to the number of ribbons removed from λ^i, which is the weight of λ^i/λ^{i-1}. Similarly, the power of q counts the number of times one arrow jumps over another arrow, which corresponds exactly to the spin of each ribbon in the horizontal n strip λ^i/λ^{i-1}. □

The ribbon lattices also have a nice visual interpretation using what is known as the *edge sequence* of a partition. We label the edges of each ribbon with red dots along all vertical edges, and blue dots along each horizontal edge. The red dots correspond to the up arrows in the lattice, and the blue dots correspond to the down arrows. Along each n-ribbon, the red dot in the upper right corner will travel to the lower left corner, and all the remaining dots will travel up and to the left (Figure 11).

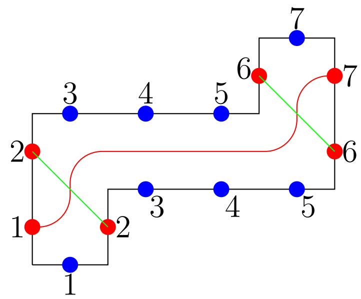


FIGURE 11. The edge sequence path of a 6-ribbon

Notice that the number of intersections of the edge sequence path is exactly equal to the spin of the ribbon, which corresponds exactly to how many times the arrows cross each other in the ribbon lattices. In an n-ribbon horizontal strip, since the top right edge of each constituent n-ribbon lies on the upper edge of the skew shape the paths just described can simply be glued together, and the resulting picture is identical to the movement of arrows in the ribbon lattices (Figure 12).

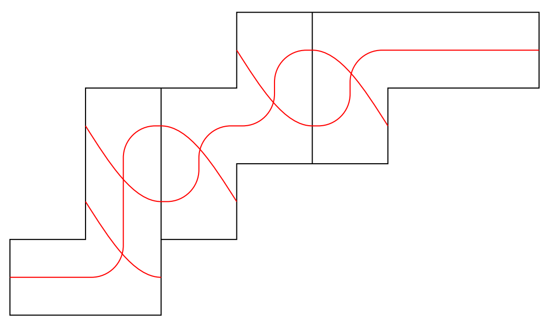


FIGURE 12. The edge sequence path of a horizontal strip

Corollary 3.8. *For any skew shape* λ/μ*, the partition function* $\mathcal{R}^{(n)}_{\lambda/\mu}(X;q)$ *is equal to the skew ribbon function* $\mathcal{G}^{(n)}_{\lambda/\mu}(X,q)$.

The structure of the ribbon lattice gives rise to combinatorial proofs of ribbon function identities. For example, splitting along any row and computing the partition function of a grid in two different ways, we obtain an analogue of Theorem 3.5:

Proposition 3.9 (Branching Rule)**.**

$$\mathcal{G}^{(n)}_{\lambda/\mu}(x_1,\ldots,x_r;q) = \sum_{T\in SSRT_n(\lambda/\gamma,k)} q^{spin(T)} x^{wt(T)}_{-(k+1)} \mathcal{G}^{(n)}_{\gamma/\mu}(x_1,\ldots,x_k;q), \tag{15}$$

$$\mathcal{G}^{(n)}_{\lambda/\mu}(x_1,\ldots,x_r;q) = \sum_{T\in SSRT_n(\gamma/\mu,k)} q^{spin(T)} x^{wt(T)} \mathcal{G}^{(n)}_{\lambda/\gamma}(x_{r-k+1},\ldots,x_r;q), \tag{16}$$

where we write $x_{-(k+1)}^{wt(T)} = x_{k+1}^{a_1} \cdots x_r^{a_k}$ *whenever* $wt(T) = (a_1, \ldots, a_k)$*, and* $SSRT_n(\lambda/\gamma, k)$ *is the set of semistandard* n*-ribbon tableaux of skew shape* λ/γ *with labels in* $[k]$*.*

4. Future Directions

Ribbon lattices have proven to be quite a useful tool for studying ribbon functions and giving combinatorial proofs of interesting identities. There are identities that are known for Schur functions that are not known for ribbon functions that can be proved using the bijection between semistandard Young tableaux and the five vertex model. For example, there is no analog of the Jacobi-Trudi identity for Schur functions that is known to hold for arbitrary n-ribbon functions. There is, however, a proof of the Jacobi-Trudi identity in [Sta99a, Sta99b] using the bijection in Proposition 3.2, and it would be worthwhile to investigate if a similar identity can be derived from ribbon lattices.

The methods in this paper are quite different from the previous methods used to prove ribbon function identities, which generally rely on the action of a Heisenberg algebra of the Fock space of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$. In the case of Schur functions, the connection between the lattice model and the quantum group representation arises from a so called Yang-Baxter equation described in [BBF09]. Lattice models satisfying a Yang-Baxter equation are called *exactly solvable*, or *integrable*. From the lattice model perspective, the Yang Baxter equation gives a consistent way of effectively permuting the weights of an admissible state without altering the partition function. In particular, it implies that the partition functions are in fact symmetric functions.

By direct computation, we were able to obtain solutions for the Yang Baxter equation for 2 and 3 ribbon lattices. The case of 1 ribbons corresponds to Schur functions, which has already been solved. The solutions can be encoded nicely in the form of so called *R-matrices* described in [BBF09]. Each R matrix depends on two rows i and j. If we denote $z = x_i/x_j$, then the R matrix for 2 ribbons corresponding to swapping rows i and j has the form

$$\left[\begin{array}{cccccccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & z^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & z & 0 & 0 & 0 & 0 & 0 & z-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z-1 \\
0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & z^2 & (q^2z-1)(z-1) & 0 & 0 & 0 & 0 & q(z^2-z) & z^2-z & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q(z^2-z) & 0 & 0 & 0 & 0 & 0 & z^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q(z-1) & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & z-1 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q(z^2-z) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]$$

We omit the R matrix corresponding to 3 ribbons. For $n \geq 4$, solving the Yang-Baxter equation is too computationally complex to be calculated in a reasonable amount of time. We conjecture that the ribbon lattices are exactly solvable for all n, that a proof for all n would likely involve a deep connection between the lattice model and the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_n})$.

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