Extended Nestohedra and their Face Numbers

Quang Dao, Christina Meng, Julian Wellman,
Zixuan Xu, Calvin Yost-Wolff, Teresa Yu

Summer 2019

Abstract

Nestohedra are a family of convex polytopes that includes permutohedra, associahedra, and graph associahedra. In this paper, we study an extension of such polytopes, called extended nestohedra. We show that these objects are indeed the boundaries of simple polytopes, answering a question of Lam and Pylyavskyy. In addition, we give formulas for their f-, h-, and γ-vectors. This includes showing that all flag extended nestohedra have nonnegative γ-vectors, thus proving Gal's conjecture for a large class of flag simple polytopes. We also relate the f- and h-vectors of the nestohedra and extended nestohedra, as well as give explicit formulas for the h- and γ-vectors in terms of descent statistics for a certain class of flag extended nestohedra. Finally, we define a partial ordering that is a lattice quotient of the weak Bruhat order on the symmetric group, and show that any linear extension of this poset gives a shelling of the stellohedron.

Contents

1 Introduction

Stasheff's **associahedron** has spurred a rich study of polytopes with combinatorial and algebraic connections. It has been generalized to the **graph associahedron**, introduced by Carr and Devadoss [CD06], which in turn has been generalized to the **nestohedron** in [Pos09; FS05]. The dual simplicial complex of the nestohedron, called the **nested complex**, was introduced by DeConcini and Procesi, along with the notion of **building sets** [DP95].

The nestohedron and nested set complex have together provided many interesting avenues of study. Among them include polytopal realizations and face numbers. The nestohedron has been geometrically realized as the boundary of a polytope through a variety of methods, including through shaving faces of a simplex [CD06] and through Minkowski sums [Pos09; FS05]. The face numbers of the nestohedron are extensively studied in [PRW08], while Gal's conjecture was proven for all flag nestohedra in [Vol10].

More recently, Lam and Pylyavskyy introduced the **extended nested complex** in their study of linear Laurent-phenomenon algebras [LP15]. In the more general context of Laurent-phenomenon algebras, they conjectured that the extended nested complex $\mathcal{N}\ \ (\mathcal{B}_\Gamma)$ is dual to the boundary of a polytope, where Γ is a directed graph [LP15, Conjecture 7.6]. Independently, Devadoss, Heath, and Vipismakul introduced dual complexes for the extended nested complex when the building sets are based on undirected graphs; they refer to such complexes as **graph cubeahedra**. They also show that graph cubeahedra can be realized by shaving the faces of an n-dimensional cube, partially proving Lam and Pylyavskyy's conjecture.

Extended nested complexes are not as well-understood as their non-extended counterparts. However, in certain cases, extended nested complexes are isomorphic to non-extended nested complexes. Manneville and Pilaud characterize such isomorphisms for graphical building sets [MP17], and show that the corresponding graphs must be **spider** and **octopus** graphs (see Definition 4.1).

We now give preliminary definitions before stating the main results of our paper. A **building set** $\mathcal{B}$ on S is a collection of nonempty subsets of S such that

1. if $I, J \in \mathcal{B}$ and $I \cap J \neq ?$, then $I \cup J \in \mathcal{B}$, and
2. $\mathcal{B}$ contains all singletons $\{i\}$ for $i \in S$ (see Definition 2.3).

A building set on S is **connected** if S itself is an element of the building set. An **extended nested collection** on a building set $\mathcal{B}$ on S is a collection $N = \{I_1, \dots, I_m, x_{i_1}, \dots, x_{i_r}\}$ of elements $I_j \in \mathcal{B}$ and x_i for $i \in S$ satisfying the following three properties.

1. For any $i \neq j$, either $I_i \subset I_j, I_j \subset I_i$, or $I_i \cap I_j = ?$.
2. For any collection $I_{i_1}, \dots, I_{i_k} \in N$ of $k \geq 2$ pairwise disjoint elements of N, their union $\bigcup_{\ell=1}^{k} I_{i_\ell}$ is not an element of $\mathcal{B}$.
3. For all $x_{i_\ell}, I_j \in N$, the set I_j does not contain i_ℓ.

A **(non-extended) nested collection** is an extended nested collection with no x_i elements (see Definitions 2.9, 2.12). If $\mathcal{B}$ is a building set on S, the **nested complex** $\mathcal{N}(\mathcal{B})$ is the simplicial complex with vertices $\{I \mid I \in \mathcal{B}\}$ and faces given by non-extended nested collections $\{I_1, \dots, I_r\}$. The **extended nested complex** $\mathcal{N}\ \ (\mathcal{B})$ is the simplicial complex with vertices $\{I \mid I \in \mathcal{B}\} \cup \{x_i \mid i \in S\}$ and faces given by extended nested collections $\{I_1, \dots, I_m\} \cup \{x_{i_1}, \dots, x_{i_r}\}$ (see Definitions 2.11, 2.14).

We now state several of our results.

Theorem 1.1 (Theorem 3.3)**.** If $\mathcal{B}$ is a building set, then $\mathcal{N}\ \ (\mathcal{B})$ can be realized geometrically as the boundary of a simplicial polytope.

For a building set $\mathcal{B}$, it is known that the nested complex $\mathcal{N}(\mathcal{B})$ is isomorphic to the boundary of a simplicial polytope, whose polar dual is a simple polytope $\mathcal{P}(\mathcal{B})$ called the **nestohedron**. Similarly, the simplicial polytope in Theorem 1.1 is polar dual to a simple polytope $\mathcal{P}\ \ (\mathcal{B})$ that we call the **extended**

nestohedron. We find a way to realize $\mathcal{P}\ \ (\mathcal{B})$ as a Minkowski sum over elements of our building set (Theorem 3.6) and use this realization to to find coordinates for the vertices of $\mathcal{P}\ \ (\mathcal{B})$ in terms of combinatorial objects.

In addition, we find cases when a non-extended nested complex is isomorphic to an extended nested complex when the corresponding building sets are not graphical.

Theorem 1.2 (Theorem 4.5)**.** Let $\mathcal{B}$ be a building set on $[n] \coloneqq \{1, \dots, n\}$ such that all elements of $\mathcal{B}$ are intervals. Define $\mathcal{B}' \coloneqq \mathcal{B} \cup \{\{n+1\}, \{n+1, n\}, \{n+1, n, n-1\}, \dots, [n+1]\}$. Then $\mathcal{B}'$ is a building set on $[n+1]$ and $\mathcal{N}\ \ (\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$.

For a simple d-dimensional polytope P, the **f-vector** and **h-vector** of P are $(f_0, f_1, \dots, f_d)$ and $(h_0, h_1, \dots, h_d)$, where f_i is the number of i-dimensional faces of P and h_i's are given by $\sum h_i(t+1)^i = \sum f_i t^i$. It is well known that the h-vectors of simple polytopes are positive and symmetric.

Theorem 1.3 (Theorem 5.3)**.** For a building set $\mathcal{B}$ on $[n]$, the f- and h-polynomials of the extended nestohedron $\mathcal{P}\ \ (\mathcal{B})$ satisfy the following formulas:

$$f_{\mathcal{P}\ \ (\mathcal{B})}(t) = \sum_{S \subseteq [n]} (t+1)^{n-|S|} f_{\mathcal{P}(\mathcal{B}|_S)}(t),$$
$$h_{\mathcal{P}\ \ (\mathcal{B})}(t) = \sum_{S \subseteq [n]} t^{n-|S|} h_{\mathcal{P}(\mathcal{B}|_S)}(t),$$

where $\mathcal{B}|_S$ is the building set restricted to elements of S, i.e. $\{I \in \mathcal{B} \mid I \subseteq S\}$.

One can compactify the h-vector using another vector called the **γ-vector** $(\gamma_0, \dots, \gamma_{\lfloor d/2 \rfloor})$, which is defined by the relation

$$\sum_{i=0}^{d} h_i t^i = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i (1+t)^{d-2i}.$$

A simplicial complex Δ is called **flag** if every collection of pairwise adjacent vertices forms a face of Δ. We say that a simple polytope is a **flag polytope** if its dual is a flag simplicial complex. Gal conjectured in [Gal05] that the γ-vector is nonnegative for any flag simple polytope.

Theorem 1.4 (Theorem 5.16)**.** If $\mathcal{B}$ is a building set with $\mathcal{P}\ \ (\mathcal{B})$ a flag extended nestohedron, then the γ-vector of $\mathcal{P}\ \ (\mathcal{B})$ is nonnegative.

A connected building set $\mathcal{B}$ is **chordal** if for any element $I = \{i_1 < \cdots < i_r\} \in \mathcal{B}$, all subsets of the form $\{i_s, i_{s+1}, \dots, i_r\}$ also belong to $\mathcal{B}$; see Definition 6.21. All extended nestohedra $\mathcal{P}\ \ (\mathcal{B})$ from chordal building sets are flag simple polytopes by Lemma 6.22, so their γ-vectors are nonnegative.

For a building set $\mathcal{B}$ on $[n]$, define the set of **$\mathcal{B}$-partial permutations**, denoted $\mathfrak{P}_n(\mathcal{B})$, as the set of partial permutations $w \in \mathfrak{S}_S$ for some $S = \{s_1, \dots, s_k\} \subseteq [n]$ such that for any $i \in [k]$, the elements $w(s_i)$ and $\max\{w(s_1), w(s_2), \dots, w(s_i)\}$ lie in the same connected component of the restricted building set $\mathcal{B}|_{\{w(s_1), \dots, w(s_i)\}}$. Define the map $\varphi_n : \mathfrak{P}_n(\mathcal{B}) \to \mathfrak{S}_{n+1}$ as follows. For a permutation $w \in \mathfrak{P}_n(\mathcal{B})$ on entry set S, let $\varphi_n(w)$ be the permutation formed by appending $[n+1] \setminus S$ to the end of w in descending order. Let $\mathfrak{S}_{n+1}(\mathcal{B}) \coloneqq \varphi_n(\mathfrak{P}_n(\mathcal{B}))$ denote the set of **extended $\mathcal{B}$-permutations**. See Definitions 6.15. This set is in bijection with vertices of the extended nestohedron $\mathcal{P}\ \ (\mathcal{B})$.

Let $\mathrm{des}(w) = |\{i \mid w(i) > w(i+1)\}|$ denote the number of descents in a permutation w. Let $\widehat{\mathfrak{S}}_{n+1}$ be the subset of permutations w of size $n+1$ without two consecutive descents and no final descent.

Theorem 1.5 (Theorem 6.11 and Theorem 6.35)**.** Let $\mathcal{B}$ be a connected chordal building set on $[n]$. Then the h-vector of the extended nestohedron $\mathcal{P}\ \ (\mathcal{B})$ is given by

$$\sum_i h_i t^i = \sum_{w \in \mathfrak{S}_{n+1}(\mathcal{B})} t^{\mathsf{des}(w)},$$

and the γ-vector of the extended nestohedron $\mathcal{P}\ \ (\mathcal{B})$ is given by

$$\sum_i \gamma_i t^i = \sum_{w \in \widehat{\mathfrak{S}}_{n+1} \cap \mathfrak{S}_{n+1}(\mathcal{B})} t^{\mathsf{des}(w)}.$$

Let $K_{1,n}$ denote the star graph consisting of a central vertex connected to n vertices. The simple polytope $\mathcal{P}(\mathcal{B}_{K_{1,n}})$ is known as the **stellohedron**. Manneville and Pilaud showed that the extended nested complex $\mathcal{N}\ \ (\mathcal{B}_{K_n})$ is isomorphic to the non-extended nested complex $\mathcal{N}(\mathcal{B}_{K_{1,n}})$, where K_n denotes the complete graph on n vertices and $K_{1,n}$ denotes the star graph on $n+1$ vertices [MP17]. Thus, the facial structure of the extended nestohedron $\mathcal{P}\ \ (\mathcal{B}_{K_n})$ is isomorphic to that of the stellohedron, and we also refer to $\mathcal{P}\ \ (\mathcal{B}_{K_n})$ as the stellohedron. Let $\mathfrak{P}_n$ denote the set of all partial permutations $w \in \mathfrak{S}_S$ for some $S \subseteq [n]$. Such permutations are in bijection with the vertices of the stellohedron. We can use the weak Bruhat order on $\mathfrak{S}_{n+1}$ and the map φ_n defined above to induce a partial order on $\mathfrak{P}_n$ as follows. Let $\pi, \sigma \in \mathfrak{P}_n$ be two partial permutations. Then $\pi \leq \sigma$ in the **partial weak Bruhat order** if and only if $\varphi_n(\pi) \leq \varphi_n(\sigma)$ in the weak Bruhat order on $\mathfrak{S}_{n+1}$ (see Definition 7.1).

Theorem 1.6 (Theorem 7.3)**.** The partial order on $\mathfrak{P}_n$ defined above is a lattice quotient of the weak Bruhat order on $\mathfrak{S}_{n+1}$.

We say that a pure n-dimensional simplicial complex Δ with r facets is **shellable** if its facets can be arranged into a linear ordering $F_1, \ldots, F_r$ such that $\left(\bigcup_{i=1}^{k-1} F_k\right) \cap F_k$ is pure of dimension $n-1$ for all $k = 2, \ldots, r$. Such an ordering is called a **shelling** of Δ.

Theorem 1.7 (Theorem 7.11)**.** Let $\mathcal{B} = \mathcal{B}_{K_n}$, and $\mathcal{N}\ \ (\mathcal{B})$ be the extended nested set complex whose facets F_π are labeled by the partial permutations $\pi \in \mathfrak{P}_n$. If a total ordering $\pi_1 \leq \cdots \leq \pi_m$ is a linear extension of the partial weak Bruhat order on $\mathfrak{P}_n$, then $F_{\pi_1}, \ldots, F_{\pi_m}$ is a shelling for $\mathcal{N}\ \ (\mathcal{B})$.

This paper is structured as follows. Section 2 contains preliminary definitions and results relating to simplicial complexes, polytopes, and building sets. In Section 3, it is shown that the extended nested complex is isomorphic to the boundary of a polytope, and in Section 4, we provide some examples of isomorphisms between extended and non-extended nested complexes. Section 5 provides recursive formulas for the f- and h-polynomials of the extended nestohedron, and we prove Gal's conjecture for all flag extended nestohedra. In Section 6, we discuss extended $\mathcal{B}$-forests and extended $\mathcal{B}$-permutations, which are both in bijection with the vertices of the extended nestohedron $\mathcal{P}\ \ (\mathcal{B})$. We then show that the h- and γ-polynomials for extended nestohedra on a special class of building sets can be written as a combinatorial formula in terms of the extended $\mathcal{B}$-permutations. Section 7 provides a partial ordering on partial permutations, and provides several nice properties of this partial ordering. We conclude with acknowledgements.

2 Background

In this section, we provide some preliminary definitions and results on simplicial complexes, polytope theory, building sets, and (extended) nestohedra.

2.1 Simplicial Complexes

A **simplicial complex** Δ on a finite set S is a collection of subsets of S such that for every $X \in \Delta$ and every $Y \subset X$, we have $Y \in \Delta$ as well. The elements of Δ are called **faces**, and the maximal elements are called **facets**. Note that, by definition, a simplicial complex is uniquely determined by its facets. The **dimension** of a face $F \in \Delta$ is $\dim F = |F| - 1$; the **dimension** of Δ is $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$. A simplicial complex is called **pure** if all of its facets have the same dimension.

For a simplicial complex Δ, the **closure** of a collection $\mathcal{S}$ of its faces is

$$\operatorname{Cl} \mathcal{S} = \{F \in \Delta \mid F \subset \sigma \text{ for some } \sigma \in \mathcal{S}\}.$$

The **(open) star** of a face σ is

$$\operatorname{St}\sigma = \{F \in \Sigma \mid \sigma \subset F\},$$

and the star of a collection $\mathcal{S}$ of faces is

$$\operatorname{St}\mathcal{S} = \bigcup_{\sigma\in\mathcal{S}} \operatorname{St}\sigma.$$

The **link** of $\mathcal{S}$ is then defined as

$$\operatorname{Lk}\mathcal{S} = \operatorname{Cl}\operatorname{St} S - \operatorname{St}\operatorname{Cl} S.$$

If $\mathcal{S}$ consists of just one face σ, then the link of σ in Δ can be described as

$$\operatorname{Lk}_\Delta \sigma = \{F \in \Delta \mid \sigma \cap F = ?, \sigma \cup F \in \Delta\}.$$

A **subcomplex** of a simplicial complex Δ is a subset of Δ that is also a simplicial complex. Note that for any collection of faces $\mathcal{S}$ of Δ, the closure and link of $\mathcal{S}$ are always subcomplexes, but the star of $\mathcal{S}$ need not be. We also call the closure of $\mathcal{S}$ the subcomplex generated by the elements of $\mathcal{S}$.

2.2 Polytope Theory

Given a finite set of points $S = \{x_1, \dots, x_k\}$ in R^n, the **convex hull** of S Conv(S) is defined to be the smallest convex set that contains X, or

$$\operatorname{Conv}(S) := \left\{ \sum_{i=1}^{k} a_i x_i \mid a_i \geq 0 \text{ for all } i = 1, \dots, k \text{ and } \sum_{i=1}^{k} a_i = 1 \right\}.$$

Such a space is called a **convex polytope**. We can also define a convex polytope as a bounded intersection of finitely many **closed half-spaces**, where a closed half-space $\mathcal{H}$ is the collection of points $(x_1, \dots, x_n)$ in R^n satisfying a linear inequality $a_1x_1 + \cdots + a_nx_n \leq b$ for some $a_1, \dots, a_n, b \in \mathsf{R}$. Since we only consider convex polytopes in our paper, we shall simply write polytope instead of convex polytope.

The **vertices** of a polytope P are the minimal set of points S such that $P = \operatorname{Conv}(S)$. The **dimension** of P is then defined to be the dimension of the affine linear span of S, considered as an affine subspace in R^n A **face** of a polytope P is an intersection of P with a closed half-space $\mathcal{H}$ such that none of the interior points of P lie on the boundary of the half-space. Note that each face of a polytope P is itself a polytope. The **boundary** of a polytope P, denoted ∂P, is the union of all proper faces of P. If P is a d-dimensional polytope, then the **vertices**, **edges**, **ridges**, and **facets** of P are the 1, 2, $(d-2)$, and $(d-1)$-dimensional faces respectively.

Given a polytope P, the **dual polytope** P^* can be defined as the set of points y in the dual space $(\mathsf{R}^n)^*$ such that $\langle y, x\rangle > 0$ for all $x \in P$, where $\langle\cdot,\cdot\rangle$ is the usual pairing. We can see that when dualizing, $\dim P^* = d$ if P is d-dimensional, and that the k-dimensional faces of P correspond to the $(d-k)$-dimensional faces of P^* for all $k = 0, \dots, d$. Furthermore, the double dual of a polytope P is isomorphic to P itself.

A polytope is called **simplicial** if all of its faces are simplices. In other words, all of its faces are the convex hull of $d+1$ points if the face is d-dimensional. A polytope is called **simple** if every vertex is adjacent to exactly d edges, where d is the dimension of the polytope. A polytope is simplicial if and only if its dual is simple.

Given a simplicial polytope P, its boundary can be viewed as a simplicial complex by taking the set of vertices for each face F of P. More generally, given a simplicial complex Δ, a **geometric realization** of Δ is a simplicial polytope P and a map $\phi : V(\Delta) \to V(P)$ of the vertex sets of Δ and P such that σ is a face of Δ if and only if $\operatorname{Conv}(\phi(\sigma))$ is a face of P.

If P is a d-dimensional polytope, then the **face number** $f_i(P)$ is the number of i-dimensional faces of P. We call the vector $(f_0(P), \dots, f_d(P))$ the **f-vector** of P, and the polynomial

$$f_P(t) := \sum_{i=0}^{d} f_i(P) t^i$$

the f**-polynomial** of P. Note that if $f = (f_{-1}, f_0, \ldots, f_d)$ is the f-vector of a polytope P with $f_{-1} := 1$, then $f' = (f_d, \ldots, f_0, f_{-1})$ is the f-vector of its dual P^*.

We can more compactly encode the face numbers of P using smaller nonnegative integers. The h**-vector** $(h_0(P), \ldots, h_d(P))$ and h**-polynomial** $h_P(t) := \sum_{i=0}^{d} h_i(P)t^i$ of a simple polytope P are determined uniquely by the relation

$$f_P(t) = h_P(t + 1).$$

From the Dehn-Sommerville relation, we know that the h-vector of a simple polytope is always symmetric; in other words, we have $h_i = h_{d-i}$ for all $i = 0, \ldots, \lfloor d/2 \rfloor$.

Definition 2.1. A simplicial complex Δ is called **flag** if a collection C of vertices of Δ forms a simplex in Δ if and only if there exists an edge in the 1-skeleton of Δ between any two vertices in C.

If Δ_P is a flag simplicial complex, then we say that its dual simplicial complex, P is a **flag polytope**. Definition 2.1 is equivalent to saying that a simple polytope P is **flag** if any collection of pairwise intersecting facets has non-empty intersection.

The γ**-vector** gives another encoding of the f- and h-vectors of a simple poytope P, but with smaller integers. The entries $\gamma_i(P)$ of the γ-vector $(\gamma_0, \gamma_1, \ldots \gamma_{\lfloor d/2 \rfloor})$ and γ-polynomial $\gamma_P(t) := \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i$ are determined by the h-polynomial:

$$h_P(t) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i (1 + t)^{d-2i} = (1 + t)^d \gamma_P \left(\frac{t}{(1 + t)^2} \right).$$

Gal conjectured the following statement about the γ-vector.

Conjecture 2.2 ([Gal05])**.** The γ-vector of any flag simple polytope has nonnegative entries.

In Section 5.2, we prove Gal's Conjecture for all flag extended nestohedra $\mathcal{P}\ (\mathcal{B})$.

2.3 Building Sets, Nested Set Complexes, and Extended Nested Set Complexes

In this subsection, we introduce definitions related to building sets and (extended) nested set complexes.

Definition 2.3. A **building set** $\mathcal{B}$ on a finite set S is a collection of subsets of S satisfying two conditions:

(B1) $\{i\} \in \mathcal{B}$ for all $i \in S$,

(B2) For any $I, J \in \mathcal{B}$ such that $I \cap J \neq$?, $I \cup J \in \mathcal{B}$.

Definition 2.4. Let $\mathcal{B}$ be a building set on S and $I \subseteq S$. The **restriction of** $\mathcal{B}$ **to** I is the building set on I

$$\mathcal{B}|_I := \{J \mid J \subseteq I, J \in \mathcal{B}\}.$$

The **contraction of** $\mathcal{B}$ **by** I is the building set on $S \setminus I$

$$\mathcal{B}/I := \{J \setminus (J \cap I) \mid J \in \mathcal{B}, J * I\}.$$

Definition 2.5. For any building set $\mathcal{B}$, let $\mathcal{B}_{\max}$ denote the set of maximal elements of $\mathcal{B}$ with respect to inclusion. Then for any $M \in \mathcal{B}_{\max}$, the restriction $\mathcal{B}|_M$ is called a **connected component** of B.

If $\mathcal{B}$ is a building set on S and $S \in \mathcal{B}$, then we say that $\mathcal{B}$ is **connected**. Note that the elements of $\mathcal{B}_{\max}$ form a disjoint union of S, with $\mathcal{B}_{\max} = \{S\}$ if $\mathcal{B}$ is connected.

We say that building sets $\mathcal{B}, \mathcal{B}'$ on S are **equivalent** if there exists a permutation $\sigma : S \to S$ giving a one-to-one correspondence $\mathcal{B} \to \mathcal{B}'$.

We can use graphs to define a very large family of building sets.

Definition 2.6. Let Γ be a directed graph without loops and multiple edges on node set S. The **graphical building set** $\mathcal{B}_\Gamma$ is defined to be $\{I \subseteq S \mid \Gamma|_I \text{ is strongly connected}\}$.

Example 2.7. Let Γ be the path graph on $[n]$, denoted P_n. If the graph is labeled from left to right in increasing order, then the building set $\mathcal{B}_\Gamma$ consists of all subsets of $[n]$ that are intervals. For example, if $n = 4$, then

$$\mathcal{B}_\Gamma = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{2,3\}, \{3,4\}, \{1,2,3\}, \{2,3,4\}, \{1,2,3,4\}\}.$$

Example 2.8. Not all building sets are graphical. For example, consider

$$\mathcal{B} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}, \{1,2,3,4\}\}.$$

If $\mathcal{B} = \mathcal{B}_\Gamma$ for some directed graph Γ, then $\{1,2\}, \{2,3\}, \{1,3\} \in \mathcal{B}$ implies that Γ must contain anti-parallel edges between 1 and 2, 2 and 3, and 3 and 1. However, $\{1,2,3,4\} \in \mathcal{B}$ implies that Γ contains an edge from 4 to some $u \in \{1,2,3\}$, and an edge from some $v \in \{1,2,3\}$ to 4. Then we would have $\{4, u, v\} \in \mathcal{B}$, but this is impossible.

Definition 2.9. A **nested collection** N of a building set $\mathcal{B}$ on S is a collection of elements $\{I_1, \ldots, I_m\}$ of $\mathcal{B} \setminus \mathcal{B}_{\max}$, such that:

(N1) For any $i \neq j$, I_i and I_j are pairwise disjoint or nested.

(N2) For any $I_{i_1}, \ldots, I_{i_k}$, pairwise disjoint, $k \geq 2$, their union is not an element of $\mathcal{B}$.

We call a nested collection **maximal** if no other nested collection properly contains it.

Example 2.10. Consider the building set $\mathcal{B}_\Gamma$ from Example 2.7. An example of a nested collection is

$$N = \{\{1\}, \{3\}, \{1,2,3\}\}.$$

It turns out N is also a maximal nested collection for $\mathcal{B}_\Gamma$. An example of something that **is not** a nested collection is

$$N' = \{\{2,3\}, \{3\}, \{4\}\},$$

since $\{3\} \cup \{4\} = \{3,4\}$ is an element of our building set.

Definition 2.11. Let $\mathcal{B}$ be a building set. The **nested complex** $\mathcal{N}(\mathcal{B})$ is defined to be the simplicial complex with vertices $\{I \mid I \in \mathcal{B} \setminus \mathcal{B}_{\max}\}$ and faces $\{I_1, \ldots, I_m\}$ for every nested collection $\{I_1, \ldots, I_m\} \subset \mathcal{B} \setminus \mathcal{B}_{\max}$.

We now extend our definitions for nested collections and nested complexes.

Definition 2.12. Let $\mathcal{B}$ be a building set on S. An **extended nested collection**

$$N = \{I_1, \ldots, I_m, x_{i_1}, \ldots, x_{i_r}\}$$

on $\mathcal{B}$ is a collection of elements of subsets $I_j \in \mathcal{B}$ and x_i for $i \in S$ such that:

1. $\{I_1, \ldots, I_m\} \subset \mathcal{B}$ form a nested collection.
2. $i_k \notin I_j$ for all $1 \leq k \leq r$ and $1 \leq j \leq m$.

The x_i elements are extensions of what [DHV11] call **design tubings** on graphs. We refer to the x_i's as **design vertices**.

We say an extended nested collection is **maximal** if no other extended nested collection properly contains it. Notice that all non-extended nested collections are also extended nested collections as an extended nested collection may not necessarily contain x_i elements.

Example 2.13. Again consider the building set $\mathcal{B}_\Gamma$ from Example 2.7. An example of an extended nested collection is

$$N = \{\{1\}, \{3\}, \{3, 4\}, x_2\}.$$

This is a maximal nested collection for our building set. An example of a collection that **is not** an extended nested collection is

$$N' = \{\{2, 3\}, \{3\}, x_1, x_2\},$$

since the number 2 appears as both an index of an x_i and as an element of one of the subsets in N'.

Definition 2.14. Let $\mathcal{B}$ be a building set on S. The **extended nested complex** $\mathcal{N}\ \ (\mathcal{B})$ is defined to be the simplicial complex with vertices $\{I \mid I \in \mathcal{B}\} \cup \{x_i \mid i \in S\}$ and faces $\{I_1, \ldots, I_m\} \cup \{x_{i_1}, \ldots, x_{i_r}\}$ where $\{I_1, \ldots, I_m, x_{i_1}, \ldots, x_{i_r}\}$ is an extended nested collection.

The extended nested complex is also referred to as the **design nested complex** in [MP17].

Notice that for a building set $\mathcal{B}$ on S, the nested complex $\mathcal{N}(\mathcal{B})$ is isomorphic to the subcomplex of the extended nested complex $\mathcal{N}\ \ (\mathcal{B})$ involving none of the x_i vertices nor the vertices corresponding to elements in $\mathcal{B}_{\max}$.

In [Pos09; FS05; CD06], it was shown that for a building set $\mathcal{B}$, the nested complex $\mathcal{N}(\mathcal{B})$ is isomorphic to the boundary of a simplicial polytope. The polar dual of this polytope is the simple polytope $\mathcal{P}(\mathcal{B})$ called the **nestohedron**. In Section 3, we show that the extended nested complex $\mathcal{N}\ \ )\mathcal{B})$ is also isomorphic to the boundary of a simplicial complex. We call the polar dual of this polytope the **extended nestohedron**, and we denote it by $\mathcal{P}\ \ (\mathcal{B})$.

We now state some basic properties of the nested complex and the extended nested complex. The first observation is that these complexes are pure. For a building set $\mathcal{B}$ on S, Zelevsinsky showed that the nested complex $\mathcal{N}(\mathcal{B})$ is pure of dimension $|S| - |\mathcal{B}_{\max}|$ (see [Zel06, Proposition 4.1]). We state and prove the result for the extended case.

Proposition 2.15. For a building set $\mathcal{B}$ on S, the extended nested complex $\mathcal{N}\ \ (\mathcal{B})$ is pure of dimension $|S|$.

Proof. Note that for an extended nested collection

$$N\ \ = \{I_1, \ldots, I_k\} \cup \{x_{i_1}, \ldots, x_{i_\ell}\}$$

to be a facet of $\mathcal{N}\ \ (\mathcal{B})$, we must have that $I_1 \cup \cdots \cup I_k \cup \{i_1, \ldots, i_\ell\} = S$, and that $N = \{I_1, \ldots, I_k\}$ is a maximal nested collection of $\mathcal{B}|_{S \setminus \{i_1, \ldots, i_\ell\}}$. We then use the result from the non-extended case to conclude that N has $|S| - \ell$ elements, and so $N\ \ $ has $|S|$ elements, completing the proof. □

Next, we prove that $\mathcal{N}(\mathcal{B})$ and $\mathcal{N}\ \ (\mathcal{B})$ only depend on the connected components of $\mathcal{B}$. To make this precise, we first recall the definition of the join of simplicial complexes.

Definition 2.16. For two simplicial complexes X, Y, their **join** $X * Y$ is the simplicial complex such that:

(a) The vertex set is equal to the disjoint union of the vertex sets of X and Y.

(b) The faces are of the form $F_X \sqcup F_Y$ where F_X, F_Y are faces of X and Y respectively.

Lemma 2.17. Let $\mathcal{B}$ be a building set, and let $\mathcal{B}_1, \ldots, \mathcal{B}_k$ be its connected components. We then have

$$\mathcal{N}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}_1) * \cdots * \mathcal{N}(\mathcal{B}_k), \qquad \mathcal{N}\ \ (\mathcal{B}) \simeq \mathcal{N}\ \ (\mathcal{B}_1) * \cdots * \mathcal{N}\ \ (\mathcal{B}_k).$$

Proof. The proof of both parts is based on the observation that a nested collection (resp. extended nested collection) on $\mathcal{B}$ is the same as the disjoint union of nested collections (resp. extended nested collections) on its connected components $\mathcal{B}_1, \ldots, \mathcal{B}_k$. □

Remark 2.18. In [Vol10, p.5 Corollary 5], the author shows that for any building set $\mathcal{B}$ on $[n]$, there exists a connected building set $\mathcal{B}'$ on $[n - |\mathcal{B}_{\max}| + 1]$ such that $\mathcal{N}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$. Thus in most cases, it suffices to consider building sets that are connected for the nested complex.

Remark 2.19. When G is one of the three graphs in Figure 1, then computer checking shows that $\mathcal{N}\ (\mathcal{B}_G)$ is not isomorphic to $\mathcal{N}(\mathcal{B}')$ for any connected building set $\mathcal{B}'$ on 5 elements. Thus, the class of extended nested complexes is not contained within the class of nested complexes.

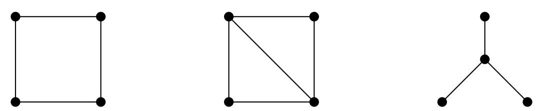


Figure 1: Graphs G for which $\mathcal{N}\ (\mathcal{B}_G)$ is not a nested complex

Next, we characterize flag extended nested complexes in terms of when a non-extended nested complex is flag.

Proposition 2.20. For any building set $\mathcal{B}$, $\mathcal{N}(\mathcal{B})$ is flag if and only if $\mathcal{N}\ (\mathcal{B})$ is flag.

Proof. The vertex set of $\mathcal{N}(\mathcal{B})$ is a subset of the vertex set of $\mathcal{N}\ (\mathcal{B})$, so we can refer to collections of elements of $\mathcal{B} \setminus \{[n]\}$ as a potential face in either one. Note that a collection of elements of $\mathcal{B} \setminus \{[n]\}$ is a face in $\mathcal{N}(\mathcal{B})$ if and only if it is a face in $\mathcal{N}\ (\mathcal{B})$. Recall that a simplicial complex is flag if and only if these minimal non-faces all only use two vertices. We will look at the minimal non-faces of these two simplicial complexes.

Suppose $\mathcal{N}\ (\mathcal{B})$ is flag. Notice that any minimal non-face F in $\mathcal{N}(\mathcal{B})$ is also a non-face in $\mathcal{N}\ (\mathcal{B})$. In particular, it is also minimal because subsets of the vertices of F in $\mathcal{N}\ (\mathcal{B})$ are still only a face if they are a face in $\mathcal{N}(\mathcal{B})$. Therefore we must have $|F| = 2$, and so $\mathcal{N}(\mathcal{B})$ is flag.

Now suppose that $\mathcal{N}(\mathcal{B})$ is flag. If F is a non-face in $\mathcal{N}\ (\mathcal{B})$, then F can be of one of the following forms:

1. F consists only of vertices from $\mathcal{B} \setminus \{[n]\}$,
2. F contains some vertex x_i, or
3. F contains the vertex $[n]$.

If we are in the first case, then by the same argument above, any minimal non-face of $\mathcal{N}\ (\mathcal{B})$ which uses only vertices which are also in $\mathcal{N}(\mathcal{B})$ must be of size 2.

Now suppose we are in the second case. If $F \setminus \{x_i\}$ is not a face, then F is not a minimal non-face. If $F \setminus \{x_i\}$ is a face, then x_i must be stopping F from being a face somehow. Therefore there exists $I \in F$ such that $i \in I$. But then $\{I, x_i\} \subseteq F$ is a non-face, so either F is not minimal, or it has size 2.

The last case, when $[n]$ is the vertex of F which doesn't appear in $\mathcal{N}(\mathcal{B})$, is analogous to the second case. □

Remark 2.21. [PRW08, Propostion 7.1] characterizes when the building set $\mathcal{B}$ has $\mathcal{N}(\mathcal{B})$ flag; in particular, for any graphical building set $\mathcal{B}_\Gamma$, the nested complex $\mathcal{N}(\mathcal{B}_\Gamma)$ is flag. Thus, we have that all of the equivalent characterizations given by [PRW08] are also equivalent to the extended nested complex $\mathcal{N}\ (\mathcal{B})$ being flag, and the complex $\mathcal{N}\ (\mathcal{B}_\Gamma)$ is flag, where $\mathcal{B}_\Gamma$ is a graphical building set.

2.4 Link Decomposition

We now provide a characterization of the link of a vertex v in the extended nested complex $\mathcal{N}\ (\mathcal{B})$. This allows us to "build up" the complex in terms of smaller complexes.

Definition 2.22. Let $\mathcal{B}$ be a building set on S, and $\mathcal{N}(\mathcal{B})$ the associated nested complex. For every $C \in \mathcal{B} \setminus \mathcal{B}_{\max}$, the **link** of C in $\mathcal{N}(\mathcal{B})$ is

$$\mathcal{N}(\mathcal{B})_C = \{N \in \mathcal{N}(\mathcal{B}) \mid N \cap \{C\} = ?, N \cup \{C\} \in \mathcal{N}(\mathcal{B})\}.$$

For any $C \in \{I \mid I \in \mathcal{B}\} \cup \{x_i \mid i \in S\}$, the **link** of C in the extended nested complex $\mathcal{N}\ \ (\mathcal{B})$ is

$$\mathcal{N}\ \ (\mathcal{B})_C = \{N\ \ \in \mathcal{N}\ \ (\mathcal{B}) \mid N\ \ \cap \{C\} = ?, N\ \ \cup \{C\} \in \mathcal{N}\ \ (\mathcal{B})\}.$$

Zelevinsky first found a formula for the link decomposition for the nested complex $\mathcal{N}(\mathcal{B})$ in [Zel06], and Aisbett provides an alternative proof in [Ais12, Lemma 3.2].

Proposition 2.23 ([Zel06], Proposition 3.2)**.** Let $\mathcal{B}$ be a building set on S. Then the link of $C \in \mathcal{B}$ in $\mathcal{N}(\mathcal{B})$ is isomorphic to $\mathcal{N}(\mathcal{B}|_C) * \mathcal{N}(\mathcal{B}/C)$.

We now state the analogous link decompositions for extended nested complexes. Since the extended nested complexes have two kinds of vertices, those labeled by elements of the building set and those labeled by design vertices, we have two different formulas for a link decomposition. Note that these formulas have appeared without proof in [MP17, Lemma 84] for graphical building sets.

Proposition 2.24. Let $\mathcal{B}$ be a building set on S, and let v be a vertex of the extended nested complex $\mathcal{N}\ \ (\mathcal{B})$ corresponding to the design vertex x_i. Then, the link of v in $\mathcal{N}\ \ (\mathcal{B})$ is given by

$$\mathcal{N}\ \ (\mathcal{B})_v \simeq \mathcal{N}\ \ (\mathcal{B}_1) * \cdots \mathcal{N}\ \ (\mathcal{B}_k),$$

where $\mathcal{B}_1, \ldots, \mathcal{B}_k$ are the connected components of $\mathcal{B}|_{S\setminus\{i\}}$.

Proposition 2.25. Let $\mathcal{B}$ be a building set on S, and let v be a vertex of the extended nested complex $\mathcal{N}\ \ (\mathcal{B})$ corresponding to an element of the building set $C \in \mathcal{B}$. Then, the link of v in $\mathcal{N}\ \ (\mathcal{B})$ is given by

$$\mathcal{N}\ \ (\mathcal{B})_v \simeq \mathcal{N}(\mathcal{B}|_C) * \mathcal{N}\ \ (\mathcal{B}/C).$$

Proof of Proposition 2.24. By definition, we have:

$$\begin{aligned}
\mathcal{N}\ \ (\mathcal{B})_{x_i} &= \{\{x_{j_1}, \ldots, x_{j_\ell}\} \cup \{I_1, \ldots, I_k\} \in \mathcal{N}\ \ (\mathcal{B}) \mid \{x_i, x_{j_1}, \ldots, x_{j_\ell}\} \cup \{I_1, \ldots, I_k\} \in \mathcal{N}\ \ (\mathcal{B})\} \\
&= \{\{x_{j_1}, \ldots, x_{j_\ell}\} \cup \{I_1, \ldots, I_k\} \in \mathcal{N}\ \ (\mathcal{B}) \mid \cup_{s=1}^k I_s \cup \{j_1, \ldots, j_\ell\} \subseteq S \setminus \{i\}\} \\
&= \left\{\{x_{j_1}, \ldots, x_{j_\ell}\} \cup \{I_1, \ldots, I_k\} \in \mathcal{N}\ \ \left(\mathcal{B}|_{S\setminus\{i\}}\right)\right\} \\
&= \mathcal{N}\ \ \left(\mathcal{B}|_{S\setminus\{i\}}\right) \\
&\simeq \mathcal{N}\ \ (\mathcal{B}_1) * \cdots * \mathcal{N}\ \ (\mathcal{B}_k),
\end{aligned}$$

where the last isomorphism follows from Lemma 2.17. □

Proof of Proposition 2.25. For a vertex v corresponding to the building set element $C \in \mathcal{B}$, the link of v in $\mathcal{N}\ \ (\mathcal{B})$ corresponds to all extended nested collections containing C. Thus, we will show that the complex of extended nested collections of $\mathcal{B}$ that contain C is isomorphic to $\mathcal{N}(\mathcal{B}|_C) * \mathcal{N}\ \ (\mathcal{B}/C)$. Let one direction of the isomorphism be given by the map

$$(N_1, N_2) \in \mathcal{N}(\mathcal{B}|_C) * \mathcal{N}\ \ (\mathcal{B}/C) \mapsto N_1 \cup N_2' \cup \{C\},$$

where

$$N_2' := \{I \mid I \in N_2 \text{ and } I \cup C \notin \mathcal{B}\} \cup \{I \cup C \mid I \in N_2 \text{ and } I \cup C \in \mathcal{B}\} \cup \{x_i \mid x_i \in N_2\}.$$

If N is an extended nested set of $\mathcal{B}$ containing C, then the inverse of the above map is given by

$$N \mapsto N_1 \cup N_2,$$

where

$$N_1 := \{I \in N \mid I \ (\ C\}, \qquad N_2 := \{I \setminus (I \cap C) \mid I \in N, I \ * \ C\} \cup \{x_i \mid x_i \in N\}.$$

Notice that $N_1 \in \mathcal{N}(\mathcal{B}|_C)$ and $N_2 \in \mathcal{N}\ \ (\mathcal{B}/C)$.

Both of these maps preserve inclusion. Thus we have an isomorphism. □

3 Polytopality

In this section, we provide two proofs of the fact that $\mathcal{N}(\mathcal{B})$ can be realized as the boundary of a polytope; this is equivalent to showing that its dual $\mathcal{P}(\mathcal{B})$ can also be realized as the boundary of a polytope. Our first proof is based on stellar subdivisions of a cross polytope, and our second proof is based on Minkowski sums.

Definition 3.1. Let Δ be a simplicial complex and F be a face of Δ. The **stellar subdivision** on the face F of Δ is defined to be

$$\Delta' = (\Delta \setminus \operatorname{Cl}\operatorname{St}_\Delta(F)) \sqcup (\{v\} * \partial F * \operatorname{Lk}_\Delta(F)).$$

In other words, we remove the subcomplex generated by all facets of Δ containing F, then add in the subcomplex generated by all facets of the form $\{v\} \cup (F \setminus \{u\}) \cup G$ where v is a new vertex, u is an element of F and G is a facet of $\operatorname{Lk}_\Delta(F)$.

Remark 3.2. If Δ has a geometric realization P, and Δ' is the result of stellar subdivision on a face F of Δ, then Δ' can be realized as $\operatorname{Conv}(P \cup \{v\})$ for any point v that lies beyond the facet hyperplanes for facets that contain F and beneath any other facet hyperplanes of P. For a precise definition of "beyond" and "beneath", see [ES74].

We now provide a geometric realization of $\mathcal{N}(\mathcal{B})$ for any building set $\mathcal{B}$.

Theorem 3.3. Let $\mathcal{B}$ be a building set on $[n]$. Then $\mathcal{N}(\mathcal{B})$ can be realized as the boundary of a polytope $\mathcal{N}_\mathcal{B}$ in the following way:

(i) Consider R^n with standard basis vectors $e_1, \ldots, e_n$. Start with the cross polytope in R^n with vertices e_i labeled $\{i\} \in \mathcal{B}$ and vertices $-e_i$ labeled x_i for all $i \in [n]$.

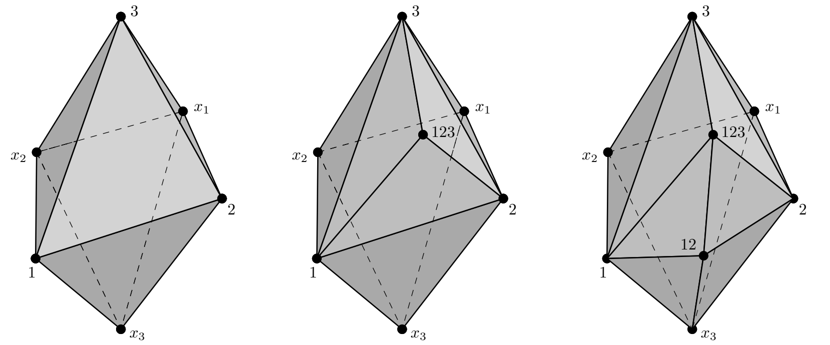
(ii) Order the non-singletons of $\mathcal{B}$ by decreasing cardinality, then for each $I \in \mathcal{B}$ a non-singleton, perform stellar subdivision on the face $\mathcal{I} = \{\{i\} \mid i \in I\}$, with the new added vertex labeled I.

(iii) The boundary of the resulting polytope $P_\mathcal{B}$ will be isomorphic to $\mathcal{N}(\mathcal{B})$.

Before providing the proof, we give an example of the process of obtaining the polytope $P_\mathcal{B}$.

Example 3.4. Consider the building set $\mathcal{B} = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,2,3\}\}$. We begin with a three-dimensional cross polytope P_0 with vertices labeled by singletons $1, 2, 3$ as well as x_1, x_2, and x_3. The polytope P_0 is illustrated in Figure 2(a).

The remaining non-singleton elements of the building set are $\{1,2\}$ and $\{1,2,3\}$. Since $\{1,2,3\}$ is larger in terms of cardinality, we start by adding a vertex corresponding to this element to the polytope. To do this, we stellarly subdivide the face $\{\{1\}, \{2\}, \{3\}\}$ of P_0, obtaining a new polytope P_1. This is shown in Figure 2(b). Next, we add a vertex corresponding to the element $\{1,2\}$ by stellarly subdividing the face $\{\{1\}, \{2\}\}$ of P_1, obtaining the final polytope $P_\mathcal{B}$, shown in Figure 2(c). The boundary of this polytope is isomorphic to $\mathcal{N}(\mathcal{B})$.



(a) P_0 (b) P_1 (c) $P_{\mathcal{B}}$

Figure 2: Process of stellar subdivision of a cross polytope to obtain $P_{\mathcal{B}}$.

Proof of Theorem 3.3. First, note that our construction is always well-defined. When we stellarly subdivide any face F of a polytope P, the faces that are removed are precisely the faces containing F, which means we do not remove any face with lower dimension than that of F, or any different face of the same dimension. This implies that the faces of the form $\mathcal{I} = \{\{i\} \mid i \in I\}$ remain faces of $P_{\mathcal{B}}$ after stellarly subdividing every face of dimension at least $\dim \mathcal{I}$, hence step (ii) of our construction is always doable.

Our strategy is to show that the facets of $P_{\mathcal{B}}$ are in bijection with the facets of $\mathcal{N}\ (\mathcal{B})$, where the bijection is the one sending x_i to the vertex x_i of $P_{\mathcal{B}}$, and $I \in \mathcal{B}$ to the vertex I of $P_{\mathcal{B}}$. Such a vertex always exists because either $I = \{i\}$ is a singleton, hence is in the original cross polytope, or I is not a singleton and is obtained after stellar subdivision on the face $\mathcal{I}$. With this identification of vertices, we can say that the facets of $P_{\mathcal{B}}$ are the same as those of $\mathcal{N}\ (\mathcal{B})$.

We prove the claim for any building set $\mathcal{B}$ on $[n]$ by downward induction on $\min\{|I| \mid I \in \mathcal{B}, |I| > 1\}$. If $\mathcal{B}$ has no non-singletons, then define this minimum to be ∞. This is also the base case of our induction, where we have $\mathcal{B} = \{\{1\}, \dots, \{n\}\}$. Here, $P_{\mathcal{B}}$ is the cross polytope, and its facets are

$$\{\{j_1\}, \dots, \{j_s\}, x_{i_1}, \dots, x_{i_r}\},$$

for any $J = \{j_1, \dots, j_s\}$ and $I = \{i_1, \dots, i_r\}$ that satisfies $I \cap J =$? and $I \cup J = [n]$. Notice that these are also the facets of $\mathcal{N}\ (\mathcal{B})$ for $\mathcal{B} = \{\{1\}, \dots, \{n\}\}$, so our claim is proved for this case.

Now, assume that our claim is true for some building set $\mathcal{B}$ on $[n]$, whose non-singletons are all of order at least m. It then suffices to show that if we add to $\mathcal{B}$ a new subset $I \subsetneq [n]$ of size at most m, then the facets of $\mathcal{N}\ (\mathcal{B} \cup \{I\})$ are the same as the facets of the polytope obtained by stellarly subdividing $P_{\mathcal{B}}$ on the face $\mathcal{I}$; denote this polytope by $P'_{\mathcal{B}}$. By definition, we obtain the facets of $P'_{\mathcal{B}}$ by removing the facets $\mathcal{F}$ containing $\mathcal{I}$ of $P_{\mathcal{B}}$, and adding in facets of the form $\{I\} \cup (\mathcal{I} \setminus \{i\}) \cup \mathcal{G}$, where $i \in I$ and $\mathcal{G}$ is a facet of $\mathrm{Lk}_{P_{\mathcal{B}}}(\mathcal{I})$. We now show that the same addition and removal of facets get us from $\mathcal{N}\ (\mathcal{B})$ to $\mathcal{N}\ (\mathcal{B} \cup \{I\})$.

First, consider the facets of $\mathcal{N}\ (\mathcal{B})$ that no longer remain facets of $\mathcal{N}\ (\mathcal{B} \cup \{I\})$. If

$$N\ := \{I_1, \dots, I_k\} \cup \{x_{j_1}, \dots, x_{j_\ell}\}$$

is such a facet, then N not being a facet of $\mathcal{N}\ (\mathcal{B} \cup \{I\})$ means that it is no longer an extended nested collection. The only condition that could prevent N from being an extended nested collection is the condition that there does not exist $I_{i_1}, \dots, I_{i_r} \in \{I_1, \dots, I_k\}$ pairwise disjoint such that $I_{i_1} \cup \dots \cup I_{i_r} = I$. This implies that there does exist such a collection whose union is I. Since the size of I is the smallest among the non-singletons of $\mathcal{B}$, each I_{i_t} must be a singleton for all $t = i_1, \dots, i_r$, hence $\{I_{i_1}, \dots, I_{i_r}\} = \mathcal{I}$. In other words, N is a facet of $\mathcal{N}\ (\mathcal{B})$ that contains $\mathcal{I}$. Conversely, any facet of $\mathcal{N}\ (\mathcal{B})$ that contains $\mathcal{I}$ will fail to be a facet of $\mathcal{N}\ (\mathcal{B} \cup \{I\})$ for the same reason. Thus, the facets removed by stellarly subdividing $P_{\mathcal{B}}$ are indeed the maximal extended nested collections of $\mathcal{B}$ that fail to remain maximal extended nested collections of $\mathcal{B} \cup \{I\}$.

Now consider any facet N of $\mathcal{N}(\mathcal{B} \cup \{I\})$ that is not a facet of $\mathcal{N}(\mathcal{B})$. This can only happen if $I \in N$. Furthermore, by Proposition 2.25, we have

$$\mathcal{N}(\mathcal{B} \cup \{I\})_I \simeq \mathcal{N}(\mathcal{B}|_I) * \mathcal{N}(\mathcal{B}/I).$$

Since $N \setminus \{I\}$ is a facet of $\mathcal{N}(\mathcal{B} \cup \{I\})_I$, it corresponds to the join of a facet of $\mathcal{N}(\mathcal{B}|_I)$ with a facet of $\mathcal{N}(\mathcal{B}/I)$. Since $\mathcal{B}|_I = \mathcal{I} \cup \{I\}$, its facets are of the form $\mathcal{I} \setminus \{i\}$ for some $i \in I$, and so $N \setminus \{I\}$ must contain $\mathcal{I} \setminus \{i\}$ for some $i \in I$. We now write $N = \{I\} \cup (\mathcal{I} \setminus \{i\}) \cup N_1$, where N_1 is an extended nested collection for $\mathcal{B}$; in fact, we have that N_1 is in $\mathrm{Lk}_{\mathcal{N}(\mathcal{B})}(\mathcal{I})$. Thus, this is one of the facets that are added in the stellar subdivision on the face $\mathcal{I}$ of $P_{\mathcal{B}}$. Conversely, any facet of the form $\{I\} \cup (\mathcal{I} \setminus \{i\}) \cup \mathcal{G}$ for some $i \in I$ and $\mathcal{G}$ a facet of $\mathrm{Lk}_{P_{\mathcal{B}}}(\mathcal{I})$ is a facet of $\mathcal{N}(\mathcal{B} \cup \{I\})$, but is not a facet of $\mathcal{N}(\mathcal{B})$ since $I \notin \mathcal{B}$. Therefore, the facets added by stellarly subdividing $P_{\mathcal{B}}$ are the maximal extended nested collections added when we go from $\mathcal{B}$ to $\mathcal{B} \cup \{I\}$. This proves the induction hypothesis. □

Remark 3.5. In [DHV11], the authors provide a polytopal realization for the extended nestohedron $\mathcal{P}(\mathcal{B})$ when $\mathcal{B} = \mathcal{B}_G$ is the building set of an undirected graph. Their argument works the same for general building set, and could be seen as the dual to our stellar subdivision approach. In particular, $\mathcal{P}(\mathcal{B})$ is obtained as the boundary of the following polytope:

(i) Take the n-dimensional cube $\mathcal{C}^n$, whose opposite facets are labeled by $\{i\}$ and x_i for every $i \in [n]$,

(ii) For each $I \in \mathcal{B}$ (ordered by decreasing cardinality), we shave face I, i.e., the face corresponding to the intersection of all facets $\{i \mid i \in I\}$.

Here, a shaving of a face F corresponding to a polytope $\mathcal{P}$ is defined as follows: consider any closed half-space $\mathcal{H}$ that intersect $\mathcal{P}$ at exactly F. Then the shaving of F corresponds is the intersection of $\mathcal{P}$ with a closed half-space $\mathcal{H}_\epsilon$ parallel to $\mathcal{H}$, and is moved a small amount ϵ toward the polytope.

We obtain a different polytopal realization of the extended nestohedron $\mathcal{P}(\mathcal{B})$, as the following Minkowski sum.

Theorem 3.6. For a building set on $[n]$, the extended nestohedron $\mathcal{P}(\mathcal{B})$ is isomorphic to the boundary of the polytope:

$$\mathcal{P} := \sum_{i \in [n]} \mathrm{Conv}(0, e_i) + \sum_{I \in \mathcal{B}} \mathrm{Conv}(\{e_S | S (I\}),$$

where $e_1, \ldots, e_n$ are the standard basis vectors of R^n, and $e_S = \sum_{i \in S} e_i$ for all $S \subseteq [n]$.

The intuition for the Minkowski sum is that we start with the cube $[0,1]^n$, and then each added term $\mathrm{Conv}(\{e_S \mid S (I\})$ corresponds to shaving face I of the cube. We again provide an example before providing the proof.

Example 3.7. Consider the building set $\mathcal{B} = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,2,3\}\}$, which is the same building set as in Example 3.4. Then the desired polytope $\mathcal{P}$ will be the Minkowski sum

$$\mathrm{Conv}(0, e_1) + \mathrm{Conv}(0, e_2) + \mathrm{Conv}(0, e_3) + \mathrm{Conv}(e_1, e_2) + \mathrm{Conv}(e_1, e_2, e_3, e_1 + e_2, e_1 + e_3, e_2 + e_3),$$

as illustrated in Figure 3.

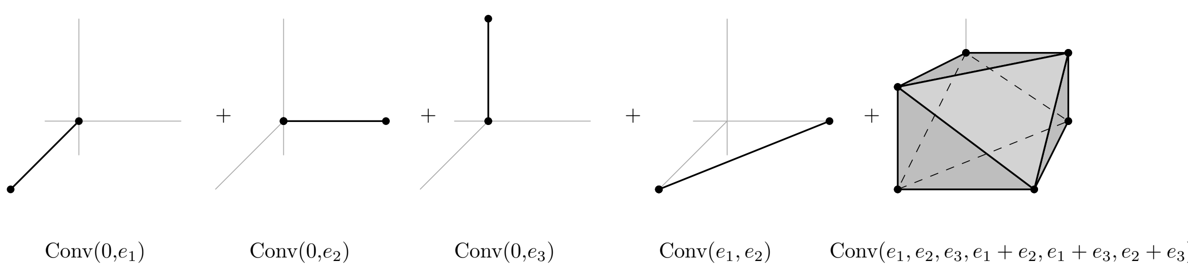


Figure 3: Decomposition of $\mathcal{P}$ into Minkowski sum.

The resulting polytope $\mathcal{P}$ is shown in Figure 4 (not drawn to scale). Labelling the vertices by the maximal extended nested collections of $\mathcal{B}$, we see that $\mathcal{P}$ is indeed the dual of the polytope constructed in Example 3.4.

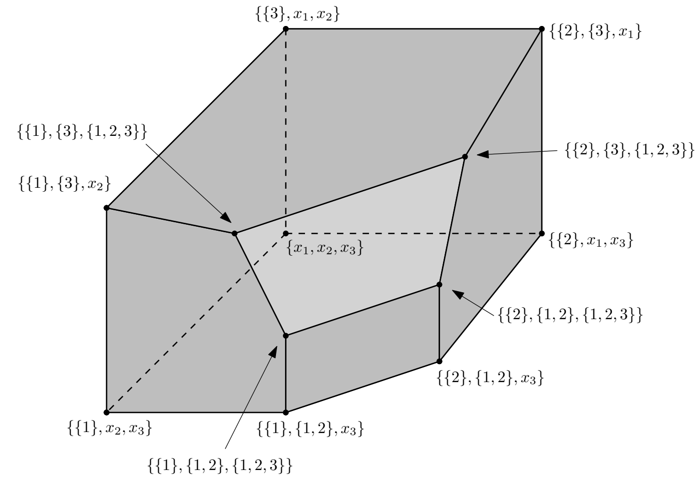


Figure 4: Polytope $\mathcal{P} = \mathcal{P}\ \ (\mathcal{B})$.

Proof of Theorem 3.6. As in [Pos09], each face of a polytope can be identified by the set of linear equations f which are maximized on that face. The face which maximizes an equation f in a Minkowski sum is the Minkowski sum of faces which are maximized at f. Our proof will involve two steps: (1) We form a map from extended nested collections of $\mathcal{B}$ to faces of $\mathcal{P}$ which is inclusion-reversing with facets corresponding to vertices, and (2) we show every linear equation is maximized exactly at a face corresponding (via our map from (1)) to an extended nested collection.

(1) View an extended nested collection $\widetilde{\sigma}$ for $\mathcal{B}$ as a nested collection $N_{\widetilde{\sigma}}$ of $\mathcal{B}$ and a set of x_i's. Define A_I for $I \in N_{\widetilde{\sigma}}$ as $A_I := \{i \in I \mid \text{ for any } J \in N_{\widetilde{\sigma}}, J \ (\ I, i \notin J\}$. Define $A_{-1} := \{i \in [n] \mid x_i \in \widetilde{\sigma}\}$ and $A_0 := [n] \setminus (A_{-1} \cup \bigcup_{I \in \mathcal{B}} A_I)$. A linear equation

$$f(\vec{t}) = a_1 t_1 + a_2 t_2 + \cdots + a_n t_n,$$

such that

- If $i \in A_{-1}$, then $a_i \leq 0$;
- If $i \in A_0$, then $a_i = 0$;
- If $i, j \in A_I$ for $I \in N_{\widetilde{\sigma}}$, then $a_i = a_j \geq 0$;
- If $i \in A_I, j \in A_J$ for $I, J \in N_{\widetilde{\sigma}}$ and $I\ (\ J$, then $a_i \geq a_j$;

will be maximized on the same face Q_I of $\text{Conv}(\{x_S \mid S\ (\ I\})$ for $I \in \mathcal{B}$ and the same face F of $\sum_{i \in [n]} \text{Conv}(0, e_i)$. Namely,

$$F = \sum_{i \notin A_{-1} \cup A_0} e_i + \sum_{i \in C_0} \text{Conv}(0, e_i),$$

for $I \in \mathcal{B}$ such that $I \cap A_{-1} \neq ?$, we have that $Q_I = x_{I \setminus (A_{-1} \cap I)}$. For $I \in \mathcal{B}$ such that $I \cap A_{-1} = ?$ and $I \cap A_0 \neq ?$,

$$Q_I = \text{Conv}(x_S \mid S = I \setminus \{\text{non-zero subsets of } (A_0 \cap I)\}).$$

For $J \in \mathcal{B}$ such that $J \cap (A_{-1} \cup A_0) = ?$, let I be the smallest element of $N_{\widetilde{\sigma}}$ containing I. Then

$$Q_J = \text{Conv}(x_S \mid S = I \setminus \{\text{an element of } A_I\}).$$

Thus f is maximized on the face $F + \sum_{I \in \mathcal{B}} Q_I$, which we denote $Q_{\widetilde{\sigma}}$. Notice that if $\widetilde{\sigma} \neq \widetilde{\pi}$ are distinct extended nested collections for $\mathcal{B}$, then $Q_{\widetilde{\sigma}} \neq Q_{\widetilde{\pi}}$. To show that the map $\widetilde{\sigma} \mapsto Q_{\widetilde{\sigma}}$ is inclusion-reversing, first notice that adding an x_j to $\widetilde{\sigma}$ corresponds to expanding the set of f which are maximized on $Q_{\widetilde{\sigma}}$ to include $a_j \leq 0$ rather than just $a_j = 0$. Next, denote the nested collection formed from adding an element I^* to $\widetilde{\sigma}$ by $\widetilde{\sigma}^*$. Then letting A_I^* be the A_Is for the nested collection $\widetilde{\sigma}^*$, we see that for a parent $I \in \widetilde{\sigma}$ of I^*, $A_I^* = A_I \cap ([n] \setminus I^*)$, and for any other $I \in \widetilde{\sigma}$, $A_I^* = A_I$. It follows that if I^* has a parent in $\widetilde{\sigma}$, then the set of f which are maximized on $Q_{\widetilde{\sigma}}^*$ consists of expanding the set of f maximized on $Q_{\widetilde{\sigma}}$ to include $a_i \geq a_j$ for $i \in A(I^*)^*$ and j in the parent of I^* rather than just $a_i = a_j$ for $i \in A(I^*)^*$ and j in the parent of I^*. If I^* does not have a parent in $\widetilde{\sigma}'$, then the set of f which are maximized on $Q_{\widetilde{\sigma}}^*$ consists of expanding the set of f maximized on $Q_{\widetilde{\sigma}}$ to include $0 \leq a_i \geq a_j$ for $i \in A(I^*)^*$ and $j \in J$ for some $J \in N_{\widetilde{\sigma}}$ rather than just $a_i = 0$ for $i \in A(I^*)^*$.

(2) We will find the face F_I of $\mathrm{Conv}(0, \{x_S | S \ (\ I\})$ and the face F of $\sum_{i \in [n]} \mathrm{Conv}(0, e_i)$ which maximizes an arbitrary linear equation of the form

$$f(\mathbf{t}) = a_1 t_1 + a_2 t_2 + \cdots + a_n t_n.$$

First, divide the a_i into sets $C_{-1}, C_0, C_1, \ldots, C_r$ as follows:

- $C_{-1} = \{a_i \mid a_i < 0\}$;
- $C_0 = \{a_i \mid a_i = 0\}$;
- If $i, j \in C_k$, then $a_i = a_j$;
- If $k < \ell$ with $i \in C_k$ and $j \in C_\ell$, then $a_i < a_j$;
- C_r is nonempty for every $r \geq 0$.

There is a unique set of subsets of $[n]$, $C_{-1}, C_0, \ldots, C_r$ which satisfy these conditions (C_r will be the set of largest a_is, and so on). The face F is

$$\sum_{i \notin C_{-1} \cup C_0} e_i + \sum_{i \in C_0} \mathrm{Conv}(0, e_i).$$

For an element of the building set I with $I \cap C_{-1} \neq$?, then $F_I = x_{I \setminus (C_{-1} \cap I)}$. For an element of the building set I with $I \cap C_{-1} =$? and $I \cap C_0 \neq$?,

$$F_I = \mathrm{Conv}(x_S \mid S = I \setminus \{\text{non-zero subsets of } (C_0 \cap I)\}).$$

For $J \in \mathcal{B}$ such that $J \cap (C_{-1} \cup C_0) =$?, let a be the smallest integer such that $J \cap C_a \neq$?, then

$$F_J = \mathrm{Conv}(x_S \mid S = I \setminus \{\text{an element of } C_a\}).$$

Thus f is maximized on the face $F + \sum_{I \in \mathcal{B}} F_I$, which we denote $F_{\widetilde{\sigma}}$. We construct the extended nested set $\widetilde{\sigma}$ which corresponds to this face as follows:

a) If $i \in C_{-1}$, then $x_i \in \widetilde{\sigma}$.

b) Add in maximal elements (which are unique) of $\mathcal{B}|_{[n] \setminus (C_0 \cup C_{-1})}$ to $\widetilde{\sigma}$. These are the maximal elements of $\widetilde{\sigma}$.

c) Recursively, when we add an element I to $(\widetilde{\sigma})$, we do the following: Partition I's elements into sets $I \cap C_1$, $I \cap C_2$, and so on. Let $I \cap C_a$ be the first nonempty subset in this partition. Add in the maximal element under I in $\mathcal{B}|_{I \setminus (I \cap C_a)}$.

Conditions a) and b) show that f is maximized on the same face of $\sum_{i\in[n]} \mathrm{Conv}(0, e_i)$ as the face $Q_{\widetilde{\sigma}}$ in the Minkowski sum. Let $Q_{\widetilde{\sigma}}|_I$ refer to the face of $\mathrm{Conv}(\{x_S|S \ (\ I\}$ in the Minkowski sum of $Q_{\widetilde{\sigma}}$. We will show $Q_{\widetilde{\sigma}}|_I = Q_I$ for all $I \in B$ using the descriptions of Q_I above.

For any $I \in \mathcal{B}$ with $I \cap C_{-1} \neq$?, $Q_{\widetilde{\sigma}}|_I = Q_I$ since condition a) implies $A_{-1} = C_{-1}$. For any $I \in \mathcal{B}$, with $I \cap C_{-1} =$? and $I \cap C_0 \neq$?, conditions b) and c) implies $Q_{\widetilde{\sigma}}|_I = Q_I$ since they imply $A_0 = C_0$. For any other $J \in \mathcal{B}$, let I be the smallest element of $\widetilde{\sigma}$ which contains J. Then, condition c) shows that

$$\begin{aligned} Q_{\widetilde{\sigma}}|_J &= \mathrm{Conv}(\{x_S \mid S = J \setminus i \text{ for } i \in A(I)\}) \\ &= \mathrm{Conv}(\{x_S \mid S = J \setminus i \text{ for } i \in C_a \text{ for minimal } a \text{ such that } C_a \cap I \neq \text{?}\}) = Q|_J, \end{aligned}$$

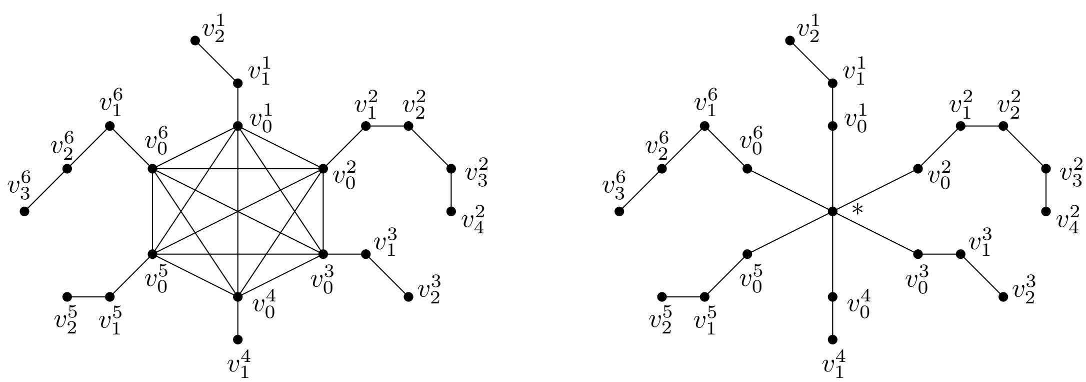
with the last equality coming from the fact that $J \cap C_a \neq$?; otherwise, there would be a smaller element of $\widetilde{\sigma}$ which contains J by condition c). It follows that f is maximized exactly on the face $Q_{\widetilde{\sigma}}$ □

4 Isomorphisms

In this section, we provide some examples of when $\mathcal{N}\ \ (\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$. First, we define some graphs whose building sets give such isomorphisms.

Definition 4.1. Let $\underline{n} := \{n_1, \ldots, n_\ell\} \in \mathsf{N}^\ell$.

- The **spider graph** $\mathfrak{X}_{\underline{n}}$ is a complete graph on vertices $\{v_0^i\}_{i\in[\ell]}$ (the body of the spider) and ℓ legs $[v_1^i, v_{n_i}^i]$ attached to vertex v_0^i.
- The **octopus graph** $\mathcal{X}_{\underline{n}}$ consists of a single vertex labeled $*$ (the head of the spider) and ℓ legs $[v_0^i, v_{n_i}^i]$ attached.



(a) $\mathfrak{X}_{\underline{n}}$ (b) $\mathcal{X}_{\underline{n}}$

Figure 5: Spider and octopus graphs for $\underline{n} = \{2, 4, 2, 1, 2, 3\}$.

Manneville and Pilaud found that, for graphical building sets, the only isomorphism between extended and non-extended nested set complexes is between spider and octopus graphs.

Theorem 4.2 ([MP17], Proposition 64)**.** Let G and G' be two connected graphs such that $\mathcal{N}\ \ (\mathcal{B}_G) \simeq \mathcal{N}(\mathcal{B}_{G'})$. Then G is a spider $\mathfrak{X}_{\underline{n}}$ and G' is an octopus $\mathcal{X}_{\underline{n}}$.

Notice that if $\underline{n} = \{0, \ldots, 0\} \in \mathsf{Z}_{\geq 0}^\ell$, then $\mathfrak{X}_{\underline{n}}$ is K_ℓ, the complete graph on ℓ vertices, and $\mathcal{X}_{\underline{n}}$ is the star graph on $\ell + 1$ vertices $K_{1,\ell}$. We therefore have the following corollary of Theorem 4.2.

Corollary 4.3 ([MP17], Example 62)**.** For all $n \in \mathsf{N}$, the simplicial complexes $\mathcal{N}\ \ (\mathcal{B}_{K_n})$ and $\mathcal{N}(\mathcal{B}_{K_{1,n}})$ are isomorphic.

Furthermore, if $\underline{n} = \{k\} \in \mathsf{Z}_{\geq 0}$, then $\mathfrak{X}_{\underline{n}}$ is P_{k+1}, the path graph on $k+1$ vertices, and $\mathcal{X}_{\underline{n}}$ is P_{k+2}, the path graph on $k+2$ vertices. Thus, we have another corollary of Theorem 4.2.

Corollary 4.4 ([MP17], Example 62)**.** For all $n \in \mathsf{N}$, the simplicial complexes $\mathcal{N}(\mathcal{B}_{P_n})$ and $\mathcal{N}(\mathcal{B}_{P_{n+1}})$ are isomorphic.

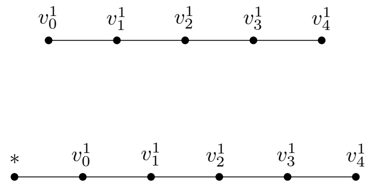


Figure 6: Spider $\mathfrak{X}_{\{4\}}$ and octopus $\mathcal{X}_{\{4\}}$ are equivalent to the path graphs P_5 and P_6 respectively.

We now consider non-graphical building sets. The following theorem concerns building sets such that every element of a building set $I \in \mathcal{B}$ is an interval, i.e. $I = \{a, a+1, \dots, b-1, b\}$ for some $a \leq b$.

Theorem 4.5. Let $\mathcal{B}$ be a building set on $[n]$ such that all elements of $\mathcal{B}$ are intervals. Define $\mathcal{B}' := \mathcal{B} \cup \{\{n+1\}, \{n, n+1\}, \{n-1, n, n+1\}, \dots, [n+1]\}$. Then $\mathcal{B}'$ is a building set on $[n+1]$ and $\mathcal{N}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$.

Proof. First we show $\mathcal{B}'$ is a building set. Notice that $\mathcal{B}'$ contains all singletons and $[n+1]$. In addition, $\mathcal{B}'$ contains all intervals which contain $n+1$. Now suppose that $I, J \in \mathcal{B}'$ with $n+1$ in at least one of I or J, and $I \cap J \neq$?. The union of two intersecting intervals is an interval, and $n+1 \in I \cup J$, so we have that $I \cup J \in \mathcal{B}'$. If $I, J \in \mathcal{B}'$ are such that $n+1 \notin I, J$ and $I \cap J \neq$?, then this implies that $I, J \in \mathcal{B}$, which is a building set. Thus, $I \cup J \in \mathcal{B} \subseteq \mathcal{B}'$.

To show $\mathcal{N}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$, we map elements of $\mathcal{B}$ to $\mathcal{B} \subset \mathcal{B}'$ by their natural inclusion, and we map x_i to $[i+1, n+1] = \{i+1, \dots, n, n+1\}$ for each $i = 1, \dots, n$. Consider a face $\{I_1, \dots, I_k\} \cup \{x_{j_1}, \dots, x_{j_\ell}\}$ of $\mathcal{N}(\mathcal{B})$. We want to show that $\{I_1, \dots, I_k\} \cup \{[j_1+1, n+1], \dots, [j_\ell+1, n+1]\}$ is a nested collection for $\mathcal{B}'$. Notice that $\{I_1, \dots, I_k\}$ is a nested collection for $\mathcal{B}'$; in particular, $\{I_1, \dots, I_k\} \subseteq \mathcal{B}'|_{[n]} = \mathcal{B}$ by definition. Furthermore, one has that $\{[j_1+1, n+1], \dots, [j_\ell+1, n+1]\}$ is a pairwise nested subset of $\mathcal{B}'$. It remains to check the following:

1. For any I_i and $[j_s+1, n+1]$, they are either nested or disjoint. This follows from the fact that $j_s \notin I_i = [a, b]$, by the definition of an extended nested collection, so either $j_s < a$ which implies $[j_s+1, n+1] \supset I_i$, or $j_s > b$ which implies $[j_s+1, n+1] \cap I_i =$?.

2. For any collection of subsets $I_{i_1}, \dots, I_{i_r}, [j_s+1, n+1]$ that are pairwise disjoint, then

$$I := I_{i_1} \cup \cdots \cup I_{i_r} \cup [j_s+1, n+1] \notin \mathcal{B}'.$$

   It suffices to show that this union is not an interval. For any t, we have that $I_{i_t} = [a, b]$ being disjoint from $[j_s+1, n+1]$ is equivalent to $j_s > b$ by the previous case. Hence, the union will contain some number less than j_s but not j_s. Therefore, $j_s \notin I$.

Conversely, given a nested collection $\{I_1, \dots, I_m\}$ of $\mathcal{B}'$, we will show that its inverse image is an extended nested collection of $\mathcal{B}$. Assume that $I_1, \dots, I_k$ doesn't contain $n+1$, any $I_{k+1}, \dots, I_m$ are of the form $[j_1+1, n+1], \dots, [j_\ell+1, n+1]$ for $\ell = m-k$ and $j_1, \dots, j_l$ pairwise distinct. Then $\{I_1, \dots, I_k\} \cup \{x_{i_1}, \dots, x_{i_l}\}$ is an extended nested collection of $\mathcal{B}$ since:

1. $\{I_1, \dots, I_k\}$ is a nested collection of $\mathcal{B} = \mathcal{B}'|_{[n]}$.

2. For any $I_r = [a, b]$ and x_{j_s}, since I_r and $[j_s+1, n+1]$ are either nested or disjoint, we have that either $j_s < a$ or $j_s > b$. In either case, we have $j_s \notin I_r$.

Thus, the map we defined gives an isomorphism $\mathcal{N}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$. □

5 Face Counting

In this section, we study the face numbers of the extended nestohedron. This includes finding recursive formulas for their f- and h-polynomials and showing that the γ-vector is nonnegative when the extended nestohedron is flag.

5.1 Formulas for the f- and h-vectors

In [Pos09, Theorem 7.11], Postnikov provides, without proof, a recursive formula for the f-polynomial of the nestohedron $\mathcal{P}(\mathcal{B})$, which is the dual of the nested complex $\mathcal{N}(\mathcal{B})$. In this subsection, we first provide a short proof for the formula for the f-polynomial of $\mathcal{N}(\mathcal{B})$, and then show how to get the original result by Postnikov for the dual $\mathcal{P}(\mathcal{B})$.

Theorem 5.1. Let $\mathcal{B}$ be a building set on $[n]$. Recall that $\mathcal{B}_{\max}$ denotes the set of maximal elements, and let $\mathcal{S}(\mathcal{B}) = \{S \ (\ [n] \mid S \cap M \ (\ M \ \forall M \in \mathcal{B}_{\max}\}$. The f-polynomials of the nested complex $\mathcal{N}(\mathcal{B})$ satisfy the following recursive formula:

$$f_{\mathcal{N}(\,?\,)}(t) = 1, \qquad f_{\mathcal{N}(\mathcal{B})}(t) = \sum_{S \in \mathcal{S}(\mathcal{B})} t^{|(\mathcal{B}|_S)_{\max}|} f_{\mathcal{N}(\mathcal{B}|_S)}(t).$$

Proof. For any nested collection $N = \{I_1, \ldots, I_k\}$ of $\mathcal{N}(\mathcal{B})$, let $S = \operatorname{Supp} N = I_1 \cup \cdots \cup I_k$. Then $S \in \mathcal{S}(\mathcal{B})$, and $\{I_j\}$ contains all maximal elements of $B|_S$, since otherwise their union would be a proper subset of S, a contradiction. We then obtain a $(k - |(\mathcal{B}|_S)_{\max}|)$-face of $\mathcal{N}(\mathcal{B}|_S)$ by removing the maximal elements of $B|_S$ from N. Conversely, for any $S \in \mathcal{S}(\mathcal{B})$ and a ℓ-face N_S of $\mathcal{N}(\mathcal{B}|_S)$, one obtain a $(\ell + |(\mathcal{B}|_S)_{\max}|)$-face of $\mathcal{N}(\mathcal{B})$ by adding the maximal elements of $B|_S$. One can verify that these two maps are inverse of to each other each other, so we obtain the recursive formula for the f-polynomial of $\mathcal{N}(\mathcal{B})$. □

Corollary 5.2. [Pos09, Theorem 7.11] The f-polynomial of $\mathcal{P}(\mathcal{B})$ satisfy the following recurrence relation:

(a) $f_{\mathcal{P}(\,?\,)}(t) = 1$.

(b) If $\mathcal{B}_1, \ldots, \mathcal{B}_k$ are the connected components of $\mathcal{B}$, then

$$f_{\mathcal{P}(\mathcal{B})}(t) = f_{\mathcal{P}(\mathcal{B}_1)}(t) \cdots f_{\mathcal{P}(\mathcal{B}_k)}(t).$$

(c) If $\mathcal{B}$ is a connected building set, then

$$f_{\mathcal{P}(\mathcal{B})} = \sum_{S \ (\ [n]} t^{n-|S|-1} f_{\mathcal{P}(\mathcal{B}|_S)}(t).$$

Proof. The first condition is true, and the second condition follows from Lemma 2.17. For the last condition, recall that the f-polynomial of a polytope $\mathcal{P}$ and its dual $\mathcal{P}^*$ are related by the formula $f_{\mathcal{P}}(t) = t^{\dim \mathcal{P}} f_{\mathcal{P}^*}(t^{-1})$. By [Zel06, Proposition 4.1], we know that $\dim \mathcal{N}(\mathcal{B}) = |S| - |\mathcal{B}_{\max}|$ if $\mathcal{B}$ is a building set on S. Hence, we have

$$f_{\mathcal{N}(\mathcal{B}|_S)}(t) = t^{|S| - |(\mathcal{B}|_S)_{\max}|} f_{\mathcal{P}(\mathcal{B}|_S)}(t^{-1}).$$

Replacing the above in the f-polynomial equation of Theorem 5.1 gives the desired conclusion. □

We now state several results about the f- and h-polynomial of an extended nestohedron $\mathcal{P}\ \ (\mathcal{B})$. It turns out that one can relate the f- and h-polynomials of $\mathcal{P}\ \ (\mathcal{B})$ in terms of the f- and h-polynomials of the nestohedron $\mathcal{P}(\mathcal{B}|_S)$ for $S \subseteq [n]$.

Theorem 5.3. For a building set $\mathcal{B}$ on $[n]$, the f- and h-polynomials of the extended nestohedron $\mathcal{P}\ \ (\mathcal{B})$ satisfy the following formulas:

$$f_{\mathcal{P}\ \ (\mathcal{B})}(t) = \sum_{S\subseteq[n]} (t+1)^{n-|S|} f_{\mathcal{P}(\mathcal{B}|_S)}(t),$$

and

$$h_{\mathcal{P}\ \ (\mathcal{B})}(t) = \sum_{S\subseteq[n]} t^{n-|S|} h_{\mathcal{P}(\mathcal{B}|_S)}(t).$$

Proof. Note that an extended nested collection $N\ \ = \{I_1,\dots,I_k\} \cup \{x_{j_1},\dots,x_{j_\ell}\}$ corresponds to a nested collection $\{I_1,\dots,I_k\}$ on its support $S = I_1 \cup \cdots \cup I_k$, and a subset $\{j_1,\dots,j_\ell\}$ of $[n]\setminus S$. Conversely, for any face N in $\mathcal{N}(B|_S)$ and any subset T of $[n]\setminus S$, we obtain an extended nested collection $N \cup \{x_j \mid j \in T\}$ of $\mathcal{B}$. Summing over all $S \subset [n]$ with this choice procedure, we have that:

$$\begin{aligned} f_{\mathcal{P}\ \ (\mathcal{B})}(t) &= \sum_{S\subseteq[n]} \left(\prod_{i\notin S}(t+1)\right) f_{\mathcal{P}(\mathcal{B}|_S)}(t) \\ &= \sum_{S\subseteq[n]} (t+1)^{n-|S|} f_{\mathcal{P}(B|_S)}(t). \end{aligned}$$

Notice that we are not over-counting since every face of $\mathcal{N}(\mathcal{B}|_S)$ must contain the maximal elements of $\mathcal{B}|_S$. By definition, we have $h_{\mathcal{P}\ \ (\mathcal{B})}(t) = f_{\mathcal{P}\ \ (\mathcal{B})}(t-1)$, hence the second equation for the h-polynomial follows from the first. □

Consider the poset $P = \{\mathcal{B}|_S \mid S \subseteq [n]\}$ given by the ordering $\mathcal{B}|_I \lessdot \mathcal{B}|_J$ if and only if $I \subset J$. Then the f- and h-polynomials are functions $P \to \mathbb{Z}[t]$, and we can use the Möbius inversion formula.

Definition 5.4 (Möbius function)**.** The **Möbius function** μ_P for a finite poset P is defined recursively as setting $\mu_P(u,u) = 1$ for each $u \in P$ and $\mu_P(u,v) = -\sum_{u\le z\le v} \mu_P(u,z)$.

The following corollary of Theorem 5.3 comes from the special case of Möbius inversion known as "inclusion-exclusion," where the poset is the Boolean lattice, i.e.

$$f(A) = \sum_{B\subseteq A} g(B) \qquad \Leftrightarrow \qquad g(A) = \sum_{B\subseteq A} (-1)^{|A|-|B|} f(B).$$

Corollary 5.5. There is a reverse relation between f- and h-polynomials of $\mathcal{P}(\mathcal{B})$ and those of $\mathcal{P}\ \ (\mathcal{B}|_S)$ for $S \subseteq [n]$ as follows:

$$f_{\mathcal{P}(\mathcal{B})}(t) = \sum_{S\subseteq[n]} (-t-1)^{n-|S|} f_{\mathcal{P}\ \ (\mathcal{B}|_S)}(t),$$

and

$$h_{\mathcal{P}(\mathcal{B})}(t) = \sum_{S\subseteq[n]} (-t)^{n-|S|} h_{\mathcal{P}\ \ (\mathcal{B}|_S)}(t).$$

Similar to Corollary 5.2, we have a formula for the f-polynomial of $\mathcal{P}\ \ (\mathcal{B})$ based on the f-polynomials of strictly smaller building sets $\{\mathcal{B}_S \mid S \subsetneq [n]\}$.

Theorem 5.6. Let $\mathcal{B}$ be a connected building set on $[n]$. Then the f- and h-polynomial of $\mathcal{P}\ \ (\mathcal{B})$ are given by

$$f_{\mathcal{P}\ \ (\mathcal{B})}(t) = \sum_{S\subsetneq[n]} \frac{(-1)^{n-|S|+1}}{t} \left((t+1)^{n-|S|+1} - 1\right) f_{\mathcal{P}\ \ (\mathcal{B}|_S)}(t),$$

and

$$h_{\mathcal{P}\ \ (\mathcal{B})}(t) = \sum_{S\subsetneq[n]} (-1)^{n-|S|+1} \left(t^{n-|S|} + t^{n-|S|-1} + \cdots + t + 1\right) h_{\mathcal{P}\ \ (\mathcal{B}|_S)}(t).$$

Proof. For the first equation, we write:

$$
\begin{aligned}
& f_{\mathcal{P}\ \ (\mathcal{B})}(t) \\
&= f_{\mathcal{P}(\mathcal{B})}(t) + \sum_{S\ (\ [n]} (t+1)^{n-|S|} f_{\mathcal{P}(\mathcal{B}|_S)}(t) && \text{(by Theorem 5.3)} \\
&= \sum_{S\ (\ [n]} \left(t^{n-|S|-1} + (t+1)^{n-|S|}\right) f_{\mathcal{P}(\mathcal{B}|_S)}(t) && \text{(by Corollary 5.2)} \\
&= \sum_{S\ (\ [n]} \left(t^{n-|S|-1} + (t+1)^{n-|S|}\right) \sum_{I \subseteq S} (-t-1)^{|S|-|I|} f_{\mathcal{P}\ \ (\mathcal{B}|_I)}(t) && \text{(by Corollary 5.5)} \\
&= \sum_{I\ (\ [n]} \sum_{I \subset S\ (\ [n]} \left(t^{n-|S|-1}(-t-1)^{|S|-|I|} + (-1)^{|S|-|I|}(t+1)^{n-|I|}\right) f_{\mathcal{P}\ \ (\mathcal{B}|_I)}(t).
\end{aligned}
$$

We now need to evaluate the terms in the parentheses. Note that the expression only depends on the size of S, hence we can write it as:

$$
\begin{aligned}
& \sum_{s=|I|}^{n-1} \binom{n-|I|}{s-|I|} \left(t^{n-s-1}(-t-1)^{s-|I|} + (-1)^{s-|I|}(t+1)^{n-|I|}\right) \\
&= \sum_{k=0}^{n-1-|I|} \binom{n-|I|}{k} t^{n-|I|-k-1}(-t-1)^k + \sum_{k=0}^{n-1-|I|} \binom{n-|I|}{k} (-1)^k (t+1)^{n-|I|}.
\end{aligned}
$$

We now evaluate the two terms separately. The first term is equal to:

$$
\frac{1}{t}\left(\sum_{k=0}^{n-|I|} \binom{n-|I|}{k} t^{n-|I|-k}(-t-1)^k - (-t-1)^{n-|I|}\right) = \frac{1}{t}\left((-1)^{n-|I|} - (-t-1)^{n-|I|}\right),
$$

and the second term is:

$$
(t+1)^{n-|I|}\left(\sum_{k=0}^{n-|I|} \binom{n-|I|}{k} (-1)^k - (-1)^{n-|I|}\right) = -(-t-1)^{n-|I|}.
$$

Adding two terms back together, we obtain:

$$
\begin{aligned}
& \frac{1}{t}\left((-1)^{n-|I|} - (-t-1)^{n-|I|}\right) - (-t-1)^{n-|I|} \\
&= (-t-1)^{n-|I|}\left(-\frac{1}{t} - 1\right) + \frac{(-1)^{n-|I|}}{t} \\
&= \frac{(-1)^{n-|I|+1}}{t}\left((t+1)^{n-|S|+1} - 1\right),
\end{aligned}
$$

which is what we want. The second equation follows from the first since $h_{\mathcal{P}\ \ (\mathcal{B})}(t) = f_{\mathcal{P}\ \ (\mathcal{B})}(t)$ and

$$
\frac{(-1)^{n-|S|+1}}{t-1}\left(t^{n-|S|+1} - 1\right) = (-1)^{n-|S|+1}\left(t^{n-|S|} + \cdots + t + 1\right)
$$

□

It is an open problem to find a combinatorial interpretation of Theorem 5.6.

Example 5.7. Let G be the path graph on 3 vertices P_3. Then

$$\mathcal{B} = \mathcal{B}_G = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}.$$

We will show that Theorem 5.6 holds for $\mathcal{P}\ \ (\mathcal{B})$ by finding the h-polynomial using both sides of the equation. First, we can find the f-polynomial and use the formula for obtaining the h-polynomial to see that

$$h_{\mathcal{P}\ \ (\mathcal{B})}(t) = t^3 + 6t^2 + 6t + 1.$$

The following are the h-polynomials for $\mathcal{P}\ \ (\mathcal{B}|_S)$ for every subset S ($[3]$.

$$\begin{aligned} &h_{\mathcal{P}\ \ (\mathcal{B}|_{\{1,2\}})}(t) = h_{\mathcal{P}\ \ (\mathcal{B}|_{\{2,3\}})}(t) = t^2 + 3t + 1, \\ &h_{\mathcal{P}\ \ (\mathcal{B}|_{\{1,3\}})}(t) = t^2 + 2t + 1, \\ &h_{\mathcal{P}\ \ (\mathcal{B}|_{\{1\}})}(t) = h_{\mathcal{P}\ \ (\mathcal{B}|_{\{2\}})}(t) = h_{\mathcal{P}\ \ (\mathcal{B}|_{\{3\}})}(t) = t + 1, \\ &h_{\mathcal{P}\ \ (\mathcal{B}|_{\varnothing})}(t) = 1. \end{aligned}$$

One can check that

$$t^3 + 6t^2 + 6t + 1 = (t+1)\left(2(t^2 + 3t + 1) + (t^2 + 2t + 1)\right) - (t^2 + t + 1)3(t+1) + (t^3 + t^2 + t + 1),$$

as desired.

We now introduce a pair of interesting invariants for building sets, called the a- and b-numbers, that first appeared in [CP15] and [PPP17] for graphical building sets. However, their definition has a natural generalization to an arbitrary building set $\mathcal{B}$, which we state here. A building set $\mathcal{B}$ is **even** if all connected components of $\mathcal{B}$ have even cardinality, and is **odd** if all connected components of $\mathcal{B}$ have odd cardinality.

Definition 5.8. For a building set $\mathcal{B}$ on $[n]$, define the **a-number** of $\mathcal{B}$ to be

$$a(\mathcal{B}) := \begin{cases} 1, & \text{if } \mathcal{B} - \mathcal{B}_{\max} = ?, \\ 0, & \text{if } \mathcal{B} \text{ is not even}, \\ -\sum_{S\ (\ [n]} a(\mathcal{B}|_S), & \text{otherwise.} \end{cases} \tag{5.8.1}$$

Define the **b-number** of $\mathcal{B}$ to be

$$b(\mathcal{B}) := \begin{cases} 1, & \text{if } \mathcal{B} - \mathcal{B}_{\max} = ?, \\ 0, & \text{if } \mathcal{B} \text{ is not odd}, \\ -\sum_{S\ (\ [n]} b(\mathcal{B}|_S), & \text{otherwise.} \end{cases} \tag{5.8.2}$$

In [PPP17], the a- and b-numbers of a graphical building set is shown to be related to the h-vector of the corresponding (extended) nested complex. In particular, they proved the following result.

Theorem 5.9. [PPP17, Corollary 4.8] For any undirected graph G with n vertices, let $\mathcal{B} = \mathcal{B}_G$ be the corresponding building set. We then have

$$\begin{aligned} a(\mathcal{B}) &= (-1)^n h_{\mathcal{P}\ \ (\mathcal{B})}(-1), \\ b(\mathcal{B}) &= (-1)^n h_{\mathcal{P}(\mathcal{B})}(-1). \end{aligned}$$

We suspect that many of the results of [PPP17] hold for general building sets, including the following:

Conjecture 5.10. For an arbitrary building set $\mathcal{B}$,

$$\begin{aligned} a(\mathcal{B}) &= (-1)^n h_{\mathcal{P}\ \ (\mathcal{B})}(-1), \\ b(\mathcal{B}) &= (-1)^n h_{\mathcal{P}(\mathcal{B})}(-1). \end{aligned}$$

Using the definitions of a- and b-numbers, the authors of [PPP17] compute the Betti numbers of the real toric manifold corresponding to the polytopes $\mathcal{P}(\mathcal{B})$ and $\mathcal{P}\ \ (\mathcal{B})$ when $\mathcal{B} = \mathcal{B}_G$ is a graphical building set. Note that if we consider the complex toric manifold corresponding to a simple polytope $\mathcal{P}$, then its Betti numbers are known to the coefficients of the h-polynomial of $\mathcal{P}$.

Theorem 5.11 ([PPP17, Theorem 1.1 and Theorem 1.2])**.** Let G be an undirected graph with $V(G) = [n]$ and $\mathcal{B} = \mathcal{B}_G$. Then the i-th Betti number of the real toric manifold associated to $\mathcal{P}(\mathcal{B})$ is given by

$$\beta^i(X^{\mathsf{R}}(\mathcal{P}(\mathcal{B}))) \sum_{\substack{S\subseteq[n],\\ |S|=2i}} |a\,(\mathcal{B}|_S)|.$$

Similarly, the i-th Betti number of the real toric manifold associated to $\mathcal{P}\ \ (\mathcal{B})$ is given by

$$\beta^i(X^{\mathsf{R}}(\mathcal{P}(\mathcal{B}))) \qquad \sum_{\substack{S\subseteq[n],\\ |S|+\kappa(\mathcal{B}|_S)=2i}} |b\,(\mathcal{B}|_S)|,$$

where $\kappa(\mathcal{B}|_S)$ is the number of connected components of $\mathcal{B}|_S$.

They use this theorem to prove the following result.

Theorem 5.12. [PPP17, Theorem 1.3] Let G be a forest, and let $L(G)$ be the line graph of G. Then we have:

$$\beta^i(X^{\mathsf{R}}(\mathcal{P}(\mathcal{B}_G))) = \beta^i(X^{\mathsf{R}}(\mathcal{P}\ \ (\mathcal{B}_{L(G)}))).$$

Here, the line graph $L(G)$ of G is the graph whose vertices are the edges of G, and whose edges are pair of edges of G that share a common endpoint.

A similar phenomenon is observed for the h-polynomial of the corresponding simple polytopes, leading us to conjecture the following.

Conjecture 5.13. Let G be a forest and $L(G)$ be the line graph of G. Then

$$h_{\mathcal{P}(\mathcal{B}_G)}(t) = h_{\mathcal{P}\ \ (\mathcal{B}_{L(G)})}(t).$$

Given the similarities between Theorem 5.12 and Conjecture 5.13, one might try to adapt the ingredients in the proof of Theorem 5.12 to prove the conjecture. In particular, it would be interesting to find t-analogues of the a and b-number, i.e., rational polynomials in t that evaluate at $t = -1$ to the a- and b-numbers. One such candidate for the t-analogue is the following.

Definition 5.14. Let $\mathcal{B}$ be a building set on $[n]$. We define the **generalized a- and b-numbers** to be

$$a(\mathcal{B}, t) = \frac{h_{\mathcal{P}\ \ (\mathcal{B})}(t)}{t^n}, \qquad b(\mathcal{B}, t) = \frac{h_{\mathcal{P}(\mathcal{B})}(t)}{t^n}.$$

This generalized definition satisfies the following identity, which is a t-analogue of [PPP17, Theorem 4.4].

Proposition 5.15. For any building set B on $[n]$, we have:

$$a(\mathcal{B}, t) = \sum_{S\subseteq[n]} b(\mathcal{B}, t), \qquad b(\mathcal{B}, t) = \sum_{S\subseteq[n]} (-1)^{n-|S|} a(\mathcal{B}, t).$$

Proof. This follows from Theorem 5.3 and Corollary 5.5. □

Another t-analogue formula that the generalized a-number satisfies is that

$$a(\mathcal{B}, t) = \sum_{S\subsetneq[n]} \frac{(-1)^{n-|S|+1}}{t^{n-|S|}} \left(t^{n-|S|} + \cdots + t + 1\right) a(\mathcal{B}|_S, t),$$

which is a straightforward consequence of the second equation in Theorem 5.6. Indeed, evaluating the equation at $t = -1$ recovers the definition of the a-number in (5.8.1), and in particular proves the first part of Conjecture 5.10. Our goal is then to find a t-analogue formula relating the a- and b-numbers that suffices to solve Conjecture 5.13.

5.2 Gal's conjecture for flag extended nestohedra

Recall that for a d-dimensional simple polytope P, the γ-vector $(\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor d/2 \rfloor})$ is defined by the h-polynomial:

$$h_P(t) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i (1+t)^{d-2i}.$$

The γ-polynomial is then $\gamma(t) = \sum \gamma_i t^i$. Gal conjectured that the γ-vector is nonnegative for flag simple polytopes (Conjecture 2.2). Volodin showed that this conjectures holds for flag nestohedra [Vol10]. In this subsection, we show the analagous result for flag extended nestohedra.

Theorem 5.16 (Gal's Conjecture for Flag Extended Nestohedra)**.** Let $\mathcal{P}\ (\mathcal{B})$ be a flag extended nestrodron. Then the γ-vector of $\mathcal{P}\ (\mathcal{B})$ is nonnegative.

This implies that Gal's conjecture holds for this large class of flag simple polytopes. To prove this result, we use a similar technique as the one used by Volodin. The general outline of the technique is as follows. For a building set $\mathcal{B}$ with a flag extended nestohedron, there exists a minimal building set $\mathcal{D} \subseteq \mathcal{B}$ such that $\mathcal{D}$ also gives a flag extended nestohedron $\mathcal{P}\ (\mathcal{D})$; this is our starting polytope, and one can show that the γ-vector of $\mathcal{P}\ (\mathcal{D})$ is nonnegative. Then, we will explain why there is a way to add back in elements of $\mathcal{B} \setminus \mathcal{D}$; each time we add back in an element, this corresponds to **shaving** a face of the starting polytope. We show that each time a face is shaved, the γ-vector remains nonnegative.

First, we provide some preliminary definitions and lemmas. A building set $\mathcal{B}$ is **flag** if $P(\mathcal{B})$ is a flag simple polytope. By Lemma 2.20, this means that $\mathcal{P}\ (\mathcal{B})$ is a flag simple polytope as well. A connected building set $\mathcal{D}$ on S is a **minimal flag building set** if it is flag and there does not exist a flag connected building set $\mathcal{C} \subsetneq \mathcal{D}$ that is also on S. Such building sets are characterized in [PRW08, Section 7.2], in which the authors show that the minimal flag building sets correspond to **plane binary trees** as follows.

Given a plane binary tree T with n leaves, label the leaves $\{1\}, \ldots, \{n\}$. For an internal node v with children u and w which are labelled by sets I and J respectively, label node v by the set $I \cup J$. Then, the root node is labelled by the set $[n]$. The building set $\mathcal{B}$ consists of all sets labelling the nodes of the tree. By this correspondence, we see that if $\mathcal{D}$ is a minimal flag building set, then for any non-singleton element $I \in \mathcal{D}$, there are exactly two inclusion-maximal elements of $\mathcal{D}|_I \setminus \{I\}$.

Example 5.17. The building set $\mathcal{B} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}\}$ is a minimal flag building set. Its corresponding plane binary tree τ is shown in Figure 7.

If we consider a non-singleton element of the building set, say $I = \{1,2,3\}$, then the set $\mathcal{D}|_I \setminus \{I\} = \{\{1,2\}, \{1\}, \{2\}, \{3\}\}$ has exactly two maximal elements: $\{1,2\}$ and $\{3\}$.

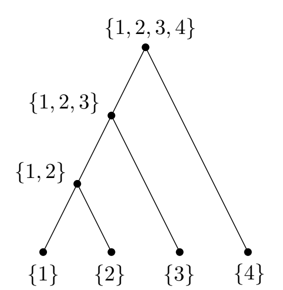


Figure 7: Plane binary tree τ corresponding to building set $\mathcal{B}$.

Definition 5.18 ([Ais14])**.** Let $\mathcal{B}$ be a building set. A **binary decomposition** or **decomposition** of a non-singleton element $I \in \mathcal{B}$ is a set $\mathcal{D} \subseteq \mathcal{B}$ such that $\mathcal{D}$ forms a minimal flag building set on I.

If $I \in \mathcal{B}$ has binary decomposition $\mathcal{D}$, then the two inclusion-maximal elements $I_1, I_2 \in \mathcal{D} \setminus \{I\}$ are the **maximal components** of I in $\mathcal{D}$.

Recall Lemma 2.17, which says that if $\mathcal{B}$ is a building set with connected components $\mathcal{B}_1, \dots, \mathcal{B}_k$, then

$$\mathcal{N}\ (\mathcal{B}) \simeq \mathcal{N}\ (\mathcal{B}_1) * \cdots * \mathcal{N}\ (\mathcal{B}_k).$$

This implies that the γ-polynomial for $\mathcal{P}\ (\mathcal{B})$ is given by

$$\gamma_{\mathcal{P}\ (\mathcal{B})}(t) = \gamma_{\mathcal{P}\ (\mathcal{B}_1)}(t) \cdots \gamma_{\mathcal{P}\ (\mathcal{B}_k)}(t),$$

so it is enough to consider connected building sets for the remainder of this section in order to prove Theorem 5.16.

The following lemmas show that given a flag building set $\mathcal{B}$ and a minimal flag building set $\mathcal{D} \subseteq \mathcal{B}$, we can add elements of $\mathcal{B} \setminus \mathcal{D}$ to $\mathcal{D}$ and remain a flag building set with each addition.

Lemma 5.19 ([Ais12], Lemma 7.2)**.** A building set $\mathcal{B}$ is flag if and only if every non-singleton element $I \in \mathcal{B}$ has a binary decomposition.

Lemma 5.20 ([Ais12], Corollary 2.6)**.** A building set $\mathcal{B}$ is flag if and only if for every non-singleton element $I \in \mathcal{B}$, there exist two elements $I_1, I_2 \in \mathcal{B}$ such that $I_1 \cap I_2 = ?$ and $I_1 \cup I_2 = I$.

Lemma 5.21 ([Ais12], Theorem 3.1, [Vol10], Lemma 6)**.** Let $\mathcal{B}$ and $\mathcal{B}'$ be connected flag building sets on $[n]$ such that $\mathcal{B} \subseteq \mathcal{B}'$. Then $\mathcal{B}'$ can be obtained from $\mathcal{B}$ by successively adding elements so that at each step, the set is a flag building set.

Given a connected flag building set $\mathcal{B}$ on $[n]$ with decomposition $\mathcal{D}$ of $[n]$, there exists an ordering $I_1, I_2, \dots, I_k$ of the elements of $\mathcal{B} \setminus \mathcal{D}$, such that $\mathcal{B}_j = \mathcal{D} \cup \{I_1, \dots, I_j\}$ is a flag building set for all $1 \leq j \leq k$. Such an ordering exists by Lemma 5.21, and is called a **flag ordering** in [Ais14].

Next, we show that each time we add an element to obtain $\mathcal{B}$ from $\mathcal{D}$, this corresponds to the geometric action of **shaving** (see Remark 3.5) a face of the extended nestohedron.

Lemma 5.22. Suppose that a connected flag building set $\mathcal{B}'$ on $[n]$ is obtained from the flag building set $\mathcal{B}$ on $[n]$ by adding an element $I \subseteq [n]$. Then $\mathcal{P}\ (\mathcal{B}')$ can be obtained from $\mathcal{P}\ (\mathcal{B})$ by shaving a codimension 2 face.

Proof. We show the dual version of this statement, i.e. that $\mathcal{N}\ (\mathcal{B}')$ can be obtained from $\mathcal{N}\ (\mathcal{B})$ by stellarly subdividing a face of dimension 2. Since $\mathcal{B}'$ is flag, by Lemma 5.20 there exist two elements $I_1, I_2 \in \mathcal{B}$ such that $I_1 \cap I_2 = ?$ and $I_1 \cup I_2 = I$. Notice that $\{I_1, I_2\}$ forms an extended nested collection of $\mathcal{B}$, since they are disjoint but $I_1 \cup I_2 = I \notin \mathcal{B}$, so there exists a dimension 2 face, or an edge, between the two vertices of $\mathcal{N}\ (\mathcal{B})$ corresponding to I_1 and I_2. Stellarly subdivide this edge, adding in a vertex corresponding to the element I, and call this new simplicial complex $\mathcal{M}$. We'll show that $\mathcal{M} \simeq \mathcal{N}\ (\mathcal{B}')$.

First notice that any facet of $\mathcal{N}\ (\mathcal{B})$ corresponding to a maximal extended nested collection N that does not contain both I_1 and I_2 remains in $\mathcal{M}$. Such maximal collections are still maximal nested collections of $\mathcal{B}'$. However, for any facet of $\mathcal{N}\ (\mathcal{B})$ corresponding to a maximal extended nested collection N such that $I_1, I_2 \in N$, stellar subdivision replaces N with two new facets,

$$N_1 = (N \setminus I_2) \cup I, \qquad N_2 = (N \setminus I_1) \cup I.$$

Since $I \in \mathcal{B}'$, an extended nested collection cannot have both I_1 and I_2. However, a collection can have either I_1 or I_2, as well as I, since $I_1, I_2 \subseteq I$. Thus, N_1 and N_2 correspond exactly to the new facets of $\mathcal{N}\ (\mathcal{B}')$. □

Let Δ^d denote the d-dimensional simplex. The following lemma provides a recursive formula for the γ-polynomial of a polytope that was obtained by shaving the face of another polytope.

Lemma 5.23 ([Vol10], Corollary 1)**.** Let Q be the simple polytope obtained from the n-dimensional simple polytope P by shaving the face G of dimension k. Then

$$\gamma_Q(t) = \gamma_P(t) + \gamma_G(t) \cdot \gamma_{\Delta^{n-k-1}}(t).$$

By the above lemma, when we shave a codimension 2 face F_0 from an extended nestohedra, then we add $\gamma_G(t) \cdot \gamma_{\Delta^{n-k-1}}(t) = t \cdot \gamma_{F_0}(t)$ to the γ-polynomial. By Lemma 5.22 and Proposition 2.25, we conclude the following.

Proposition 5.24. Let $\mathcal{P}\ (\mathcal{B}), \mathcal{P}\ (\mathcal{B}')$ be flag extended nestohedra such that $\mathcal{B}'$ is the result of adding I to $\mathcal{B}$, and both $\mathcal{B}'$ and $\mathcal{B}$ are on $[n]$. Then,

$$\begin{aligned} \gamma_{\mathcal{P}\ (\mathcal{B}')}(t) &= \gamma_{\mathcal{P}\ (\mathcal{B})}(t) + t\gamma_{\mathcal{P}(\mathcal{B}'|_I)}(t)\gamma_{\mathcal{P}\ (\mathcal{B}'/I)}(t) \\ &= \gamma_{\mathcal{P}\ (\mathcal{B})}(t) + t\gamma_{\mathcal{P}(\mathcal{B}|_I)}(t)\gamma_{\mathcal{P}\ (\mathcal{B}/I)}(t). \end{aligned}$$

We are now able to prove our main result of this subsection.

Proof of Theorem 5.16. Let $\mathcal{B}$ be a connected building set on $[n]$. We strongly induct on n, the number of singletons in the building set, as well as induct on the number of elements k of $\mathcal{B} \setminus \mathcal{D}$, where $\mathcal{D}$ is the decomposition of $[n]$ in $\mathcal{B}$.

Our base case of $n = 1$ and $k = 0$ is easily covered with $\mathcal{B} = \{\{1\}\}$. We now cover the other base cases of $n > 1$ and $k = 0$. By [PRW08] Proposition 7.3, the minimal flag building sets are binary trees. For any binary tree $\mathcal{B}$, there exists a plane binary tree $\mathcal{B}'$ constructed by relabeling the singletons in $\mathcal{B}$ such that $\mathcal{N}\ (\mathcal{B}) \simeq \mathcal{N}\ (\mathcal{B}')$ and each element of $\mathcal{B}'$ is an interval. By Theorem 4.5, there exists a $\mathcal{C}$ such that $\mathcal{N}\ (\mathcal{B}') \simeq \mathcal{N}(\mathcal{C})$. Since $\mathcal{N}\ (\mathcal{B})$ is flag, we have that $\mathcal{N}(\mathcal{C})$ is also flag. Thus by [Vol10], we have that the polynomials $\gamma_{\mathcal{N}(\mathcal{C})}(t) = \gamma_{\mathcal{N}\ (\mathcal{B})}(t)$ have nonnegative coefficients.

Suppose $\mathcal{B} \subsetneq \mathcal{B}'$ are connected flag building sets on $[n]$, with $\mathcal{B}' = \mathcal{B} \cup \{I\}$. Assume for our inductive hypothesis that $\gamma_{\mathcal{P}\ (\mathcal{B})}(t)$ has nonnegative coefficients, and for any flag building set $\mathcal{C}$ on $[m]$ with $m < n$, that $\gamma_{\mathcal{P}\ (\mathcal{C})}(t)$ also has nonnegative coefficients.

By Proposition 5.24, it is enough to show that the polynomials $\gamma_{\mathcal{P}(\mathcal{B}|_I)}(t)$ and $\gamma_{\mathcal{P}\ (\mathcal{B}/I)}(t)$ have nonnegative coefficients. Notice that since $\mathcal{B}'$ is a flag building set on $[n]$ and $I \in \mathcal{B}'$, then the building sets $\mathcal{B}'|_I = \mathcal{B}|_I$ and $\mathcal{B}'/I = \mathcal{B}/I$ are also flag. By [Vol10], the polynomial $\gamma_{\mathcal{P}(\mathcal{B}|_I)}$ has nonnegative coefficients, as it is the γ-polynomial of a non-extended flag nestohedron. The building set $\mathcal{B}/I$ is isomorphic to a flag building set $\mathcal{C}$ on $[m]$, with $m < n$. In addition, the polynomial $\gamma_{\mathcal{P}\ (\mathcal{B}/I)}$ has nonnegative coefficients by our inductive hypothesis on n, the number of singletons. Thus, we have that the polynomial $\gamma_{\mathcal{P}\ (\mathcal{B}')}$ has nonnegative coefficients, as desired.

□

6 Chordal Building Sets and the γ-Vector

Although the previous section shows that the γ-vector is nonnegative for flag extended nestohedra, our proof for the result does not provide a combinatorial interpretation for the γ-vector. Before Volodin's result on the nonnegativity of the γ-vector for arbitrary flag nestohedra, Postnikov, Reiner, and Williams [PRW08] proved Gal's conjecture for flag nestohedra $\mathcal{P}(\mathcal{B})$ when $\mathcal{B}$ is a chordal building set. They do this by providing a combinatorial interpretation for the γ-vector for these polytopes.

In this section, we give a combinatorial interpretation of the h- and γ-vectors of the extended nestohedra for chordal building sets by following the method used in [PRW08]. Along the way, we will define some combinatorial objects related to building sets that share some nice properties with extended nested complexes and extended nestohedra.

Most of our definitions, results, and proofs are analogous to ones given in [PRW08] for the non-extended case. We will indicate the correspondences accordingly.

6.1 Extended $\mathcal{B}$-Forests

We first discuss extended $\mathcal{B}$-forests, which are combinatorial objects associated with extended nested set complexes. These are extensions of **$\mathcal{B}$-forests** and **$\mathcal{B}$-trees**, which were originally defined by Postnikov in [Pos09]. Our goal of this subsection is to define these forests and then show the following result.

Proposition 6.1. For a connected building set $\mathcal{B}$ on $[n]$, the h-polynomial of the extended nestohedron $\mathcal{P}\ \ (\mathcal{B})$ is given by

$$h_{\mathcal{P}\ \ (\mathcal{B})}(t) = \sum_{S \subseteq [n]} t^{n-|S|} \sum_{F} t^{\mathsf{des}(F)},$$

where the second sum is over $\mathcal{B}|_S$-forests.

We now begin to define extended $\mathcal{B}$-forests. A **rooted tree** is a tree with a distinguished node, called the **root**. We can view a rooted tree T as a partial order on the tree's nodes, with the partial order $i <_T j$ if j lies on the unique path from i to the root. We can also view a rooted tree as a directed graph, with all edges directed towards the root.

Let F be a forest of rooted trees $\{T^{(1)}, \ldots, T^{(k)}\}$ on $S \subseteq [n]$, meaning that

$$S = \bigsqcup_{i=1}^{k} \{\text{nodes of } T^{(i)}\}.$$

Then F is called a **rooted forest**, and its roots are the roots of each tree $T^{(i)}$.

If i is a node of F and $i \in T^{(j)}$, then let

$$F_{\leq i} := \{\ell \mid \ell \text{ is a descendant of } i \text{ in } T^{(j)}\}.$$

Note that $i \in F_{\leq i}$. We can think of F as the Hasse diagram for a poset, where nodes i and j are **incomparable** in the forest F if either

1. i and j are in two separate trees of F, or

2. i and j are in the same tree of F, but neither is a descendant of the other.

Definition 6.2 (cf. [Pos09], Definition 7.7)**.** Let $\mathcal{B}$ be a connected building set on $[n]$ and $S \subseteq [n]$. Define a $\mathcal{B}|_S$**-forest** as a rooted forest F on vertex set S such that

(F1) For any $i \in S$, the node set $F_{\leq i} \in \mathfrak{B}|_S$.

(F2) For $k \geq 2$ incomparable nodes $i_1, \ldots, i_k \in S$, we have that $\bigcup_{j=1}^{k} F_{\leq i_j} \notin \mathcal{B}$.

(F3) The sets $F_{\leq i}$, for all roots i of F, are exactly the maximal elements of the building set $\mathcal{B}$.

The union of $\mathcal{B}|_S$-forests over all $S \subseteq [n]$ is the set of **extended $\mathcal{B}$-forests**.

Example 6.3. Consider $\mathcal{B}_\Gamma$ with $\Gamma = P_4$. Let $S = \{1, 2, 4\}$. Then an example of an $\mathcal{B}|_S$-forest is the forest with two trees, T_1, T_2, shown in Figure 8, with the nodes 4 and 2 the roots of each tree respectively.

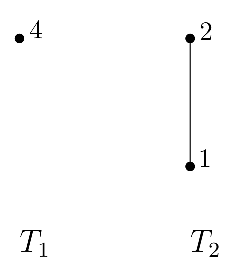


Figure 8: Extended $\mathcal{B}$-forest on S, consisting of two rooted trees T_1 and T_2.

This is an extended $\mathcal{B}$-forest on S because there are two trees, one for each connected component of $\mathcal{B}|_S$, and the node set $F_{\leq 2} = \{1, 2\}$ is an element of $\mathcal{B}$. In addition, for any pair of incomparable nodes, such as 2 and 4, we have that

$$F_{\leq 2} \cup F_{\leq 4} = \{1, 2, 4\} \notin \mathcal{B}.$$

By condition (F3), the number of connected components (i.e., the number of trees) of a $\mathcal{B}|_S$-forest equals the number of connected components of the building set $\mathcal{B}|_S$. When $S = [n]$, a $\mathcal{B}|_S$ forest consists of a single rooted tree. In this case, the tree is actually a $\mathcal{B}$-tree. In [PRW08], the authors provide the following result about the h-polynomial of non-extended nestohedra in terms of $\mathcal{B}$-trees.

Proposition 6.4. [PRW08, Corollary 8.4] For a connected building set $\mathcal{B}$ on $[n]$, the h-polynomial of the non-extended nestohedron $\mathcal{P}(\mathcal{B})$ is given by

$$h_{\mathcal{P}(\mathcal{B})}(t) = \sum_{T} t^{\mathsf{des}(T)},$$

where the sum is over $\mathcal{B}$-trees T.

Comparing the definitions of an extended maximal nested collection and an extended $\mathcal{B}$-forest, it is not hard to see that we have the following bijection.

Proposition 6.5. For a connected building set $\mathcal{B}$ on $[n]$, the map sending a forest of rooted trees F on node set $S \subseteq [n]$ to the collection of elements

$$\{F_{\leq i} \mid i \in S\} \cup \{x_i \mid i \notin S\}$$

gives a bijection between extended $\mathcal{B}$-forests and maximal extended nested sets.

Example 6.6. The extended $\mathcal{B}$-forest of Example 6.3 corresponds to the maximal extended nested set

$$N = \{\{1\}, \{4\}, \{1, 2\}, x_4\}.$$

We now define a statistic on posets that is used in the formula for the h-polynomial in terms of extended $\mathcal{B}$-forests.

Definition 6.7. Given a poset F on $[n]$, define the **descent set** $\mathrm{Des}(F)$ to be the set of ordered pairs (i, j) for which $i \lessdot_F j$ is a covering relation in F but $i >_{\mathbb{Z}} j$, where $>_{\mathbb{Z}}$ is the standard partial order on the integers. Define the **descent number** to be $\mathrm{des}(F) := |\mathrm{Des}(F)|$.

Example 6.8. Consider the poset F on [8] shown in Figure 9. The descent set of F is

$$\mathrm{Des}(F) = \{(7, 1), (6, 1), (8, 3)\},$$

and the descent number is $\mathrm{des}(F) = 3$.

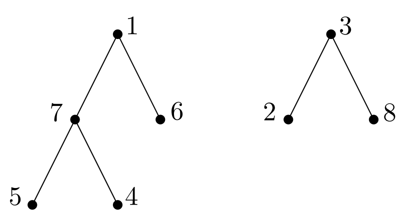


Figure 9: Poset F.

Recall from Theorem 5.6 that the recursive formula for the h-polynomial of extended nestohedra is given by

$$h_{\mathcal{P}\ \ (\mathcal{B})}(t) = \sum_{S \subseteq [n]} t^{n-|S|} h_{\mathcal{P}(\mathcal{B}|_S)}(t).$$

We are now ready to prove our result about h-polynomials of extended nestohedra in terms of extended $\mathcal{B}$-forests.

Proof of Proposition 6.1. Let $\{F_S\}$ denote the set of $\mathcal{B}|_S$-forests for $S \subseteq [n]$. We first show that

$$h_{\mathcal{P}(\mathcal{B}|_S)}(t) = \sum_{F \in \{F_S\}} t^{\mathsf{des}(F)}.$$

Suppose $\mathcal{B}|_S$ consists of connected components $\mathcal{B}_1, \ldots, \mathcal{B}_k$. Then, by Lemma 2.17, we have that

$$\mathcal{P}(\mathcal{B}|_S) = \mathcal{P}(\mathcal{B}_1) * \cdots * \mathcal{P}(\mathcal{B}_k),$$

implying that the h-polynomial of $\mathcal{P}(\mathcal{B}|_S)$ can be given by

$$h_{\mathcal{P}(\mathcal{B}|_S)}(t) = h_{\mathcal{P}(\mathcal{B}_1)}(t) \cdots h_{\mathcal{P}(\mathcal{B}_k)}(t).$$

By Proposition 6.4, we can rewrite the polynomial using $\mathcal{B}_i$-trees:

$$h_{\mathcal{P}(\mathcal{B}|_S)}(t) = \sum_{T \in \{\mathcal{B}_1\text{-}\mathsf{trees}\}} t^{\mathsf{des}(T)} \cdots \sum_{T \in \{\mathcal{B}_k\text{-}\mathsf{trees}\}} t^{\mathsf{des}(T)}.$$

Notice that each extended $\mathcal{B}$-forest on S consists of exactly one $\mathcal{B}_i$-tree for every i. In addition, F is an extended $\mathcal{B}$-forest on S consisting of trees $T_1, \ldots, T_k$, with T_i a $\mathcal{B}_i$-tree, then

$$\mathrm{des}(F) = \mathrm{des}(T_1) + \cdots + \mathrm{des}(T_k) \implies t^{\mathsf{des}(F)} = t^{\mathsf{des}(T_1)} \cdots t^{\mathsf{des}(T_k)}.$$

This shows that

$$h_{\mathcal{P}(\mathcal{B}|_S)}(t) = \sum_{T \in \{\mathcal{B}_1\text{-}\mathsf{trees}\}} t^{\mathsf{des}(T)} \cdots \sum_{T \in \{\mathcal{B}_k\text{-}\mathsf{trees}\}} t^{\mathsf{des}(T)} = \sum_{F \in \{F_S\}} t^{\mathsf{des}(F)}.$$

Plugging into our recursive formula for the h-polynomial of the extended nestohedron gives us our result:

$$h_{\mathcal{P}\ (\mathcal{B})}(t) = \sum_{S \subseteq [n]} t^{n-|S|} h_{\mathcal{P}(\mathcal{B}|_S)}(t) = \sum_{S \subseteq [n]} t^{n-|S|} \sum_{F \in \{F_S\}} t^{\mathsf{des}(F)}.$$

□

Before continuing with our study of the h-polynomial, we use extended $\mathcal{B}$-forests to describe the vertices of the polytope defined in the construction of the extended nestohedron in Theorem 3.6. By following the maps of facets defined in Theorem 3.6, one can show through direct computation that the coordinates of the extended nested collection can be written as follows.

Proposition 6.9. The coordinates $v = (v_1, \ldots, v_n)$ of a vertex corresponding to a maximal extended nested collection N in the extended nestohedron $\mathcal{P}\ (\mathcal{B})$ is given by the following:

$$v_k = \begin{cases} 0, & \text{if } x_k \in N, \\ |\{I \in \mathcal{B} \mid k \in I\}| - |F_{\leq k}| + 1, & \text{otherwise,} \end{cases}$$

where F is the extended $\mathcal{B}$-forest corresponding to N.

Example 6.10. Consider the building set $\mathcal{B} = \mathcal{B}_{K_3} = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$, and the maximal extended nested collections

$$N_1 = \{\{2\}, \{2,3\}, x_1\}, \qquad N_2 = \{\{3\}, \{1,3\}, \{1,2,3\}\}.$$

The extended $\mathcal{B}$-forests corresponding to N_1 and N_2 are F_1 and F_2 respectively, shown in Figure 10. We will show how to determine the coordinates of their corresponding vertices in the extended nestohedron $\mathcal{P}\ (\mathcal{B})$.

First consider N_1 and corresponding vertex $v = (v_1, v_2, v_3)$. Since $x_1 \in N_1$, we have that $v_1 = 0$. For the remaining coordinates, we have that

$$v_2 = |\{\{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}| - |\{\{2\}\}| + 1 = 4,$$
$$v_3 = |\{\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}| - |\{\{2\}, \{3\}\}| + 1 = 3,$$

so $v = (0, 4, 3)$. Now consider N_2 and corresponding vertex $u = (u_1, u_2, u_3)$. Then,

$$u_1 = |\{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}| - |\{\{1\}, \{3\}\}| + 1 = 3,$$
$$u_2 = |\{\{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}| - |\{\{1\}, \{2\}, \{3\}\}| + 1 = 2,$$
$$u_3 = |\{\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}| - |\{\{3\}\}| + 1 = 4.$$

Thus, $u = (3, 2, 4)$.

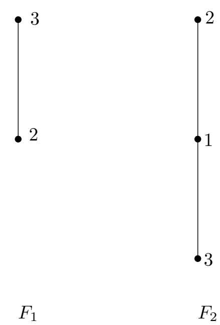


Figure 10: Extended $\mathcal{B}$-forests corresponding to N_1 and N_2.

6.2 $\mathcal{B}$-Partial Permutations and Extended $\mathcal{B}$-Permutations

Next, we study permutations called $\mathcal{B}$-partial permutations and extended $\mathcal{B}$-permutations, which are analogous to $\mathcal{B}$-permutations defined in [PRW08]. We will show the following result, which is that when considering a special class of building sets, we can formulate the h-polynomial of the extended nestohedron in terms of extended $\mathcal{B}$-permutations.

Theorem 6.11. For a chordal building set $\mathcal{B}$, the h-polynomial of the extended nestohedron $\mathcal{P}\ (\mathcal{B})$ is

$$h_{\mathcal{P}\ (\mathcal{B})}(t) = \sum_{w \in \mathfrak{S}_{n+1}(\mathcal{B})} t^{\mathsf{des}(w)}.$$

To show this result, we will introduce $\mathcal{B}$-partial permutations and extended $\mathcal{B}$-permutations, and then show how these two sets relate to extended $\mathcal{B}$-forests.

A **partial permutation** of $[n]$ is a permutation $w \in \mathfrak{S}_S$, for some $S \subseteq [n]$, where $\mathfrak{S}_S$ denotes the symmetric group acting on S. If $S =$ **?**, then we denote the unique partial permutation with S as its entry set by (). Let $\mathfrak{P}_n$ denote the set of partial permutations on $[n]$. Notice that

$$\mathfrak{P}_n = \bigcup_{S \subseteq [n]} \mathfrak{S}_S.$$

Example 6.12. The set of partial permutations of [2], $\mathfrak{P}_2$, consists of the following permutations:

$$(1, 2), (2, 1), (1), (2), ().$$

We now begin to show how partial permutations relate to extended $\mathcal{B}$-forests. Like [PRW08], who define a surjective map $\Psi_{\mathcal{B}} : \mathfrak{S}_n \to \{\mathcal{B}\text{-trees}\}$, we recursively define a surjective map $\Psi_{\mathcal{B}}$ from all partial permutations to the extended $\mathcal{B}$-forests.

Definition 6.13. Let $\mathcal{B}$ be a connected building set on $[n]$, and $S \subseteq [n]$ with $S = \{s_1 < \cdots < s_k\}$. Given a permutation $w = (w(s_1), w(s_2), \ldots, w(s_k)) \in \mathfrak{S}_S \subseteq \mathfrak{P}_n$, one recursively constructs a $\mathcal{B}|_S$-forest $F = F(w)$ as follows.

Let $\mathcal{B}^{(1)}, \ldots, \mathcal{B}^{(r)}$ be the connected components of the building set $\mathcal{B}|_S$. Restricting w to each of the sets $\mathcal{B}^{(i)}$ gives a subword of w, say $w_i = (w(s_{i_1}), \ldots, w(s_{i_k}))$. For each $i = 1, \ldots, r$, we construct a rooted tree $T^{(i)}$ using w_i.

Let the root of $T^{(i)}$ be the node $w(s_{i_k})$. Let $\mathcal{B}_1^{(i)}, \ldots, \mathcal{B}_{i_r}^{(i)}$ be the connected components of the restriction $\mathcal{B}^{(i)}_{\{w(s_{i_1}),\ldots,w(s_{i_{k-1}}))\}}$. Restricting w_i to each of the sets $\mathcal{B}_{i_j}^{(i)}$ again gives a subword of w_i, to which one recursively applies the construction, attaching each of these subsequent trees to the root node $w(s_{i_k})$.

Example 6.14. Consider the building set $\mathcal{B} = \mathcal{B}_\Gamma$, where $\Gamma = P_8$, and let $S = \{1, 2, 3, 5, 7\}$. We will show how to find the extended $\mathcal{B}$-forest $F = \Psi_{\mathcal{B}}(w)$, where $w = (3, 7, 1, 5, 2)$.

The three restricted components of the building set $\mathcal{B}|_S$ are

$$\mathcal{B}^{(1)} = \mathcal{B}|_{\{1,2,3\}}, \quad \mathcal{B}^{(2)} = \mathcal{B}|_{\{5\}}, \quad \mathcal{B}^{(3)} = \mathcal{B}|_{\{7\}}.$$

Thus, F will consist of 3 rooted trees, $T^{(1)}, T^{(2)}$, and $T^{(3)}$. Since $\mathcal{B}^{(2)}$ and $\mathcal{B}^{(3)}$ each only consist of one singleton element, we know that each of these trees will just be a single node labelled 5 and 7 respectively.

To construct $T^{(1)}$, consider the subword $w_1 = (3, 1, 2)$, which is obtained from w by restricting to $\{1, 2, 3\}$. The last entry of w_1 is 2, so the root of $T^{(1)}$ is the node labelled 2. Restricting $B^{(1)}$ to $\{1, 3\}$, we have two connected components, $\mathcal{B}|_{\{1\}}$ and $\mathcal{B}|_{\{3\}}$. The two subsequent trees obtained from these connected components are just single nodes, labelled 1 and 3. We connect these two trees to the root node, 2, to obtain $T^{(1)}$. Thus, we obtain the forest F, shown in Figure 11 as the extended $\mathcal{B}$-forest obtained under the map.

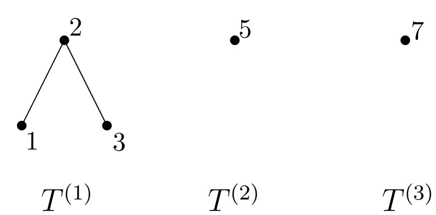


Figure 11: Extended $\mathcal{B}$-forest $F = \Psi_{\mathcal{B}}(w)$.

This brings us to the definition of $\mathcal{B}$-partial permutations. Such permutations provide a nice section of the surjection $\Psi_{\mathcal{B}}$.

Definition 6.15 (cf. [PRW08], Definition 8.7)**.** Let $\mathcal{B}$ be a building set on $[n]$ and $S = \{s_1, \ldots, s_k\} \subseteq [n]$. Define the set $\mathfrak{P}_n(\mathcal{B}|_S)$ of **partial $\mathcal{B}$-permutations on** S as the set of partial permutations $w \in \mathfrak{S}_S$ such that for any $i \in [k]$, the elements $w(s_i)$ and $\max\{w(s_1), w(s_2), \ldots, w(s_i)\}$ lie in the same connected component of the restricted building set $\mathcal{B}|_{\{w(s_1),\ldots,w(s_i)\}}$.

Let $\mathfrak{P}_n(\mathcal{B}) \coloneqq \bigcup_{S\subseteq[n]} \mathfrak{P}_n(\mathcal{B}|_S)$ be the set of **$\mathcal{B}$-partial permutations**.

Note that when $S = [n]$, the set of partial $\mathcal{B}$-permutations on S is exactly the set of non-extended **$\mathcal{B}$-permutations** (see [PRW08, Definition 8.7]. Next, we provide the following lemma which characterizes when a partial permutation is a $\mathcal{B}$-partial permutation; this is an extension of [PRW08, Lemma 8.8], which characterizes when a permutation is a $\mathcal{B}$-permutation.

Lemma 6.16 (cf. [PRW08], Lemma 8.8)**.** Suppose $S = \{s_1 < \cdots < s_k\} \subseteq [n]$ and $\mathcal{B}$ is a connected building set on $[n]$. A permutation $w \in \mathfrak{S}_S$ is a $\mathcal{B}$-partial permutation on S if and only if it can be constructed via Algorithm 1.

Algorithm 1: $\mathcal{B}$-partial permutation procedure.

Data: Connected building set $\mathcal{B}$ on $[n]$ and $S = \{s_1 < \cdots < s_k\} \subset [n]$
Result: $\mathcal{B}$-partial permutation $w = (w(s_1), w(s_2), \dots, w(s_k)) \in \mathfrak{S}_S$

$m := s_k$;
$\mathcal{C} :=$ connected component of $\mathcal{B}|_S$ containing m;
Pick $w(s_k)$ from $\mathcal{C}$;
for *i=k-1,k-2,...,1* **do**
  $m = \max\{S \setminus \{w(s_k), w(s_{k-1}), \dots, w(s_{i+1})\}\}$;
  $\mathcal{C} =$ connected component of $\mathcal{B}|_{S\setminus\{w(s_k),w(s_{k-1}),\dots,w(s_{i+1})\}}$ containing m;
  Pick $w(s_i)$ from $\mathcal{C}$;
end
return w;

Example 6.17. Again consider $\mathcal{B}_\Gamma$ with $\Gamma = P_4$ and $S = \{1, 2, 4\}$. We will construct w, a $\mathcal{B}$-partial permutation on S.

The restricted building set $\mathcal{B}|_S$ consists of two connected components, $\mathcal{B}|_{\{1,2\}}$ and $\mathcal{B}|_{\{4\}}$. Here, $s_k = 4$, so we must have that $w(4) = 4$. We then have two options for $w(s_{k-1}) = w(2)$, since there is only one connected component of $\mathcal{B}|_{S\setminus\{w(4)\}} = \mathcal{B}|_{\{1,2\}}$, and it contains two elements (including the maximal element of $S \setminus \{w(4)\}$. Let $w(2) = 2$. Then we must have that $w(1) = 1$. Thus, the partial permutation $w = (1, 2, 4)$ is a $\mathcal{B}$-partial permutation on S.

Let F be a collection of rooted trees on $S \subseteq [n]$, with $S = \{s_1 < \cdots < s_k\}$. Recall that one can view F itself as a poset. Then, the **lexicographically minimal linear extension** of F is the permutation $w \in \mathfrak{S}_S$ such that $w(s_1)$ is a leaf of a tree of F and minimal in the usual order of Z, $w(s_2)$ is the minimal leaf of $F \setminus \{w(s_1)\}$ (the forest F with the vertex $w(s_1)$ removed), $w(s_3)$ is the minimal leaf of $F \setminus \{w(s_1), w(s_2)\}$, etc.

One can also construct the lexicographically minimal linear extension of F in the following way. The following lemma generalizes [PRW08, Lemma 8.9], which shows the result when the poset is a single rooted tree.

Lemma 6.18 (cf. [PRW08], Lemma 8.9)**.** Let w be the lexicographically minimal linear extension of a rooted forest F on node set $S = \{s_1 < \cdots < s_k\} \subseteq [n]$. Then the permutation w can be constructed from F, as follows: $w(s_k)$ is the root of the connected component of F that contains the maximal vertex of this forest in the usual order on Z; $w(s_{k-1})$ is the root of the connected component of $F \setminus \{w(s_k)\}$ that contains the maximal vertex of this new forest, etc.

In general, $w(s_i)$ is the root of the connected component of the forest

$$F \setminus \{w(s_k), \dots, w(s_{i+1})\}$$

that contains the vertex $\max(w(s_1), \dots, w(s_i))$.

Proof. We induct on the number of vertices of the forest F. The base case, when F consists of a single vertex, is clear. Assume for the induction hypothesis that for any forest F' on $k-1$ nodes, its lexicographically minimal linear extension w' can be constructed as by the procedure above.

Let F' be the forest obtained from F by removing the minimal (in the usual order of Z) leaf ℓ. If w is the lexicographically minimal linear extension of F, then $w = (\ell, w')$, where w' is the lexicographically minimal linear extension of F' (where w and w' are written as lists). By the induction hypothesis, w' was constructed by the procedure from F'. Notice that when constructing F, for all $i > 1$, the vertex ℓ cannot be the root of the connected component of $F \setminus \{w(s_k), \dots, w(s_{i+1})\}$ that contains the maximal vertex. Thus, the backward procedure described above to obtain w from F produces the same permutation as $w = (\ell, w')$. □

We can now now show a correspondence between extended $\mathcal{B}$-forests and $\mathcal{B}$-partial permutations.

Proposition 6.19. Let $\mathcal{B}$ be a connected building set on $[n]$. The set $\mathfrak{P}_n(\mathcal{B})$ of $\mathcal{B}$-partial permutations is exactly the set of lexicographically minimal linear extensions of the extended $\mathcal{B}$-forests. This implies that the set of $\mathcal{B}$-partial permutations and the set of extended $\mathcal{B}$-forests are in bijection.

Proof. Let $w \in \mathfrak{S}_S \subseteq \mathfrak{P}_n$ with $S = \{s_1 < \cdots < s_k\} \subseteq [n]$, and let $F = F(w)$ be the corresponding extended $\mathcal{B}$-forest constructed, using Definition 6.13. Notice that for all $i = k, \ldots, 1$, the connected components of the forest $F|_{\{w(s_1),\ldots,w(s_i)\}}$ correspond to the connected components of the building set $\mathcal{B}|_{w(s_1),\ldots,w(s_i)\}}$, where corresponding components between the forest and the building set have the same vertex sets. By Lemma 6.18, the partial permutation w is the lexicographically minimal linear extension of F if and only if w is a $\mathcal{B}$-partial permutation constructred via Algorithm 1. □

Example 6.20. The $\mathcal{B}$-partial permutation on $S = \{1, 2, 4\}$ given in Example 6.17 is the lexicographically minimal linear extension of the extended $\mathcal{B}$-forest from Example 6.3.

We now describe the special class of building sets $\mathcal{B}$ for which the extended $\mathcal{B}$-forests and corresponding $\mathcal{B}$-partial permutations agree on their descent numbers. This will allow us to write the h-polynomial of the extended nestohedron $\mathcal{P}$ $(\mathcal{B})$ as the descent generating function of the $\mathcal{B}$-partial permutations.

Definition 6.21. [PRW08, Definition 9.2] A building set $\mathcal{B}$ on $[n]$ is **chordal** if it satisfies the following condition: for any $I = \{i_1 < \cdots < i_r\} \in \mathcal{B}$ and $s = 1, \ldots, r$, the subset $\{i_s, i_{s+1}, \ldots, i_r\}$ also belongs to $\mathcal{B}$.

Note that if $\mathcal{B}$ is a chordal building set on $[n]$ and $S \subseteq [n]$, then $\mathcal{B}|_S$ is also chordal. The name for these building sets is justified by the fact that a graphical building set $\mathcal{B}_\Gamma$ is chordal if and only if Γ is chordal and is labelled in a particular way (see [PRW08], Proposition 9.4 for details). Thus, many nice families of building sets are chordal, such as the building sets $\mathcal{B}_\Gamma$ when Γ is a path graph, complete graph, or star graph.

It turns out that the extended nestohedron $\mathcal{P}$ $(\mathcal{B})$ is a flag simple polytope for $\mathcal{B}$ chordal, by the following lemma.

Lemma 6.22. If $\mathcal{B}$ is a chordal building set, then the extended nestohedron $\mathcal{P}$ $(\mathcal{B})$ is a flag simple polytope.

Proof. By [PRW08, Proposition 9.7], the non-extended nestohedron $\mathcal{P}(\mathcal{B})$ is a flag simple polytope. By Lemma 2.20, $\mathcal{P}$ $(\mathcal{B})$ is flag as well. □

By Theorem 5.16, we know that the γ-vector for this polytope is nonnegative, so it is plausible to give its γ-vector a combinatorial interpretation. In order to do so, we first have to find a combinatorial interpretation of the h-vector. We now give some definitions and technical results to relate extended $\mathcal{B}$-forests and $\mathcal{B}$-partial permutations for chordal building sets. This will allow us to prove our result about the h-polynomial in terms of $\mathcal{B}$-partial permutations.

Definition 6.23. Let $S = \{s_1 < \cdots < s_k\}$. A **descent** of a permutation $w \in \mathfrak{S}_S$ is a pair $(w(s_i), w(s_{i+1}))$ such that $w(s_i) > w(s_{i+1})$. Let $\mathrm{Des}(w)$ be the set of all descents in w, and $\mathrm{des}(w) := |\mathrm{Des}(w)|$.

Proposition 6.24 (cf. [PRW08], Proposition 9.5)**.** Let $\mathcal{B}$ be a connected chordal building set on $[n]$. Then, for any extended $\mathcal{B}$-forest F and the corresponding $\mathcal{B}$-partial permutation w, one has $\mathrm{Des}(w) = \mathrm{Des}(T)$.

Proof. Let F be an extended $\mathcal{B}$-forest with node set $S = \{s_1 < \cdots < s_k\}$, and let w the corresponding $\mathcal{B}$-partial permutation, which was constructed from F using the procedure given in Lemma 6.18. Fix $i \in \{1, 2, \ldots, k-1\}$, and consider the forest

$$F \setminus \{w(s_k), w(s_{k-1}), \ldots, w(s_{i+1})\}.$$

This forest consists of the subtrees $T_1, \ldots, T_r, T_1', \ldots, T_s'$, where the trees $T_1, \ldots, T_r$ have roots that were children of the node $w(s_{i+1})$ in the original forest F, while the trees $T_1', \ldots, T_s'$ are the remaining trees.

We will show that there is exactly one descent edge between $w(s_{i+1})$ and one of the roots of $T_1, \dots, T_r$ if and only if $w(s_i) > w(s_{i+1})$.

Let $m = \max\{w(s_1), \dots, w(s_i)\}$, and first suppose that $m \in T_j$ for some $1 \le j \le r$; without loss of generality, suppose $m \in T_1$. By Lemma 6.18, it must be that $w(s_i)$ is the root of T_1. We show that all of the vertices of the trees $T_2, \dots, T_r$ are less than $w(s_{i+1})$. If $I = F_{\le w(s_{i+1})} \subseteq S$, then notice that $I \in \mathcal{B}$, by the definition of an extended $\mathcal{B}$-forest. Then, if $w(s_{i+1})$ is the maximum element of I, it is definitely true that the vertices of $T_2, \dots, T_r$ are all less than $w(s_{i+1})$, since the node sets of $T_2, \dots, T_r$ are contained within I.

If $w(s_{i+1})$ is not the maximal element of I, then the set $I' = I \cap \{w(s_{i+1}) + 1, w(s_{i+1}) + 2, \dots, n\}$ is an element of $\mathcal{B}$, since $\mathcal{B}$ is chordal, and I' is nonempty since $m \in I'$. Notice that the vertex set of T_1, denote it by J, must be an element of the building set by the definition of the extended $\mathcal{B}$-forest, and since $m \in I'$ as well as J, it follows that $I' \subseteq J$. Thus, all of the nodes of the trees $T_2, \dots, T_r$ are less than $w(s_{i+1})$. This implies that the only possible descent edge between $w(s_{i+1})$ and a child would have to be between $w(s_{i+1})$ and $w(s_i)$, the root of T_1. Notice that this edge is a descent edge if and only if $w(s_i) > w(s_{i+1})$ which is exactly when a descent occurs in permutation w at indices i and $i+1$.

Now suppose that m is in one of the subtrees that is not a descendant of $w(s_{i+1})$, say T_1'. This would imply that $w(s_{i+1})$ is greater than all $w(s_1), \dots, w(s_i)$. If not, this would imply that $m > w(s_{i+1})$, and that $w(s_{i+1})$ would not have been chosen to be the $(i+1)$-st index of the permutation w, as per the algorithm of Lemma 6.18. Since $w(s_{i+1}) > w(s_j)$ for all $j = 1, \dots, i$, none of the edges connecting $w(s_{i+1})$ with the roots of $T_1, \dots, T_r$ could be descent edges and $w(s_i) < w(s_{i+1})$. □

Proposition 6.24 and Proposition 6.1 imply the following corollary.

Corollary 6.25. For a chordal building set $\mathcal{B}$ on $[n]$, the h-polynomial of $\mathcal{P}\ (\mathcal{B})$ equals

$$h_{\mathcal{P}\ (\mathcal{B})}(t) = \sum_{S \subseteq [n]} \sum_{w \in \mathfrak{P}_n(\mathcal{B}|_S)} t^{\mathsf{des}(w) + n - |S|},$$

where $\mathrm{des}(w)$ is the number of descents of w.

In order to simplify the formula for the h-polynomial given in Corollary 6.25 and not have a double sum, we define the following map from partial permutations to permutations.

Define the map $\varphi_n : \mathfrak{P}_n \to \mathfrak{S}_{n+1}$ as follows. For a permutation $w \in \mathfrak{S}_S \subseteq \mathfrak{P}_n$ with $S \subseteq [n]$, let $\varphi_n(w)$ be the permutation formed by appending $[n+1] \setminus S$ to the end of w in descending order. The map φ_n is an injection into $\mathfrak{S}_{n+1}$. Let $\mathfrak{S}_{n+1}(\mathcal{B}) := \varphi_n(\mathfrak{P}_n(\mathcal{B}))$ be the set of **extended $\mathcal{B}$-permutations**.

Notice that for $w \in \mathfrak{P}_n(\mathcal{B}_S)$,

$$\mathrm{des}(w) + n - |S| = \mathrm{des}(\varphi_n(w)),$$

since the number of descents occurring in the subword of $\varphi_n(w)$ beginning with the entry $n+1$ is exactly equal to $n - |S|$.

This allows us to rewrite the h-polynomial of $\mathcal{P}\ (\mathcal{B})$ as the descent-generating function over the set of extended $\mathcal{B}$-permutations when $\mathcal{B}$ is chordal:

$$h_{\mathcal{P}\ (\mathcal{B})}(t) = \sum_{w \in \mathfrak{S}_{n+1}(\mathcal{B})} t^{\mathsf{des}(w)},$$

thus proving Theorem 6.11

6.3 γ-Vector of Chordal Extended Nestohedra

Recall that the γ-vector of a d-dimensional simple polytope is given by the h-polynomial: $h(t) = \sum h_i t^i = \sum \gamma_i t^i (1+t)^{d-2i}$. If $\mathcal{B}$ is a chordal building set, then the extended nestohedron $\mathcal{P}\ (\mathcal{B})$ is flag by Lemma 6.22. We showed in the previous section that flag extended nestohedra have nonnegative

γ-vectors. It is therefore possible to give γ-vectors combinatorial interpretations for such polytopes. In this section, we find such an interpretation for the γ-vector of chordal extended nestohedra. To do so, we use the technique used in [PRW08], in which the analogous result for chordal nestohedra is shown.

The general outline of the technique is as follows. Suppose P is a d-dimensional simple polytope that has a h-polynomial that has a combinatorial interpretation of the form $h_P(t) = \sum_{a\in A} t^{f(a)}$, where $f(a)$ is some statistic on the set A. Then, we show that A can be partitioned into classes such that

$$\sum_{a\in C} t^{f(a)} = t^r(1+t)^{d-2r},$$

for each class $C \subseteq A$ and for some $r \in \mathsf{N}$. If $\widehat{A} \subseteq A$ denotes the set of representatives of each class where $f(a)$ takes its minimal value, then

$$\gamma_P(t) = \sum_{a\in\widehat{A}} t^{f(a)}.$$

For us, A is the set of extended $\mathcal{B}$-permutations, and $f(a)$ is the descent number for permutations. Like [PRW08] who define operations on $\mathcal{B}$-permutations, we will define a series of operations on extended $\mathcal{B}$-permutations that will allow us to partition such permutations, and then we will able to give the γ-polynomial as the descent-generating function over a subset of extended $\mathcal{B}$-permutations.

First, we give several preliminary definitions related to the "topography" of permutations. For $w \in \mathfrak{S}_{n+1}$, a **final descent** is when $w(n) > w(n+1)$; a **double descent** is pair of consecutive descents, i.e. a triple $w(i) > w(i+1) > w(i+2)$. A **peak** of w is an entry $w(i)$ for $1 \leq i \leq n+1$ such that $w(i-1) < w(i) > w(i+1)$, where here and for the rest of this section (unless otherwise specified), we set $w(0) = w(n+2) = 0$, so peaks can occur at indices 1 and $n+1$. A **valley** of w is an entry $w(i)$ for $1 < i < n$ such that $w(i-1) > w(i) < w(i+1)$. The **peak-valley sequence** of w is the subsequence in w formed by all peaks and valleys. An entry $w(i)$ is an **intermediary entry** if $w(i)$ is neither a peak nor a valley. We say that $w(i)$ is an **ascent-intermediary entry** if $w(i-1) < w(i) < w(i+1)$, and it is a **descent-intermediary entry** if $w(i-1) > w(i) > w(i+1)$.

One can graphically represent a permutation $w \in \mathfrak{S}_{n+1}$ as a "mountain range" M_w in the following way. Plot the points $(0,0), (1, w(1)), (2, w(2)), \ldots, (n+1, w(n+1)), (n+2, 0)$ on R^2, and then connect points by straight line intervals. Now, peaks in w correspond to local maxima of M_w, valleys correspond to local minima, and ascent-intermediary (resp. descent-intermediary) entries correspond to points that are on increasing (resp. decreasing) slopes of M_w.

Example 6.26. Consider the permutation $w = (2,4,1,6,5,3) \in \mathfrak{S}_6$. The mountain range M_w is shown in Figure 12. Notice that w has a final descent $(3,0)$ and two double descents, $(6,5,3)$ and $(5,3,0)$. It has peaks at 4 and 6, and a valley at 1. The descent-intermediary entries are 5 and 3, while the only ascent intermediary entry is 2.

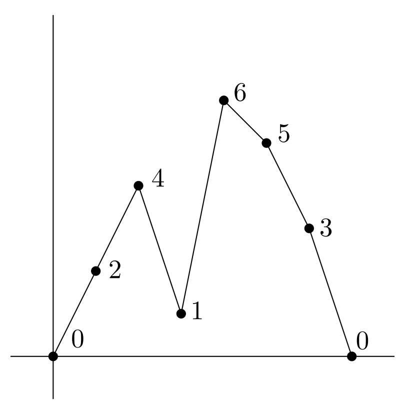
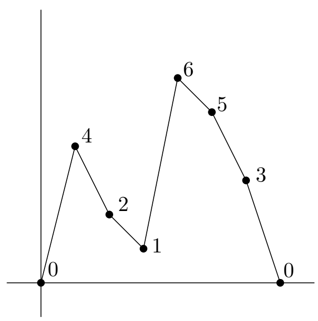


Figure 12: Mountain range M_w.

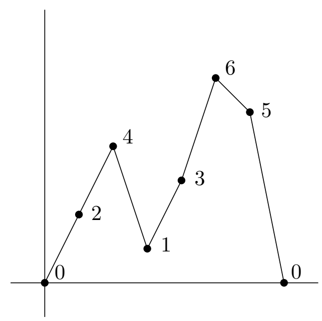
We now begin to define some operations on permutations. The **leap operations** L_a and L_a^-, as well as powers of leap operations, were first introduced [PRW08, Section 11.2], and they allow us to define operations on extended $\mathcal{B}$-permutations later on. The leap operations are defined as follows. The permutation $L_a(w)$ is obtained from permutation w by removing an intermediary node a from the i-th position in w and inserting a in the position between $w(j)$ and $w(j+1)$ where j is the minimal index such that $j > i$ and a is between $w(j)$ and $w(j+1)$. Similarly, the permutation $L_a^-(w)$ is obtained from removing a from the i-th position and inserting a between $w(k)$ and $w(k+1)$ where k is the maximum index such that $k < i$ and a be between $w(k)$ and $w(k+1)$.

Informally, if $w \in \mathfrak{S}_{n+1}$ and a is an intermediary entry, then the permutation $L_a(w)$ is obtained from w by moving an intermediary point a on the mountain range M_w directly to the right until it hits the next slope of M_w. Likewise, the permutation $L_a^-(w)$ is obtained from w by moving a directly to the left until it hits the next slope of M_w.

Example 6.27. Again consider the permutation $w = (2, 4, 1, 6, 5, 3) \in \mathfrak{S}_6$ from Example 6.26. The permutation $L_2(w)$ is obtained by moving node 2 to the right until hitting the next slope, so the resulting permutation is $L_2(w) = (4, 2, 1, 6, 5, 3)$. The mountain range for the this permutation is shown in Figure 13(a). The permutation $L_3^-(w)$ is obtained by moving node 3 to the left until hitting the next slope, so the resulting permutation is $L_3^-(w) = (2, 4, 1, 3, 6, 5)$. The mountain range for this permutation is shown in Figure 13(b).



(a) $L_2(w)$



(b) $L_3^-(w)$

Figure 13: Examples of leaps on w.

Next, we define powers of the leap operations:

$$L_a^r := \begin{cases} (L_a)^r, & \text{for } r \in \mathsf{Z}_{\geq 0}, \\ (L_a^-)^{-r}, & \text{for } r \in \mathsf{Z}_{\leq 0}. \end{cases}$$

In words, if r is positive, then for a permutation w and intermediary entry a, the permutation $L_a^r(w)$ is obtained from w by moving a to the right until it hits the r^{th} slope from its original slope; if r is negative, then the permutation $L_a^r(w)$ is obtained from w by moving a to the left until it hits the $-r^{\text{th}}$ slope from its original slope.

Notice that $L_a^r(w)$ is only defined whenever r is in a certain integer interval, since there are finitely many slopes to the left and to the right of a's current slope. Let $[r_{\min}, r_{\max}]$ denote this interval. In addition, if a is an ascent-intermediary entry in w, then a is still ascent-intermediary in $L_a^r(w)$ for even r, but it is descent-intermediary for odd r. Similarly, if a is a descent-intermediary entry in w, then a is still descent intermediary in $L_a^r(w)$ for even r, but it is ascent-intermediary for odd r.

We now want to ensure that if $w \in \mathfrak{S}_{n+1}(\mathcal{B})$ is an extended $\mathcal{B}$-permutation, then there exists some r such that $L_a^r(w)$ is an extended $\mathcal{B}$-permutation as well. To do so, we give the following reformulation of extended $\mathcal{B}$-permutations for chordal building sets.

For a permutation $w \in \mathfrak{S}_{n+1}$ and $a \in [n+1]$ with $w(i) = a$, let

$$\{w \nwarrow a\} := \{w(j) \mid j \leq i, w(j) \geq a\}$$

be the set of entries in w that occur before a but are greater than or equal to a; this set includes a itself. In the graph M_w, the set $\{w \nwarrow a\}$ is the set of entries of w that are located above and to the left of the point corresponding to a. In addition, if $n+1 = w(k)$, then let w denote the subword $(w(1), \dots, w(k-1))$.

By Definition 6.15 and our definition for $\mathfrak{S}_{n+1}(\mathcal{B})$, the set $\mathfrak{S}_{n+1}(\mathcal{B})$ is the set of permutations w such that for all $i = 1, \dots, |w|$, there exists $I \in \mathcal{B}$ such that both $w(i)$ and $\max\{w(1), \dots, w(i)\}$ are elements of I, and $I \subseteq \{w(1), \dots, w(i)\}$. If $\mathcal{B}$ is chordal, then $I' := I \cap [w(i), \infty]$ is an element of $\mathcal{B}$. This implies that $w(i)$ and $\max\{w(1), \dots, w(i)\} \in I'$ and $I' \subseteq \{w(1), \dots, w(i)\}$. Also notice that $\max\{w(1), \dots, w(i)\} = \max\{w \nwarrow w(i)\}$. Thus, we can reformulate our definition for $\mathfrak{S}_{n+1}(\mathcal{B})$ if $\mathcal{B}$ is a chordal building set. A similar reformulation for non-extended $\mathcal{B}$-permutations is given in [PRW08, Lemma 11.8].

Definition 6.28. Let $\mathcal{B}$ be a chordal building set on $[n]$. Then the set of extended $\mathcal{B}$-permutations $\mathfrak{S}_{n+1}(\mathcal{B})$ is the set of permutations $w \in \mathfrak{S}_{n+1}$ such that for any $a \in w$, the elements a and $\max\{w \nwarrow a\}$ are in the same connected component of $\mathcal{B}|_{\{w \nwarrow a\}}$. In other words, there exists $I \in \mathcal{B}$ such that for all $a \in I$, we have that $\max\{w \nwarrow a\} \in I$ and $I \subseteq \{w \nwarrow a\}$.

If $w \in \mathfrak{S}_{n+1}(\mathcal{B})$ and a is an intermediary entry of w, then there are 2 possible reasons why the permutation $u = L_a^r(w)$ may no longer be an element of $\mathfrak{S}_{n+1}(\mathcal{B})$:

A) if $a \in u$, and the entries a and $\max\{u \nwarrow a\}$ are in different connected components of $\mathcal{B}|_{\{u \nwarrow a\}}$, i.e. there does not exist an element $I \in \mathcal{B}|_{\{u \nwarrow a\}}$ such that $a, \max\{u \nwarrow a\} \in I$, or

B) if another entry $b \neq a$ is in u and $\max\{u \nwarrow b\}$ are in different connected components of $\mathcal{B}|_{\{u \nwarrow b\}}$.

We call these two types of failures **A-failure** and **B-failure**. Note that these terms have slightly different definitions given in [PRW08], as they are using them in the context of non-extended $\mathcal{B}$-permutations.

Next, we have the following technical lemma on when A- and B-failures can and cannot occur.

Lemma 6.29 (cf. [PRW08], Lemma 11.9)**.** Suppose $w \in \mathfrak{S}_{n+1}(\mathcal{B})$ and a is an intermediary entry of w.

1) For left leaps $u = L_a^r(w), r < 0$, one can never have a B-failure.

2) For the maximal left leap $u = L_a^{r_{\min}}(w)$, one cannot have an A-failure.

3) For the maximal right leap $u = L_a^{r_{\max}}(w)$, one cannot have an A-failure.

4) Let $u = L_a^r(w)$ and $u' = L_a^{r+1}(w)$ be two adjacent leaps such that a is descent-intermediary in u (implying that a is ascent-intermediary in u'). Then there is an A-failure in u if and only if there is an A-failure in u'.

Proof. 1) Let $b \neq a$ be an entry of w. Then, since $w \in \mathfrak{S}_{n+1}(\mathcal{B})$, there exists a subset $I \in \mathcal{B}$ such that $b, \max\{w \nwarrow b\} \in I$ and $I \subseteq \{w \nwarrow b\}$. The same subset I works for u since $\{u \nwarrow b\} = \{w \nwarrow b\}$ or $\{u \nwarrow b\} = \{w \nwarrow b\} \cup \{a\}$, and $a \neq \max\{u \nwarrow b\}$, since this would imply a became a peak under the operation, which is impossible.

2) In this case, a is greater than all preceding entries in u, so $a = \max\{u \nwarrow a\}$. Thus, the singleton element $\{a\}$ satisfies the necessary conditions for an element of $\mathcal{B}$.

3) Notice that a becomes an element after $n+1$, i.e. $a \notin u$, so it is impossible to consider an A-failure for a.

4) All of the entries between a in u and a in u' are less than a, so $\{u \nwarrow a\} = \{u' \nwarrow a\}$. Thus, u has an A-failure if and only if u' has an A-failure.

□

This next lemma guarantees that for an extended $\mathcal{B}$-permutation w and intermediary entry a, there exists at least some $r \in [r_{\min}, r_{\max}]$ such that $L_a^r(w)$ is still an extended $\mathcal{B}$-permutation.

Lemma 6.30 (cf. [PRW08], Lemma 11.7)**.** Let $\mathcal{B}$ be a chordal building set on $[n]$. Suppose that $w \in \mathfrak{S}_{n+1}(\mathcal{B})$.

1) If a is an ascent-intermediary entry in w, then there exists an odd positive integer $r > 0$ such that $L_a^r(w) \in \mathfrak{S}_{n+1}(\mathcal{B})$ and $L_a^s(w) \notin \mathfrak{S}_{n+1}(\mathcal{B})$ for all $0 < s < r$.

2) If a is an descent-intermediary entry in w, then there exists an odd negative integer $r < 0$ such that $L_a^r(w) \in \mathfrak{S}_{n+1}(\mathcal{B})$ and $L_a^s(w) \notin \mathfrak{S}_{n+1}(\mathcal{B})$ for all $r < s < 0$.

Proof. 1) We will show that there exists a permutation u without a B-failure, and then that among permutations without B-failures, there exists one without an A-failure. First suppose that there exists an entry $b \neq a$ in the permutation w such that $b \in w$, and that b and $m = \max\{w \nwarrow b\}$ are in different connected components of $\mathcal{B}|_{\{w \nwarrow b\} \setminus \{a\}}$. It cannot be that $a \notin \{w \nwarrow b\}$, since this would imply that b and m are in different connected components of $\mathcal{B}|_{\{w \nwarrow b\}}$, contradicting the fact that w is an extended $\mathcal{B}$-permutation. Thus, we have that $a \in \{w \nwarrow b\}$, implying that $b < a$ and that a is to the left of b in w. Let b be the leftmost entry of w that satisfies the hypothesis for this case. Consider the permutation $u = L_a^t(w)$ for some $t > 0$. If entry a is to the right of b in permutation u, then u would have a B-failure; if entry a stays to the left of b in u, then there would not be a B-failure. Since a is ascent-intermediary and $b < a$, we have that a stays to the left of b in $L_a^1(w)$, so a permutation u without a B-failure exists in this case.

Let $u = L_a^t(w)$ be the maximal right leap such that entry a stays to the left of b, so $t > 0$ is maximized; we have that $a, b \in u$. Notice that all entries in u that are between the indices of a and b are less than a; otherwise, t would not be maximal. Then, $m = \max\{u \nwarrow a\} = \max\{w \nwarrow b\}$. Since $w \in \mathfrak{S}_{n+1}(\mathcal{B})$, there exists an $I \in \mathcal{B}$ such that $b, m \in I$ and $I \subseteq \{w \nwarrow b\}$. Notice that we chose b to not be in the same connected component as m in $\mathcal{B}|_{\{w \nwarrow b\} \setminus \{a\}}$, so we necessarily have that $a \in I$ as well. Then, $I' := I \cap [a, \infty]$ is an element of $\mathcal{B}$, with $a, m \in I'$ and $I' \subseteq \{u \nwarrow a\}$, implying that there is no A-failure in u and so $u \in \mathfrak{S}_{n+1}(\mathcal{B})$.

If no such entry b in w as above exists, then none of the permutations $L_a^r(w)$ have B-failures. By 3) of Lemma 6.29, the permutation $L_a^{r_{\max}}(w)$ does not have an A-failure, so it is an extended $\mathcal{B}$-permutation.

In both cases, there exists a positive integer r such that $L_a^r(w) \in \mathfrak{S}_{n+1}(\mathcal{B})$ and for all $0 < s < r$, only A-failures are possible for $L_a^s(w)$. Pick r to be minimal. Then, r must be such that a is descent-intermediary in $L_a^r(w)$; otherwise, we could have chosen $r - 1$ by 4) of Lemma 6.29. Thus, r must be odd.

2) By parts 1) and 2) of Lemma 6.29, there necessarily exists a negative r such that $L_a^r(w) \in \mathfrak{S}_{n+1}(\mathcal{B})$, namely $r = r_{\min}$. Choose r with minimal possible absolute value such that $L_a^r(w) \in \mathfrak{S}_{n+1}(\mathcal{B})$. Notice that r must necessarily be odd. If not, then a would be descent intermediary in $L_a^r(w)$ and there would be no A-failure; by 4) of Lemma 6.29, this would imply that $L_a^{r+1}(w)$ would also have no A-failure, so $L_a^{r+1}(w) \in \mathfrak{S}_{n+1}(\mathcal{B})$ with $|r+1| < |r|$, contradicting minimality of the absolute value of r.

□

We are now able to define an operation that turns one extended $\mathcal{B}$-operation into another. The analogous version of this in [PRW08] is called the $\mathcal{B}$-hop operation, denoted $\mathcal{B}H_a$.

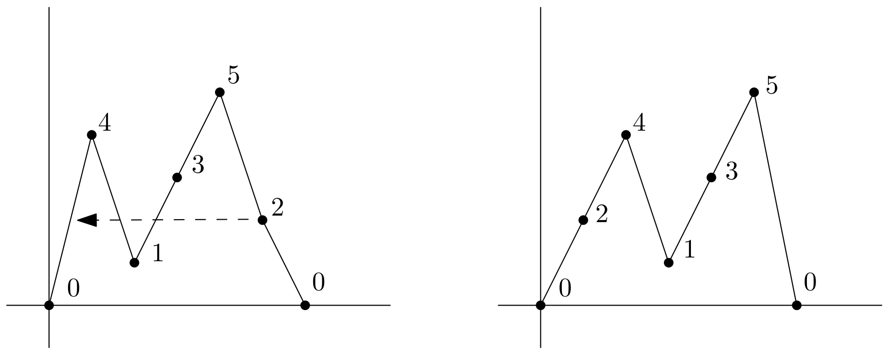
Definition 6.31 (cf. [PRW08] Definition 11.10)**.** An **extended $\mathcal{B}-$hop operations** $\mathcal{B}H_a$ is defined as follows. For $w \in \mathfrak{S}_{n+1}(\mathcal{B})$ with an ascent-intermediary (resp., descent-intermediary) entry a, the permutation $\mathcal{B}H_a(w)$ is the right leap $u = L_a^r(w)$, with $r > 0$ (resp., the left leap $u = L_a^r(w)$ with $r < 0$) with minimal possible $|r|$ such that $u \in \mathfrak{S}_{n+1}(\mathcal{B})$.

Informally, the permutation $\mathcal{B}H_a(w)$ is obtained from w by moving the point a on the graph M_w to the right if a is ascent-intermediary in w, or to the left if a is descent-intermediary, until a hits a slope such that the new permutation is an extended $\mathcal{B}$-permutation. It is possible that the point a passes through several slopes.

Example 6.32. Consider the building set

$$\mathcal{B} = \{\{1\},\{2\},\{3\},\{4\},\{1,4\},\{3,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}\}.$$

An example of an extended $\mathcal{B}$-permutation is $w = (4,1,3,5,2)$. The extended $\mathcal{B}$-hop on w for the entry 2 is $\mathcal{B}H_2\,(w) = L_2^{-3}(w)$; we require three left leaps for this extended $\mathcal{B}$-hop operation, since neither $L_2^{-1}(w)$ nor $L_2^{-1}(w)$ are extended $\mathcal{B}$-permutations. Thus, $\mathcal{B}H_2\,(w)$ corresponds to moving the node 2 to the left, passing through two slopes.



(a) w (b) $\mathcal{B}H_2\,(w)$

Figure 14: Extended $\mathcal{B}$-hop on entry 2 of w.

Lemma 6.33. For any extended $\mathcal{B}$-permutation w and intermediary entry a of w, the extended $\mathcal{B}$-hop operation $\mathcal{B}H_a\,(w)$ is well-defined. If a is an ascent-intermediary entry in w, then a is descent-intermediary in $\mathcal{B}H_a\,(w)$; if a is descent-intermediary in w, then a is ascent-intermediary in $\mathcal{B}H_a\,(w)$. In addition, $(\mathcal{B}H_a\,)^2(w) = w$.

Proof. All follows from Lemma 6.30. □

This next result shows that the extended $\mathcal{B}$-hop operations pairwise commute with each other. A similar statement for non-extended $\mathcal{B}$-hop operations is given in [PRW08], and the proof for our extended case follows in the same way as the non-extended case.

Lemma 6.34 (cf. [PRW08] Lemma 11.12)**.** Let w be an extended $\mathcal{B}-$permutation with two intermediary entries a and b. Then $\mathcal{B}H_a\,(\mathcal{B}H_b\,(w)) = \mathcal{B}H_b\,(\mathcal{B}H_a\,(w))$.

For a set of extended $\mathcal{B}$-permutations with the same peak-valley sequence, the extended $\mathcal{B}$-hop operations $\mathcal{B}H_a$ generate the action of the group $(\mathsf{Z}/2\mathsf{Z})^m$, where m is the number of intermediary entries in any permutation of this set. We say that two extended $\mathcal{B}-$permutations are **extended $\mathcal{B}$-hop equivalent** if they can be obtained from each other via a series of extended $\mathcal{B}$-hop operations $\mathcal{B}H_a$ for various intermediary entries a. This allows us to partition the set of extended $\mathcal{B}$-permutations into extended $\mathcal{B}$-hop equivalence classes. Notice that each class has exactly one permutation with no descent-intermediary entries since, if $w \in \mathfrak{S}_{n+1}(\mathcal{B})$ had a descent-intermediary entry a, then one can always apply the extended $\mathcal{B}$-hop operation to make it an ascent-intermediary entry.

Let $\widehat{\mathfrak{S}}_{n+1}$ denote the set of permutations in $\mathfrak{S}_{n+1}$ that do not have any final descents or double descents. Notice that this is equivalent to the set of permutations that have no descent-intermediary entries. For $\mathcal{B}$ a chordal building set on $[n]$, let $\widehat{\mathfrak{S}}_{n+1}(\mathcal{B}) := \widehat{\mathfrak{S}}_{n+1} \cap \mathfrak{S}_{n+1}(\mathcal{B})$.

We are now able to give the combinatorial interpretation for the γ-vector for chordal building sets.

Theorem 6.35 (cf. [PRW08], Theorem 11.6)**.** For a connected chordal building set $\mathcal{B}$ on $[n]$, the γ-polynomial of the extended nestohedron $\mathcal{P}_{\mathcal{B}}$ is the descent-generating function for the permutations in $\widehat{\mathfrak{S}}_{n+1}(\mathcal{B})$:

$$\gamma_{\mathcal{P}\ (\mathcal{B})}(t) = \sum_{w \in \widehat{\mathfrak{S}}_{n+1}(\mathcal{B})} t^{\mathsf{des}(w)}.$$

Proof. For an extended $\mathcal{B}$-permutation $w \in \widehat{\mathfrak{S}}_{n+1}(\mathcal{B})$, the descent generating function of the extended $\mathcal{B}$-hop-equivalence class C of w is

$$\sum_{u \in C} t^{\mathsf{des}(u)} = t^{\mathsf{des}(w)}(t-1)^{n-2\,\mathsf{des}(w)}.$$

Each extended $\mathcal{B}$-hop equivalence class has exactly one representative without descent-intermediary entry in the set $\widehat{\mathfrak{S}}_{n+1}$. Thus the h-polynomial of the extended nestohedron $\mathcal{P}_{\mathcal{B}}$ is

$$h_{\mathcal{P}\ (\mathcal{B})}(t) = \sum_{w \in \mathfrak{S}_{n+1}(\mathcal{B})} t^{\mathsf{des}(w)} = \sum_{w \in \widehat{\mathfrak{S}}_{n+1}(\mathcal{B})} t^{\mathsf{des}(w)}(t-1)^{n-2\,\mathsf{des}(w)}.$$

By definition of the γ-polynomial, we are done. □

Thus, we have a nice combinatorial interpretation of the γ-polynomial of extended chordal nestohedra in terms of the descent number of our special class of extended $\mathcal{B}$-permutations.

7 A Weak Order on Partial Permutations

In this section, we define a partial order on partial permutations, $\mathfrak{P}_n$. We show that this poset can be realized as the lattice quotient of the weak Bruhat order on the symmetric group $\mathfrak{S}_{n+1}$, which provides several nice properties about our partial order on $\mathfrak{P}_n$. We then by showing that any linear extension of the partial order on $\mathfrak{P}_n$ gives a shelling of the stellohedron. In addition, we define partial orders on maximal non-extended and extended nested collections whose Hasse diagrams can be realized by an acyclic directed graph on the 1-skeleta of nestohedra and extended nestohedra. When this poset is a lattice, it has further nice properties.

First, we give some preliminary definitions on partially ordered sets (posets). Let P be a poset. If $u \le v$ in P and $u \le z \le v$ implies that $u = z$ or $z = v$, then $u \lessdot v$ is a **cover relation**. A poset P is a **lattice** if for any pair of elements $x, y \in P$, there exist a unique least upper bound and a unique greatest lower bound for x and y which is also contained in P. The former is called the **join** of x and y, denoted $x \vee y$, and the latter is called the **meet** of x and y, denoted $x \wedge y$. An **interval** $[u, v]$ of the poset P is a subposet of elements $z \in P$ such that $u \le z \le v$. For any poset P, the **dual poset**, denoted P^*, is the poset with $u \le v$ in P^* if and only if $v \le u$ in P.

Given a lattice L, a **congruence** Θ of L is an equivalence relation on the elements of L that respects meets and joins in the following sense: If $a_1 \equiv a_2$ and $b_1 \equiv b_2$, then $a_1 \wedge b_1 \equiv a_2 \wedge b_2$ and similarly for joins. Given a lattice L and congruence Θ, define a partial order on the equivalence classes by $[a]_\Theta \le [b]_\Theta$ if and only if there exists $x \in [a]_\Theta$ and $y \in [b]_\Theta$ such that $x \le_L y$. The set of equivalence classes under this partial order is called the **lattice quotient** of L with respect to Θ, and is denoted by L/Θ.

Next, we define the weak Bruhat order on the symmetric group $\mathfrak{S}_n$. For any permutation $\pi \in \mathfrak{S}_n$, let

$$\operatorname{inv}(\pi) := \{(i, j) \mid 1 \le i < j \le n \text{ and } \pi(i) > \pi(j)\},$$

denote the **inversion set** of π. The **weak Bruhat order** on $\mathfrak{S}_n$ is the partial order given by containment of inversion sets: $\pi \le \sigma$ if and only if $\operatorname{inv}(\pi) \subseteq \operatorname{inv}(\sigma)$. It is well-known that the weak Bruhat order on $\mathfrak{S}_n$ is a lattice.

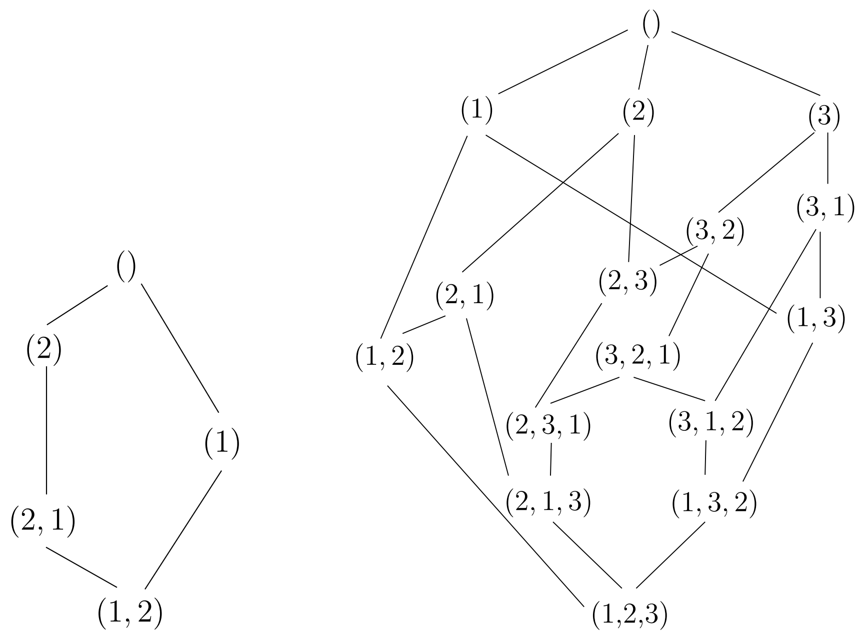
Recall that a **partial permutation** $\pi \in \mathfrak{P}_n$ is an ordered sequence $\pi = (a_1, \ldots, a_r)$ with $a_i \in [n]$ for all i, and $r = 0, \ldots, n$. If $r = 0$, then π is the empty permutation, which we denote by $()$. Recall the injective map

$$\varphi_n : \mathfrak{P}_n \to \mathfrak{S}_{n+1}.$$

Let $\widetilde{\pi} := \varphi_n(\pi)$. We can use the weak Bruhat order on $\mathfrak{S}_{n+1}$ to induce a partial order on $\mathfrak{P}_n$ as follows.

Definition 7.1. Let $\pi, \sigma \in \mathfrak{P}_n$ be two partial permutations. We say that $\pi \le \sigma$ in the **partial weak Bruhat order** if and only if $\widetilde{\pi} \le \widetilde{\sigma}$ in the weak Bruhat order on $\mathfrak{S}_{n+1}$.

Example 7.2. Consider $\pi = (2,1,4,3)$, $\sigma = (4,2) \in \mathfrak{P}_4$. Then, $\widetilde{\pi} = (2,1,4,3,5)$ and $\widetilde{\sigma} = (4,2,5,3,1)$. Since $\widetilde{\pi} \leq \widetilde{\sigma}$ in the weak Bruhat order on $\mathfrak{S}_5$, we have that $\pi \leq \sigma$ in $\mathfrak{P}_4$.



(a) $\mathfrak{P}_2$ (b) $\mathfrak{P}_3$

Figure 15: Partial weak Bruhat order on $\mathfrak{P}_2$ and $\mathfrak{P}_3$.

We now show our main result on the partial weak order.

Theorem 7.3. The partial order on $\mathfrak{P}_n$ defined above is a lattice quotient of the weak Bruhat order on $\mathfrak{S}_{n+1}$.

Proof. Let $\phi : \mathfrak{S}_{n+1} \to \mathfrak{P}_n$ denote the map given by restricting π to the entries appearing before $n+1$. Note that $\widetilde{\phi(\pi)} \geq \pi$, and that if $\pi \leq \sigma$ in $\mathfrak{S}_{n+1}$, then $\phi(\pi) \leq \phi(\sigma)$ in $\mathfrak{P}_n$.

Define the congruence Θ of $\mathfrak{S}_{n+1}$ by $a \equiv b$ if and only if $\phi(a) = \phi(b)$. It suffices to show that for any $\pi, \sigma \in \mathfrak{S}_{n+1}$, we have that $\phi(\pi \wedge \sigma) = \phi(\pi) \wedge \phi(\sigma)$ and $\phi(\pi \vee \sigma) = \phi(\pi) \vee \phi(\sigma)$. Note that since the weak Bruhat order on $\mathfrak{S}_{n+1}$ is a lattice, the $\pi \wedge \sigma$ and $\pi \vee \sigma$ are uniquely defined, but we do not yet know what meets and joins are in $\mathfrak{P}_n$.

By the definitions of meet and join:

$$\pi \wedge \sigma \leq \pi, \sigma \leq \pi \vee \sigma,$$

and so $\phi(\pi \wedge \sigma)$ and $\phi(\pi \vee \sigma)$ are indeed lower and upper bounds for $\phi(\pi)$ and $\phi(\sigma)$. Now suppose that $\rho \in \mathfrak{P}_n$ is such that $\phi(\pi), \phi(\sigma) \leq \rho$. Then:

$$\pi \leq \widetilde{\phi(\pi)} \leq \widetilde{\rho} \text{ and } \sigma \leq \widetilde{\phi(\sigma)} \leq \widetilde{\rho}.$$

We conclude that $\widetilde{\rho} \geq \pi \vee \sigma$, by the definition of join. Therefore, we have $\rho = \phi(\widetilde{\rho}) \geq \phi(\pi \vee \sigma)$, and so $\phi(\pi \vee \sigma)$ is the least upper bound (join) of $\phi(\pi)$ and $\phi(\sigma)$.

Next, $\phi(\pi \wedge \sigma)$ is a permutation on the set $[n] \setminus \{i \mid \pi(i) \geq \pi(n+1), \sigma(i) \geq \sigma(n+1)\}$, which is the union of the two sets which $\phi(\pi), \phi(\sigma)$ act on. Then note that $\phi(\pi \hat{\wedge} \sigma) = \widetilde{\phi(\pi)} \wedge \widetilde{\phi(\sigma)}$. Therefore if $\rho \leq \phi(\pi), \phi(\sigma)$, then $\rho \leq \phi(\pi \hat{\wedge} \sigma)$, so $\rho \leq \phi(\pi \wedge \sigma)$. This completes the proof that ϕ respects both meets and joins; it is also surjective, so it gives a lattice quotient. $\square$

The following is an immediate corollary.

Corollary 7.4. The partial weak Bruhat order on $\mathfrak{P}_n$ is a lattice.

Remark 7.5. In [BM18], Barnard and McConville define a partial order L_G on maximal nested collections for the graphical building set $\mathcal{B}_G$ that we extend in the following subsection. We note that the partial weak Bruhat order on $\mathfrak{P}_n$ is the dual poset of the poset L_G, where G is the star graph $K_{1,n}$. In particular, the partial permutation π corresponds to the maximal nested collection of $\mathcal{B}_G$ with $\mathcal{B}$-permutation $(a_1, \ldots, a_{n+1})$, where $\widetilde{\pi} = (a_{n+1}, \ldots, a_1)$.

We also note that although the weak Bruhat order on $\mathfrak{S}_{n+1}$ is semi-distributive, congruence normal, and congruence uniform, the partial weak Bruhat order on $\mathfrak{P}_n$ does not have any of these properties. However, Theorem 7.3 provides us with some nice corollaries about the partial order on $\mathfrak{P}_n$. The first comes from the fact that $\mathfrak{S}_n$ is a **crosscut-simplicial lattice**. McConville showed that the lattice quotient of a crosscut-simplicial lattice is also crosscut-simplicial [McC17, Theorem 1]. In addition, a crosscut-simplicial lattice has every interval either contractible or homotopy equivalent to a sphere. Thus, we have the following result.

Corollary 7.6. The partial order on $\mathfrak{P}_n$ is a crosscut-simplicial lattice. In particular, every interval is either contractible or homotopy equivalent to a sphere.

Next, using the congruence Θ, define the projection $\pi_\downarrow : \mathfrak{S}_{n+1} \to \mathfrak{S}_{n+1}$ by mapping an element $w \in \mathfrak{S}_{n+1}$ to the minimal element in the equivalence class of w under Θ. One can identify $\mathfrak{P}_n = \mathfrak{S}_{n+1}/\Theta$ with the induced subposet $Q = \pi_\downarrow(\mathfrak{S}_{n+1})$. Then, we have the following corollary by [Rea02, Proposition 26].

Corollary 7.7. If $x = \vee^Q Y$ for some $Y \subseteq Q$, then $x = \vee^{\mathfrak{S}_{n+1}} Y$. If $x = \vee^{\mathfrak{S}_{n+1}} Y$ for some $Y \subseteq \mathfrak{S}_{n+1}$, then $\pi_\downarrow(x) = \vee^Q \pi_\downarrow(Y)$.

Recall the surjective map $\Psi_\mathcal{B}$ from all partial permutations to extended $\mathcal{B}$-forests (Definition 6.13). Using this map, we can construct a maximal extended nested set of $\mathcal{B}$ from a partial permutation $w = (w(s_1), w(s_2), \ldots, w(s_r)) \in \mathfrak{P}_n$, where $S = \{s_1 < \cdots < s_r\} \subseteq [n]$. For $i = 1, \ldots, r$, let

$$\mathcal{B}_i \coloneqq \mathcal{B}|_{\{w(s_1), \ldots, w(s_{r-i+1})\}}.$$

Then the elements of the building set that make up the corresponding maximal extended nested collection is

$$N_w \coloneqq \{C_{w,k} \in \mathcal{B}_k \mid w(s_{r-k+1}) \in C_{w,k} \text{ and } C_{w,k} \text{ maximal with respect to inclusion, for } k = 1, \ldots, r\},$$

and the maximal extended nested collection itself is

$$F_w \coloneqq N_w \cup \{x_j \mid j \in [n] \setminus S\}.$$

Lemma 7.8. Let $\mathcal{B} = \mathcal{B}_{K_n}$. For two partial permutations $\pi, \sigma \in \mathfrak{P}_n$, let F_π and F_σ be the corresponding maximal extended nested collections associated to π and σ. If $\pi \lessdot \sigma$ in the partial weak Bruhat order, then F_π and F_σ differ by exactly one element.

Proof. Suppose $\pi, \sigma \in \mathfrak{P}_n$ such that $\pi \lessdot \sigma$ in the partial weak Bruhat order. Then either

1. $\pi = (a_1, a_2, \ldots, a_r)$ and $\sigma = (a_1, \ldots, a_{i-1}, a_{i+1}, a_i, \ldots, a_r)$ with $a_i < a_{i+1}$ for some $1 \le i \le r-1$, or

2. $\pi = (a_1, a_2, \ldots, a_{r-1}, a_r)$ and $\sigma = (a_1, a_2, \ldots, a_{r-1})$.

If we are in the first case, then there are only two pairs of elements F_π and F_σ that could possibly differ: the pair $C_{\pi,r-i}$ and $C_{\sigma,r-i}$, and the pair $C_{\pi,r-i+1}$ and $C_{\sigma,r-i+1}$. For all other corresponding pairs, the considered restricted building sets as well as the element needed to be included in the subset added the extended nested collection are the exact same.

When $k = r - i$, the building set considered for finding $C_{\pi,k}$ and $C_{\sigma,k}$ is

$$\mathcal{B}|_{\{a_1,\ldots,a_{i-1},a_i,a_{i+1}\}} = \mathcal{B}|_{\{a_1,\ldots,a_{i-1},a_{i+1},a_i\}}.$$

Notice that $C_{\pi,k}$ must contain a_{i+1} and $C_{\sigma,k}$ must contain a_i. Since the original building set is $\mathcal{B}_{K_n}$, every subset of $[n]$ is an element of the building set. Thus, the maximal element of the restricted building set containing a_{i+1} or a_i is in fact the same element, so

$$C_{\pi,k} = C_{\sigma,k} = \{a_1, \ldots, a_i, a_{i+1}\}.$$

For $k = r - i + 1$, the building set considered for F_π is $\mathcal{B}|_{\{a_1,\ldots,a_{i-1},a_i\}}$, and the element $C_{\pi,k}$ must contain a_i, so $C_{\pi,k} = \{a_1, \ldots, a_{i-1}, a_i\}$. For F_σ, the considered building set is $\mathcal{B}|_{\{a_1,\ldots,a_{i-1},a_{i+1}\}}$, we have that $C_{\sigma,k} = \{a_1, \ldots, a_{i-1}, a_{i+1}\}$. Notice that $C_{\pi,k} \neq C_{\sigma,k}$. Thus, the only difference between F_π and F_σ is that $C_{\pi,r-i+1} \in F_\pi$, whereas $C_{\sigma,r-i+1} \in F_\sigma$; all other elements of the maximal extended nested collections are the same.

If we are in the latter case, then

$$F_\pi = N_\pi \cup \{i \mid i \in [n] \setminus \{a_1, \ldots, a_r\}.$$

Notice that

$$\begin{aligned} F_\sigma &= N_\sigma \cup \{i \mid i \in [n] \setminus \{a_1, \ldots, a_{r-1}\}\} \\ &= (N_\pi \setminus \{C_{\pi,r}\}) \cup \{i \mid i \in [n] \setminus \{a_1, \ldots, a_{r-1}\}\}, \end{aligned}$$

since for each $k = 1, \ldots, r-1$, we have that $C_{\pi,k} = C_{\sigma,k}$. Thus, the only difference between F_π and F_σ is that $C_{\pi,r} \in F_\pi$, whereas $x_{a_r} \in F_\sigma$. □

7.1 Shellings of the Stellohedron

The partial order defined on partial permutations provides us with shellings for the dual of the extended stellohedron. Throughout this subsection, let Δ be a pure d-dimensional finite simplicial complex. We will be considering linear orderings $F_1, F_2, \ldots$ of its facets. Given such an ordering, we define $\Delta_k := \bigcup_{i=1}^{k} C_i$ for $k \geq 1$ and let $\Delta_0 =$?.

Definition 7.9. An n-dimensional simplicial complex Δ with r facets is **shellable** if its facets can be arranged into a linear ordering $F_1, F_2, \ldots, F_r$ such that $\Delta_{k-1} \cap F_k$ is pure of dimension $n - 1$ for all $2 \leq k \leq r$. Such an ordering is called a **shelling** of Δ.

For shellings of the dual of the permutohedron, $\mathcal{N}(\mathcal{B}_{K_n})$, Björner proved the following (in the more general setting of a the weak order in an arbitrary Weyl group).

Theorem 7.10 ([Bjö84, Theorem 2.1])**.** Let $\mathcal{B} = \mathcal{B}_{K_n}$, and $\mathcal{N}(\mathcal{B})$ be nested complex for $\mathcal{B}$, whose facets F_π are labeled by the permutations $\pi \in \mathfrak{S}_n$. If a total ordering $\pi_1 \leq \cdots \leq \pi_{n!}$ is a linear extension of the weak Bruhat order on $\mathfrak{S}_n$, then $F_{\pi_1}, \ldots, F_{\pi_{n!}}$ is a shelling for $\mathcal{N}(\mathcal{B})$.

We will prove this theorem for the extended case, where the analogues of $\mathcal{N}(\mathcal{B}_{K_n})$, permutations $\mathfrak{S}_n$, and weak Bruhat order are (respectively) $\mathcal{N}$ $(\mathcal{B}_{K_n})$, partial permutations $\mathfrak{P}_n$, and the partial weak order (see Definition 7.1).

Theorem 7.11. Let $\mathcal{N}$ $(\mathcal{B}_{K_n})$ be the extended nested set complex for $\mathcal{B}_{K_n}$, whose facets F_π are labeled by the partial permutations $\pi \in \mathfrak{P}_n$. If a total ordering $\pi_1 \leq \cdots \leq \pi_m$ is a linear extension of the partial weak Bruhat order on $\mathfrak{P}_n$, then $F_{\pi_1}, \ldots, F_{\pi_m}$ is a shelling order for $\mathcal{N}$ $(\mathcal{B}_{K_n})$.

We will use the following equivalent condition for an ordering of facets to give a shelling.

Lemma 7.12 ([Bjö84], Proposition 1.2)**.** An ordering $F_1, F_2, \dots, F_r$ of the facets of an n-dimensional simplicial complex Δ is a shelling if and only if for all $i \leq j$, there exists $\ell \leq j$ such that $F_i \cap F_j \subseteq F_\ell \cap F_j$ and $|F_\ell \cap F_j| = n - 1$.

In addition, we make use of the following result, which was given in the more general context of any graph associahedron in [BM18].

Lemma 7.13 ([BM18], Corollary 2.19)**.** Let $G = K_{1,n}$. For any (non-extended) nested collection N of $\mathcal{B}_G$, the set of maximal nested collections that contain N is an interval in L_G.

Recall Corollary 4.3, which states that $\mathcal{N}\ \ (\mathcal{B}_{K_n}) \simeq \mathcal{N}(\mathcal{B}_{K_{1,n}})$. Thus, any non-extended nested collection N of the star graph building set corresponds exactly to an extended nested collection $N\ \ $ of the complete graph building set, and the maximal nested collections and maximal extended nested collections that contain N and $N\ \ $ respectively are in bijection. In addition, by Remark 7.5, the partial weak Bruhat order on $\mathfrak{P}_n$ is dual to the poset L_G when $G = K_{1,n}$, so an interval $[u, v]$ in L_G is an interval $[v, u]$ in the partial weak Bruhat order on $\mathfrak{P}_n$. Thus, we have the following result.

Lemma 7.14. Consider $\mathcal{B} = \mathcal{B}_{K_n}$, and suppose $\pi_i, \pi_j \in \mathfrak{P}_n$ are partial permutations. Then the subposet of elements $\sigma \in \mathfrak{P}_n$ such that $F_{\pi_i} \cap F_{\pi_j} \subseteq F_\sigma$ forms an interval of the partial weak Bruhat order. In particular, there exists a unique minimal partial permutation ρ such that $F_\pi \cap F_\sigma \subseteq F_\rho$.

We are now able to prove our main shelling result.

Proof of Theorem 7.11. By Lemma 7.12, it suffices to show that for all partial permutations $\pi_i \quad \pi_j$, there exists a partial permutation $\pi_\ell \leq \pi_i$ such that $F_{\pi_i} \cap F_{\pi_j} \subseteq F_{\pi_\ell}$ and $|F_{\pi_\ell} \cap F_{\pi_i}| = n - 1$. Then fix any two partial permutations $\pi_i \quad \pi_j$. By Lemma 7.14, there exists a minimal partial permutation ρ such that $F_{\pi_i} \cap F_{\pi_j} \subseteq F_\rho$. Since $\pi_i \quad \pi_j$, we can conclude that $\pi_i \neq \rho$. By Lemma 7.13, the set of partial permutations whose corresponding facets contain $F_{\pi_i} \cap F_{\pi_j}$ is an interval. In addition, there exists some partial permutation $\pi_\ell \mid \pi_i$ such that $F_{\pi_i} \cap F_{\pi_j} \subseteq F_{\pi_\ell}$. Since $\pi_\ell \mid \pi_i$ is a cover relation, by Lemma 7.8 we have $|F_{\pi_\ell} \cap F_{\pi_i}| = n - 1$, as desired.

□

7.2 Partial Orders on Maximal (Extended) Nested Collections

Motivated by poset-theoretic results by Hersh related to generic cost vectors, we show in this subsection that all nestohedra and extended nestohedra have the property that there exists a "generic" cost vector that gives an acyclic directed graph isomorphic to the Hasse diagram of a poset. This will allow us to use Hersh's results in the case that this poset is a lattice to give nice properties of this lattice. We will first provide some definitions related to such vectors.

For a simple polytope $P \subseteq \mathsf{R}^d$, a **generic cost vector** $\mathbf{c} \in \mathsf{R}^d$ is a vector such that $\mathbf{c} \cdot u \neq \mathbf{c} \cdot v$ for distinct vertices u, v of P. Given such a vector $\mathbf{c}$, we obtain the acyclic directed graph, denoted $G(P, \mathbf{c})$, on the 1-skeleton of P by orienting each edge $e_{u,v}$ from u to v whenever $\mathbf{c} \cdot u < \mathbf{c} \cdot v$.

We now extend the poset L_G defined by Barnard and McConville for graph associahedra to all nestohedra and extended nestohedra. In the process, we will show that there exists a generic cost vector $\mathbf{c}$ such that $G(P, \mathbf{c})$ is the Hasse diagram of this poset. When extending the poset to all nestohedra, we omit many details; they follow very naturally from [BM18].

First, we define the following function on maximal nested collections.

Lemma 7.15 (cf. [BM18], Lemma 2.4)**.** Let $\mathcal{B}$ be a building set on $[n]$, and let N be a maximal nested collection. For each $I \in N \cup \{[n]\}$, there exists a unique element $\mathrm{top}_N(I) \in [n]$ not contained in any element of $\{J \in N \mid J \ (\ I\}$. In particular, this function is a bijection between elements of $N \cup \{[n]\}$ and $[n]$.

Since maximal nested collections correspond to the vertices of the corresponding nestohedron, one can describe the coordinates of the vertices of the nestohedron in terms of the maximal nested collection; see [Pos09, Proposition 7.9]. Recall that $\mathcal{B}$-trees are in bijection with maximal nested collections.

Lemma 7.16. Let $\mathcal{B}$ be a building set on $[n]$. If N is a maximal nested collection and T is the corresponding $\mathcal{B}$-tree, then the point $\mathbf{v}_N = (v_1, \dots, v_n)$ is a vertex of the nestohedron $\mathcal{P}(\mathcal{B})$ where

$$v_i := |\{I \in \mathcal{B} \mid i \in I \subseteq T_{\leq i}\}|.$$

We now define a partial order on maximal nested collections. Let $\mathcal{B}$ be a building set on $[n]$, with maximal nested collection N. Suppose that I is a non-maximal element of N. There exists a unique building set element $J \neq I$ such that $M = N \setminus \{I\} \cup \{J\}$ is also a maximal nested collection of $\mathcal{B}$. Define a **flip** as the relation $N \to M$ if $\mathrm{top}_N(I) < \mathrm{top}_M(J)$. We say that $N \leq M$ if there exists a sequence of flips of maximal nested collections of the form $N \to \cdots \to M$.

Theorem 7.17 (cf. [BM18], Lemma 2.8)**.** The set of maximal nested collections is partially ordered by the relation $\leq$ defined above.

Proof. Let $\mathbf{c} = (n, n-1, n-2, \dots, 1)$. If N and M are maximal nested collections such that $M = N \setminus \{I\} \cup \{J\}$ for building set elements $I \neq J$, then $\mathbf{c} \cdot (\mathbf{v}_M - \mathbf{v}_N) > 0$, where $\mathbf{v}_M, \mathbf{v}_N$ are the vertices of $\mathcal{P}(\mathcal{B})$ corresponding to M and N respectively. Thus, $N \to M$ on maximal nested collections is induced by the vector $\mathbf{c}$, so the relation is acyclic. Thus, the transitive closure of such relations is a partial order. □

Corollary 7.18. For any nestohedron $\mathcal{P}(\mathcal{B})$, there exists a generic cost vector $\mathbf{c}$ such that $G(\mathcal{P}(\mathcal{B}), \mathbf{c})$ is the Hasse diagram of a poset.

We now prove an analogous statement for extended nestohedra. For a building set $\mathcal{B}$ on $[n]$ and maximal extended nested collection N, let

$$\mathrm{Supp}(N) := \{i \in [n] \mid x_i \notin N\}.$$

Lemma 7.19. Let $\mathcal{B}$ be a building set on $[n]$, and let N be a maximal extended nested collection. For each $I \in N$ such that $I \subseteq \mathrm{Supp}(N)$, there exists a unique element $\mathrm{top}_N(I) \in \mathrm{Supp}(N)$ not contained in any element of $\{J \in N \mid J \subsetneq I\}$. In particular, this function is a bijection between non-design vertex elements of N and $\mathrm{Supp}(N)$.

Recall Proposition 6.9, which gives the coordinates of the vertex of extended nestohedron $\mathcal{P}$ $(\mathcal{B})$ corresponding to a maximal extended nested collection.

We now define a partial order on maximal extended nested collections. Let $\mathcal{B}$ be a building set on $[n]$, with maximal nested collection N. Suppose that $I \in \mathcal{B}$ is a non-maximal element of N. There exists a unique building set element $J \neq I$ such that $M = N \setminus \{I\} \cup \{J\}$ is also a maximal extended nested collection of $\mathcal{B}$, with $\mathrm{Supp}(N) = \mathrm{Supp}(M)$. Like in the non-extended case, a flip is the relation $N \to M$ if $\mathrm{top}_N(I) < \mathrm{top}_M(J)$ and $\mathrm{Supp}(N) = \mathrm{Supp}(M)$ of if $\mathrm{Supp}(N) = \mathrm{Supp}(M) \cup \{i\}$ for some $i \notin \mathrm{Supp}(M)$. We say that $N \leq M$ if there exists a sequence of flips of maximal extended nested collections of the form $N \to \cdots \to M$.

Theorem 7.20. The set of maximal extended nested collection is partially ordered by the relation $\leq$ defined above.

Proof. The edges of the extended nestohedron $\mathcal{P}$ $(\mathcal{B})$ are of one of two forms, depending on the maximal extended nested collections N and M corresponding to the vertices that the edge connects: either $\mathrm{Supp}(N) = \mathrm{Supp}(M)$, or $\mathrm{Supp}(N) = \mathrm{Supp}(M) \cup \{i\}$ for some $i \notin \mathrm{Supp}(M)$. If we are in the first case, then $M = N \setminus \{I\} \cup \{J\}$ for some $I, J \in \mathcal{B}$ with $I \neq J$. Let $i = \mathrm{top}_N(I)$ and $j = \mathrm{top}_M(J)$. Then using Proposition 6.9, we see that $\mathbf{v}_N$ and $\mathbf{v}_M$, the vertices of $\mathcal{P}$ $(\mathcal{B})$ corresponding to N and M respectively, agree on every coordinate except the i-th and j-th coordinates. In fact, $\mathbf{v}_M - \mathbf{v}_N = k(-e_i + e_j)$, where k is equal to the number of building set elements contained in $I \cup J$ and contain both i and j.

If we are in the second case, then notice that $\mathbf{v}_N$ and $\mathbf{v}_M$ differ in only the i-th coordinate, with $\mathbf{v}_M - \mathbf{v}_N = ke_i$, where $k < 0$.

Let $\mathbf{c} = (-n, -n+1, \cdots, -1)$. If N and M are maximal extended nested collections such that there is an edge connecting their corresponding vertices in $\mathcal{P}\ \ (\mathcal{B})$, then $\mathbf{c} \cdot (\mathbf{v}_M - \mathbf{v}_N) > 0$. Thus, $N \to M$ on maximal extended nested collections is induced by the vector $\mathbf{c}$, so the relation is acyclic. Thus, the transitive closure of such relations is a partial order. □

Corollary 7.21. For any extended nestohedron $\mathcal{P}\ \ (\mathcal{B})$, there exists a generic cost vector $\mathbf{c}$ such that $G(\mathcal{P}\ \ (\mathcal{B}), \mathbf{c})$ is the Hasse diagram of a poset.

Let $L(\mathcal{B})$ and $L\ \ (\mathcal{B})$ denote the partial orders defined on maximal nested collections and maximal extended nested collections respectively. We now have the following result by [Her18, Theorem 1.4].

Proposition 7.22. If $L(\mathcal{B})$ or $L\ \ (\mathcal{B})$ is a lattice, then each open interval (u, v) in the lattice has order complex which is homotopy equivalent to a ball or a sphere of some dimension. Therefore, the Möbius function $\mu(u, v)$ only takes values $0, 1$, and -1.

8 Acknowledgements

This research was carried out as part of the 2019 REU program at the School of Mathematics at University of Minnesota, Twin Cities. The authors are grateful for the support of NSF RTG grant DMS-1148634 for making the REU possible. The authors would like to thank Vic Reiner and Sarah Brauner for their mentorship and guidance.

References

[Ais12] Natalie Aisbett. "Inequalities between gamma-polynomials of graph-associahedra". In: *Electron. J. Combin.* 19.2 (2012), Paper 36, 17. issn: 1077-8926.

[Ais14] Natalie Aisbett. "Frankl-Füredi-Kalai inequalities on the γ-vectors of flag nestohedra". In: *Discrete Comput. Geom.* 51.2 (2014), pp. 323–336. issn: 0179-5376. doi: 10.1007/s00454-013-9567-0. url: https://doi.org/10.1007/s00454-013-9567-0.

[Bjö84] Anders Björner. "Some combinatorial and algebraic properties of Coxeter complexes and Tits buildings". In: *Advances in Mathematics* 52.3 (1984), pp. 173–212.

[BM18] Emily Barnard and Thomas McConville. "Lattices from graph associahedra and subalgebras of the Malvenuto-Reutenauer algebra". In: *arXiv preprint arXiv:1808.05670* (2018).

[CD06] Michael Carr and Satyan L. Devadoss. "Coxeter complexes and graph-associahedra". In: *Topology and its Applications* 153.12 (2006), pp. 2155–2168.

[CP15] Suyoung Choi and Hanchul Park. "A new graph invariant arises in toric topology". In: *Journal of the Mathematical Society of Japan* 67.2 (2015), pp. 699–720.

[DHV11] Satyan L. Devadoss, Timothy Heath, and Wasin Vipismakul. "Deformations of bordered surfaces and convex polytopes". In: *Notices Amer. Math. Soc.* 58.4 (2011), pp. 530–541. issn: 0002-9920.

[DP95] C. De Concini and C. Procesi. "Wonderful models of subspace arrangements". In: *Selecta Math. (N.S.)* 1.3 (1995), pp. 459–494. issn: 1022-1824. doi: 10.1007/BF01589496. url: https://doi.org/10.1007/BF01589496.

[ES74] Günter Ewald and Geoffrey C Shephard. "Stellar subdivisions of boundary complexes of convex polytopes". In: *Mathematische Annalen* 210.1 (1974), pp. 7–16.

[FS05] Eva Maria Feichtner and Bernd Sturmfels. "Matroid polytopes, nested sets and Bergman fans". In: *Portugaliae Mathematica* 62.4 (2005), pp. 437–468.

[FY04] Eva Maria Feichtner and Sergey Yuzvinsky. "Chow rings of toric varieties defined by atomic lattices". In: *Inventiones mathematicae* 155.3 (2004), pp. 515–536.

[Gal05] Światosław R. Gal. "Real root conjecture fails for five- and higher-dimensional spheres". In: *Discrete Comput. Geom.* 34.2 (2005), pp. 269–284. issn: 0179-5376. doi: 10.1007/s00454-005-1171-5. url: https://doi.org/10.1007/s00454-005-1171-5.

[Her18] Patricia Hersh. "Posets arising as 1-skeleta of simple polytopes, the nonrevisiting path conjecture, and poset topology". In: *arXiv:1802.04342* (2018).

[LP15] Thomas Lam and Pavlo Pylyavskyy. "Linear Laurent phenomenon algebras". In: *International Mathematics Research Notices* 2016.10 (2015), pp. 3163–3203.

[McC17] Thomas McConville. "Crosscut-simplicial lattices". In: *Order* 34.3 (2017), pp. 465–477. issn: 0167-8094. doi: 10.1007/s11083-016-9409-9. url: https://doi.org/10.1007/s11083-016-9409-9.

[MP17] Thibault Manneville and Vincent Pilaud. "Compatibility fans for graphical nested complexes". In: *J. Combin. Theory Ser. A* 150 (2017), pp. 36–107. issn: 0097-3165. doi: 10.1016/j.jcta.2017.02.004. url: https://doi.org/10.1016/j.jcta.2017.02.004.

[Pos09] Alexander Postnikov. "Permutohedra, associahedra, and beyond". In: *International Mathematics Research Notices* 2009.6 (2009), pp. 1026–1106.

[PPP17] Boram Park, Hanchul Park, and Seonjeong Park. "Graph invariants and Betti numbers of real toric manifolds". In: *arXiv preprint arXiv:1801.00296* (2017).

[PRW08] Alexander Postnikov, Victor Reiner, and Lauren Williams. "Faces of generalized permutohedra". In: *Documenta Mathematica* 13.207-273 (2008), p. 51.

[Rea02] Nathan Reading. "Order dimension, strong Bruhat order and lattice properties for posets". In: *Order* 19.1 (2002), pp. 73–100. issn: 0167-8094. doi: 10.1023/A:1015287106470. url: https://doi.org/10.1023/A:1015287106470.

[Sta97] Richard P. Stanley. *Enumerative combinatorics. Vol. 1*. Vol. 49. Cambridge Studies in Advanced Mathematics. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original. Cambridge University Press, Cambridge, 1997, pp. xii+325. isbn: 0-521-55309-1.

[Vol10] V. D. Volodin. "Cubical realizations of flag nestohedra and a proof of Gal's conjecture for them". In: *Uspekhi Mat. Nauk* 65.1(391) (2010), pp. 183–184. issn: 0042-1316. doi: 10.1070/RM2010v065n01ABEH004667. url: https://doi.org/10.1070/RM2010v065n01ABEH004667.

[Zel06] Andrei Zelevinsky. "Nested Complexes and their Polyhedral Realizations". In: *Pure and Applied Mathematics Quarterly* 2.3 (2006), pp. 655–671.