Homomesy and Rowmotion on the Trapezoid Poset

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Summer 2019

Abstract

In this report, we proved that the action of rowmotion on the trapezoid poset is homomesic for the down-degree statistic for the case $T_{3,n}$ and $T_{4,n}$. We also conjectured that the bijection between P-partitions of the rectangle and trapezoid poset, introduced in [HPPW18], commutes with rowmotion on those posets.

1 Introduction and Background

A *partially ordered set* (henceforth abbreviated a *poset*) is a set $\mathcal{P}$ with a binary relation $\leqslant$ that is reflexive, anti-symmetric, and transitive. Two elements $x, y \in \mathcal{P}$ are *comparable* if we have $x \leqslant y$ or $y \leqslant x$, and *incomparable* otherwise. We say y *covers* x if $y \geqslant x$ and there does not exist $z \in \mathcal{P}$ such that $y > z > x$; equivalently, we say that x *is covered by* y. The *Hasse diagram* of $\mathcal{P}$ is an undirected graph with vertex set $\mathcal{P}$, and an edge between y and x if they have a cover relation. A *graded poset* is a poset P with a rank function $rank : P \to \mathbb{Z}$ such that if $x < y$, $rank(x) < rank(y)$ and if y covers x, $rank(y) = rank(x) + 1$.

Given a poset $\mathcal{P}$, an *order ideal* I of $\mathcal{P}$ is a subset of $\mathcal{P}$ that is downward closed, i.e: if $x \in I$ and $y \leqslant x$ in $\mathcal{P}$, then $y \in I$ as well. A *filter ideal* I' of $\mathcal{P}$ is the complement of an order ideal, or equivalently, a subset of $\mathcal{P}$ that is upward closed. Denote the set of order ideals of $\mathcal{P}$ to be $J(\mathcal{P})$, which is also a poset with relation given by inclusion between order ideals.

The *dual poset* of a poset $\mathcal{P}$ is the poset $\mathcal{P}^*$ that has the same underlying set as $\mathcal{P}$, but with the binary relation reversed. Note that an order ideal of $\mathcal{P}$ corresponds to a filter ideal of $\mathcal{P}^*$ and vice versa.

Given an order ideal $I \in \mathcal{P}$, we can take the maximal elements of I to get an *antichain* A, i.e: a set of elements of $\mathcal{P}$ that are incomparable to one another. In general, the set of order ideals of P is in bijection with the set of antichains, with the reverse map sending A to the order ideal *generated by* A, i.e: the ideal $\{x \in \mathcal{P} \mid x \leqslant y \text{ for some } y \in A\}$. The *down-degree* of I is defined to be the number of maximal elements of I.

A *linear extension* of a poset $\mathcal{P}$ is a bijection $\rho : \mathcal{P} \to \{1, 2, \ldots, |\mathcal{P}|\}$ that is order-preserving, i.e: $\rho(x) < \rho(y)$ for all $x < y$ in $\mathcal{P}$. A *P-partition of $\mathcal{P}$ of height m* is a order preserving map from $\mathcal{P}$ to $[m] = \{0, \ldots, m\}$. Denote $\mathrm{PP}^m(\mathcal{P})$ the set of all P-partitions of $\mathcal{P}$ of height m. Note that there is a bijection between $\mathrm{PP}^m(\mathcal{P})$ and chains of order ideals of $\mathcal{P}$ of length m, given by sending a P-partition $\pi : \mathrm{PP}^m(\mathcal{P})$ to the chain $\pi^{-1}(\{0\}) \subset \pi^{-1}(\{0, 1\}) \subset \cdots \subset \pi^{-1}(\{0, \ldots, m-1\})$. With this bijection, it's clear that a P-partition of height 1 corresponds to an order ideal.

In our paper, we will mostly study the rectangle poset

$$R_{a,b} = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leqslant i \leqslant a, 1 \leqslant j \leqslant b\}$$

and the trapezoid poset

$$T_{a,a+b} = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leqslant i \leqslant j \leqslant a + b - 1\}$$

with the relation $(i, j) \leqslant (i', j') \iff i \leqslant i'$ and $j \leqslant j'$. It was shown in [Sta86] that the number of P-partitions of height m is the same for the rectangle and trapezoid poset; in other

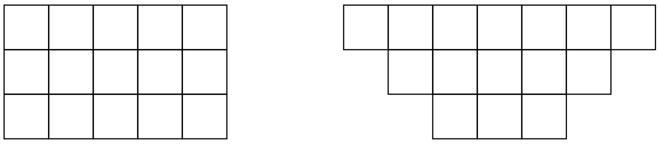


Figure 1: $R_{a,b}$ and $T_{a,b}$ for $a = 3$, $b = 5$

words, we have: $\mathrm{PP}^m(R_{a,b}) = \mathrm{PP}^m(T_{a,a+b})$ for all a, b, m. Such a pair (P, Q) of posets satisfying $\mathrm{PP}^m(P) = \mathrm{PP}^m(Q)$ for all m is called a *doppelgänger pair*. It turns out that the rectangle and trapezoid poset (conjecturally) have a deeper connection than just being a doppelgänger pair; for a survey of these conjectures, see [Hop19].

2 Rowmotion and Minuscule Doppelgängers

2.1 Rowmotion on order ideals and piecewise linear rowmotion

Definition 2.1. Let $\mathcal{P}$ be a poset, and $I \in J(\mathcal{P})$ an order ideal of $\mathcal{P}$. Then the rowmotion of I, denoted $\rho(I)$ is the order ideal generated by the minimal elements that are not in I, i.e.

$$\rho(I) = \langle a \in \mathcal{P} : a \in \min\{\mathcal{P} - I\}\rangle$$

Rowmotion can be viewed as composition of 'toggles' of poset elements. For $p \in \mathcal{P}$ and order ideal I, we denote $\tau_p(I)$ as toggling p on I which is defined as follows:

$$\tau_p(I) = \begin{cases} I \cup p & \text{if } p \notin I \text{ and } I \cup p \in J(\mathcal{P}), \\ I \setminus p & \text{if } p \in I \text{ and } I \setminus p \in J(\mathcal{P}), \\ I & \text{otherwise.} \end{cases}$$

Then rowmotion is just performing toggles row by row from top to bottom.

Proposition 2.2. *[PR15] $\rho(I) = \tau_{p_1} \circ \tau_{p_{n-1}} \circ \cdots \circ \tau_{p_n}(I)$ where $\{p_1, \cdots, p_n\}$ is a linear extension of the poset $\mathcal{P}$.*

Rowmotion is generalized by Eisenstein and Propp to a piecewise linear action on P-partitions (or equivalently, order polytopes), which toggles are refined by a tropical exchange relation:

$$\tau(p) = \max\{a : a < p\} + \min\{b : b > p\} - p$$

We can identify a rowmotion as a height 1 P-partition where elements in the ideal are labeled as 0 and 1 otherwise. Then the classical Rowmotion is equivalent to piecewise linear rowmotion on this P-partition, thus we do not distinguish the notation between classical and piecewise linear rowmotion.

2.2 Bijections between plane partitions of minuscule doppelgängers

In this section we describe the bijection φ, given in [HPPW18], between P-partitions of the rectangle $R_{a,b}$ and the trapezoid $T_{a,a+b}$. Note that this bijection applies to all minuscule doppelgängerpairs shown in Figure 3, but we will mostly focus on the case of the rectangle and trapezoid. The construction is based on *k-jeu-de-taquin* slides, which are a K-theoretic analogue of the usual jeu-de-taquin. The discussion below is an adaptation of [HPPW18, Section 6.2].

Definition 2.3. An *increasing tableaux* of height ℓ on a poset $\mathcal{P}$ is a function $T : \mathcal{P} \to [\ell]$ such that whenever $x < y$ in $\mathcal{P}$, we have $T(x) < T(y)$. We say that x *covers* (resp. is covered) $a \in \mathbb{N}$ if there exists $y \in \mathcal{P}$ covered by (resp. covering) x with $T(y) = a$. Denote the set of all increasing tableaux of height l to be $\mathrm{IT}^{[l]}(\mathcal{P})$.

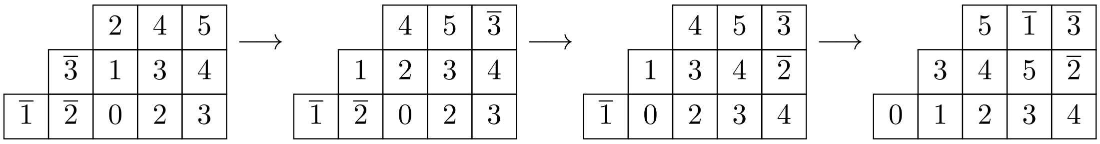


Figure 2: The map φ for an order ideal $I \in J(T_{3,3})$

When $\mathcal{P}$ is a ranked poset with all maximal chains of the same length $\mathrm{ht}(\mathcal{P})$, [DPS17, Theorem 4.1] shows that there is a bijection $\mathrm{PP}^{[h]}(\mathcal{P}) \simeq \mathrm{IT}^{[h+\mathrm{ht}(\mathcal{P})]}(\mathcal{P})$. Since all of the miniscule Doppelgängers pairs are of this form, it suffices to find a bijection φ between increasing tableaux of such pairs. To define φ, we first need some preliminary definitions.

Definition 2.4. The *swap* of two numbers a, b in an increasing tableaux T is the function $\mathrm{swap}_{a,b}(T)$ such that for all $x \in P$:

$$\mathrm{swap}_{a,b}(T)(x) = \begin{cases} a & \text{if } T(x) = b \text{ and } x \text{ covers } a \\ b & \text{if } T(x) = a \text{ and } x \text{ is covered by } b \\ T(x) & \text{else} \end{cases}$$

After performing a swap, the resulting tableaux can be considered to still be increasing, but with the order of the numbers a and b switched. Next, we can describe K-jeu-de-taquin as a sequence of swaps, which turns a number a into the maximal number.

Definition 2.5. Suppose T is an increasing tableaux of height ℓ. Then define the *k-jeu-de-taquin slide* of $a \in \ell$ to be the tableaux

$$\mathrm{jdt}_a(T) := \left(\prod_{b=a+1}^{\ell} \mathrm{swap}_{a,b} \right) (T).$$

The resulting tableaux will still be increasing with the ordering that a is now the maximal value. Alternatively, we could make the tableaux increasing by replacing all instances of a with ℓ and decrease all $b \in [a+1, \ell]$ by one. However, unless stated otherwise, it is assumed that we don't relabel the entries.

We now define the bijection φ for the case of rectangle and trapezoid. For the other doppelgängers pairs, see [HPPW18].

Definition 2.6. Given an increasing tableaux $T \in IT^{[l]}(R_{a,b})$ of the rectangle, one obtains an increasing tableaux $\varphi(T) \in IT^{[l]}(T_{a,b})$ as follows:

(i) Create a larger poset $\mathcal{P} = R_{a,b} \cup \{(i,j) \mid 0 \leqslant i \leqslant a-1, -a+1 \leqslant j \leqslant 0, i+j \leqslant a-1\}$, and consider the tableaux $T \cup \overline{U}$ on $\mathcal{P}$ that is T on the part $R_{a,b}$, U is the minimal tableaux associated to the triangle shape, and $\overline{U}$ is U with all entries having a bar on top. Order the entries so that $\overline{1} < \overline{2} < \cdots < \overline{2a-3} < 1 < 2 < \cdots < l$.

(ii) Do K-jdt slide of $\bar{i}$ for all $i = 0, \ldots, 2a-3$.

(iii) Take the resulting increasing tableaux on the trapezoidal part $T_{a,b}$ of $\mathcal{P}$ to be the result $\varphi(T)$.

See Figure 2 for an example of φ. We are now equipped to state the main conjecture for this section:

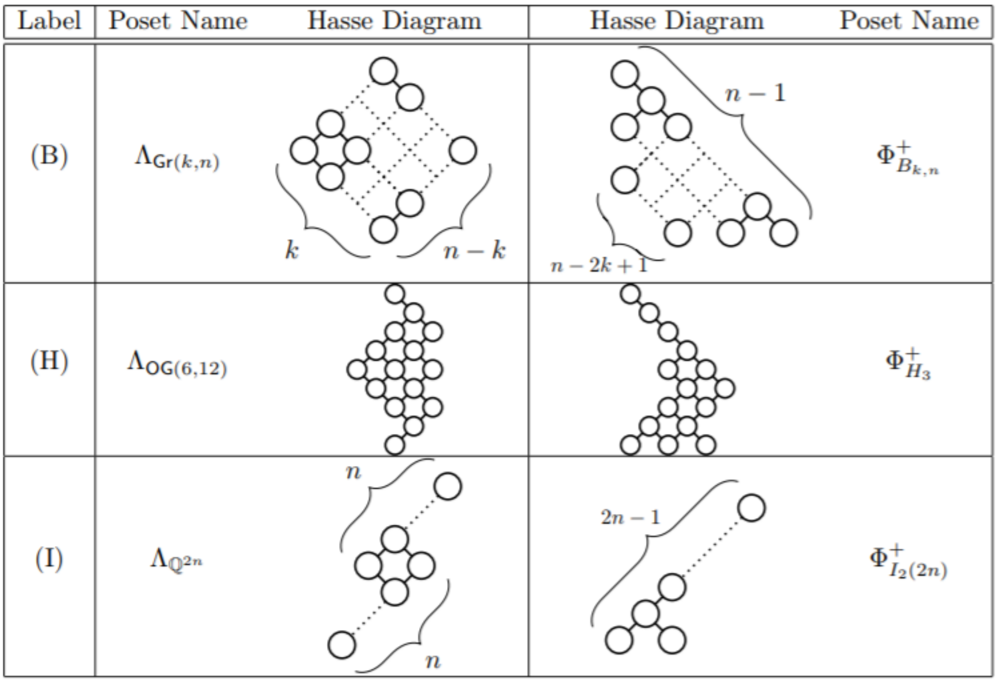


Figure 3: [HPPW18, Figure 1] Relevant miniscule Doppelgängers pairs

Conjecture 2.7. *For any of the minuscule doppelgängers pair*

$$(P, Q) \in \{(R_{a,b}, T_{a,a+b}), (OG(6,12), H_3), (\mathbb{Q}^{2n}, I_2(2n))\}$$

φ commutes with rowmotion on order ideals of the pair. In other words, we have a commutative diagram:

$$\begin{array}{ccc} J(P) & \xrightarrow{\varphi} & J(Q) \\ \Big\downarrow \text{row} & & \Big\downarrow \text{row} \\ J(P) & \xrightarrow{\varphi} & J(Q) \end{array}$$

Although Conjecture 2.7 considers three miniscule doppelgängers pairs, the only hard part comes from the case of the rectangle and trapezoid. In particular, we have the following:

Theorem 2.8. *Conjecture 2.7 is true for* $(P, Q) \in \{(R_{a,b}, T_{a,a+b}), (OG(6,12), H_3), (\mathbb{Q}^{2n}, I_2(2n))\}$ *for all n, and all $a, b \leqslant 8$.*

Proof. The cases $(R_{a,b}, T_{a,a+b})$ for $a, b \leqslant 8$ and $(OG(6,12), H_3)$ are checked by computer. For the last case, note that each poset only has 2 rowmotion orbits, one of size $2n-2$ and the other of size 2. It's easy to see that for $I \in J(OG(6,12))$ in orbit of size 2, $\varphi(I)$ is also in orbit of size 2, hence rowmotion commutes with φ for such I. For any ideal I in the other rowmotion orbit of $J(OG(6,12))$, I consists of all elements of $J(OG(6,12))$ of rank $\leq m$ for some m. In this case, it can be seen that $\varphi(I)$ is the set of elements of $J(H_3)$ of rank $\leq m$ for some m. Observe that for a graded poset P with all maximal elements of P having rank $m+1$, and the ideal $I \subseteq P$ consisting of all elements $\leq m$, the minimal elements of $P \setminus I$ are the elements of rank $m+1$ thus $\text{row}(I)$ is either the ideal consisting of all elements of a poset P of rank $\leq m+1$ or the empty ideal. Since $OG(6,12)$ and H_3 both have the same maximal rank, we conclude that φ commutes with rowmotion on $(OG(6,12), H_3)$ □

3 Toggle Symmetry and Homomesy of Down-Degree

In this section, we will prove our second result about rowmotion on the trapezoid poset: that it's *homomesic* with respect to the *down-degree statistic* for the trapezoid $T_{3,n}$ and $T_{4,n}$. We first provide the relevant definitions.

Definition 3.1 ([PR15])**.** A statistic f on a set S is said to be *homomesic* with respect to an invertible operator $\Phi : S \to S$ if for all Φ orbits $\mathcal{O}$,

$$\frac{1}{\#\mathcal{O}} \sum_{T \in \mathcal{O}} f(T)$$

is invariant of $\mathcal{O}$, and equal to

$$\frac{1}{\#S} \sum_{T \in S} f(T).$$

Propp and Roby had rowmotion on the rectangle in mind when they formulated the term "homomesy." We will be concerned with when down-degree exhibits homomesy with respect to rowmotion for the rectangle and trapezoid. To approach this, we will focus on toggles as introduced by [CF95]. We define toggles as in [Hop17]. For an ideal I and an antichain A, we denote

$$\mathcal{T}_A^+(I) := \begin{cases} 1 & \text{if } A \notin I \text{ and } A \cup I \text{ is an ideal} \\ 0 & \text{otherwise} \end{cases}$$
$$\mathcal{T}_A^-(I) := \begin{cases} 1 & \text{if } A \in I \text{ and } I \setminus A \text{ is an ideal} \\ 0 & \text{otherwise} \end{cases}$$
$$\mathcal{T}_A(I) := \mathcal{T}_A^+(I) - \mathcal{T}_A^-(I)$$

We commonly use p to denote a single element antichain associated to the element p of the poset. We call a distribution μ on ideals in a poset *toggle-symmetric* if for any fixed p,

$$\mathbb{E}[\mu; \mathcal{T}_p(I)] = 0.$$

Some clear examples of toggle-symmetric distributions are the uniform distribution on all ideals or a uniform distribution on ideals contained in a single rowmotion orbit. The distribution defined by choosing a plane partition J_m uniformly at random, and then picking an ideal in J_m uniformly at random is a toggle symmetric distribution. ((ANDY: why is this?)) The distribution defined by choosing a plane partition J_m uniformly at random from a rowmotion orbit, and then picking an ideal in J_m uniformly at random is also toggle-symmetric.

In slightly greater generality, we call a distribution μ on ideals in a poset *toggle on anitchains-symmetric* if for any fixed antichain A,

$$\mathbb{E}[\mu; \mathcal{T}_A(I)] = 0.$$

The uniform distribution on all ideals and a uniform distribution on ideals contained in a single rowmotion orbit are both toggle on anitchains-symmetric.

Remark 3.2. The distribution defined by choosing a plane partition J_m uniformly at random, and then picking an ideal in J_m uniformly at random is toggle on anitchains-symmetric. is not toggle on anitchains-symmetric (has sam proved this somewhere?).

In [CHHM17], Chan, Haddadan, Hopkins, and Moci show for any toggle-symmetric distribution μ on $\mathcal{R}_{a,b}$, $\mathbb{E}[\mu; \text{ddeg}] = \mathbb{E}[uni_{J(\mathcal{R}_{a,b})}; \text{ddeg}]$. In particular, this showed that thedown-degree statistic is homomesic with respect to the action of rowmotion on the set of order ideals of the rectangle. To do this, they use the "rook" approach.

3.1 The rook approach

We will be concerned with rook arrangements on the trapezoid. For readers interested in rook arrangements on other shapes, see [Hop17]. Before we proceed we need a way of of labeling the elements of our posets. For the trapezoid, we use the Cartesian coordinates with the minimal element being $(0,0)$ and the maximal elements having coordinates $(i, a+b-1-2i)$ for $0 \leq i < a$, see 4 for an example. For notational convenience, (i, λ_i) will refer to the element $(i, a+b-1-2i)$.

Definition 3.3. A *rook* on the (i,j) square of a trapezoid $T_{a,b}$ is a linear equation

$$R_{i,j} : \mathbb{R}^{J(T_{a,b})} \to \mathbb{R}$$

$$R_{i,j}(a \cdot I) = a \cdot \left(\sum_p \left(c_p^- \mathcal{T}_p^-(I) + c_p^+ \mathcal{T}_p^+(I) \right) + \sum_{i=0}^{a-2} c^-_{\{(i',\lambda_{i'}),(i'+1,\lambda_{i'+1})\}} T^-_{\{(i',\lambda_{i'}),(i'+1,\lambda_{i'+1})\}}(I) \right)$$

where

$$c^-_{(i',j')} = \begin{cases} 1 & \text{if } i' \geq i \text{ and } i'+j' \geq i+j \\ -1 & \text{if } i' < i \text{ and } i'+j' < i+j \text{ and } j' > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$c^+_{(i',j')} = \begin{cases} 1 & \text{if } i' \leq i \text{ and } i'+j' \leq i+j \\ -1 & \text{if } i' > i \text{ and } i'+j' > i+j \text{ and } j' > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$c^+_{\{(i',\lambda_{i'}),(i'+1,\lambda_{i'+1}\}} = \begin{cases} -1 & \text{if } i' \geq i \text{ and } i'+a+b-2-2i' \geq i+j \\ 0 & \text{otherwise} \end{cases}.$$

For each rook placed on the trapezoid, it *affects* an element p of our poset $c_p^- + c_p^+$ times. We say that the rook *attacks* a square p if $c_p^- + c_p^+ \neq 0$. As we see in Figure 4, a rook on the trapezoid will attack the squares that lie in the same row and column, with the exception that once we reach the leftmost or upmost part of the row/column, the rook starts attacking squares (or pairs of squares) that lie on the diagonal.

Our definition of a rook satisfy the following nice property:

Proposition 3.4 ([Hop17])**.** *For any order ideal $I \in J(T_{a,b})$, we have $R_{i,j}(I) = 1$.*

For the case when $a = b$, Hopkins shows that all toggle-symmetric distributions have the same expected down-degree using the rook approach.

Theorem 3.5 ([Hop17], Theorem 4.2)**.** *For any toggle-symmetric distribution μ on the trapezoid $T_{a,a}$,*

$$\mathbb{E}[\mu; \mathrm{ddeg}] = \mathbb{E}[uni_{J(T_{a,b})}; \mathrm{ddeg}] = \frac{ab}{a+b}$$

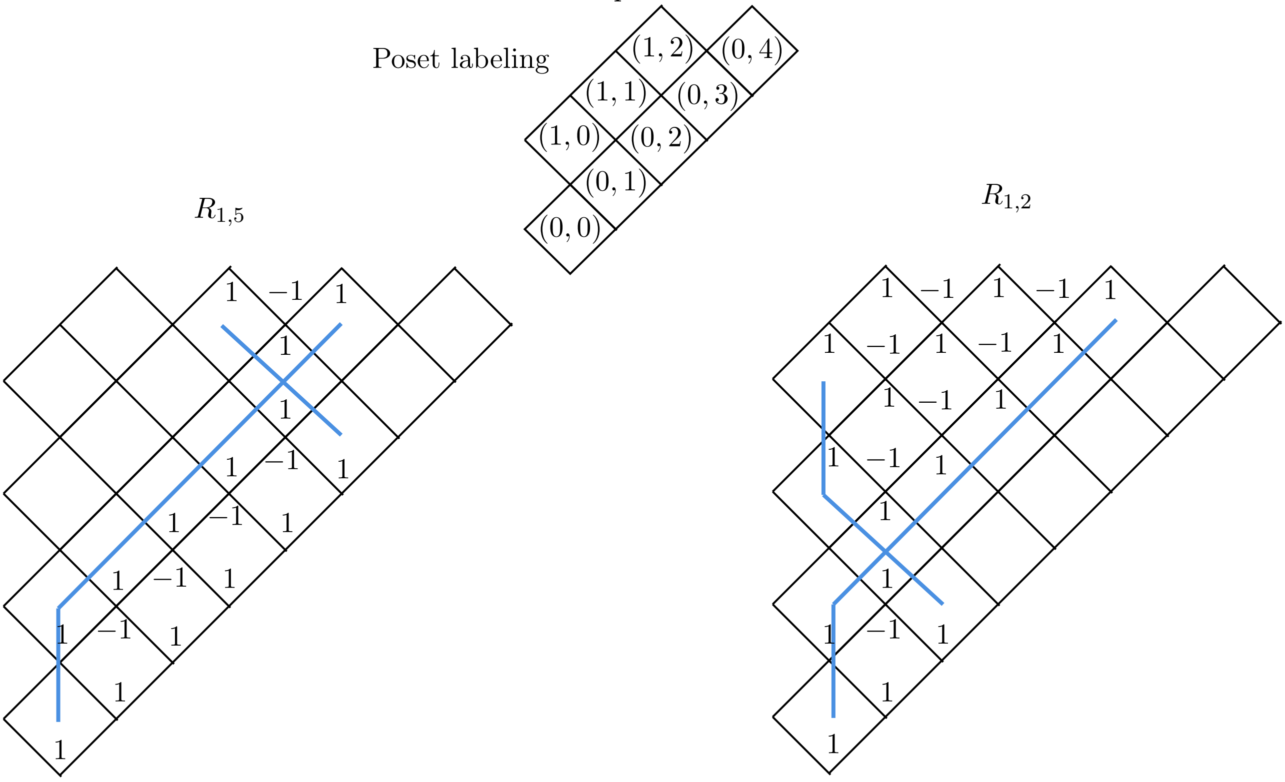
For a general trapezoid and a toggle-symmetric distribution μ, it is not necessarily true that we $\mathbb{E}[\mu; \mathrm{ddeg}] = \mathbb{E}[uni_{J(T_{a,b})}; \mathrm{ddeg}]$ (see [Hop17]). However, we can use the rook approach to to find the difference between $\mathbb{E}[\mu; \mathrm{ddeg}]$ and $\mathbb{E}[uni_{J(T_{a,b})}; \mathrm{ddeg}] = ab/(a+b)$:

Lemma 3.6. *For any toggle-symmetric distribution μ on the trapezoid $T_{a,b}$,*

$$\mathbb{E}[\mu; \mathrm{ddeg}] = \frac{ab}{a+b} + \frac{a-b}{a+b} \left(\sum_{i=0}^{a} (a-1-i) \mathbb{E}[\mu; \mathcal{T}^+_{(i,0)}] - \sum_{i=0}^{a-2} i \mathbb{E}[\mu; \mathcal{T}^-_{(i,\lambda_i),(i+1,\lambda_{i+1})}] \right)$$

Figure 4: Rook Arrangements

In the below figure we write a number in a V-shaped nook with element p of our poset above it to denote to represent c_p^+ and a number in an upside down V-shaped nook ((ANDY: Λ-shaped nook)) with an element of P below it to denote c_p^-. A number in a V-shaped nook at the top of our trapezoid refers to c_A^- were A is the antichain of the two elements bordering the nook. The blue line shows the squares the rook attacks.



Proof. Consider the linear map $\varphi : \mathbb{R}^{J(T_{a,b})} \to \mathbb{R}$

$$\varphi := (2a - a^2)R_{(0,0)} + (b - a)\sum_{j=1}^{a} R_{(0,j)} + b\sum_{i=1}^{a-1} R_{(i,0)} + a\sum_{j=a+1}^{a+b-1} R_{(0,j)}.$$

We will compute $\mathbb{E}[\mu; \varphi]$ in two ways. On one hand, for any ideal I,

$$\varphi(I) = (2a - a^2)R_{(0,0)}(I) + (b - a)\sum_{j=1}^{a} R_{(0,j)}(I) + b\sum_{i=1}^{a-1} R_{(i,0)}(I) + a\sum_{j=a+1}^{a+b-1} R_{(0,j)}(I) = ab$$

thus $\mathbb{E}[\mu; \varphi] = ab$.

On the other hand, we see that the rook arrangement which φ defines attacks every element of $T_{a,b}$ $a+b$ times except for elements of form $(i, 0)$, which are attacked $(a+b)-(a-b)\cdot(a-1-i)$ times. Thus for some constants c_p,

$$\varphi = (a + b) \cdot \mathrm{ddeg} - \sum_{i=0}^{a}(a - b)(a - 1 - i)\mathcal{T}^+_{(i,0)} + \sum_{i=0}^{a-2}(a - b)\mathcal{T}_{(i,\lambda_i),(i+1,\lambda_{i+1})} + \sum_{p} c_p\mathcal{T}_p$$

Over a toggle-symmetric distribution μ, $\mathbb{E}[\mu; \sum_p c_p\mathcal{T}_p] = 0$. Using linearity of expectation

$$\mathbb{E}\left[\mu; \sum_p c_p\mathcal{T}_p\right] = (a+b)\cdot\mathbb{E}[\mu; \mathrm{ddeg}] - \sum_{i=0}^{a}(a-b)(a-1-i)\mathbb{E}\left[\mu; \mathcal{T}^+_{(i,0)}\right] + \sum_{i=0}^{a-2}(a-b)\mathbb{E}\left[\mu; \mathcal{T}^-_{(i,\lambda_i),(i+1,\lambda_{i+1})}\right]$$

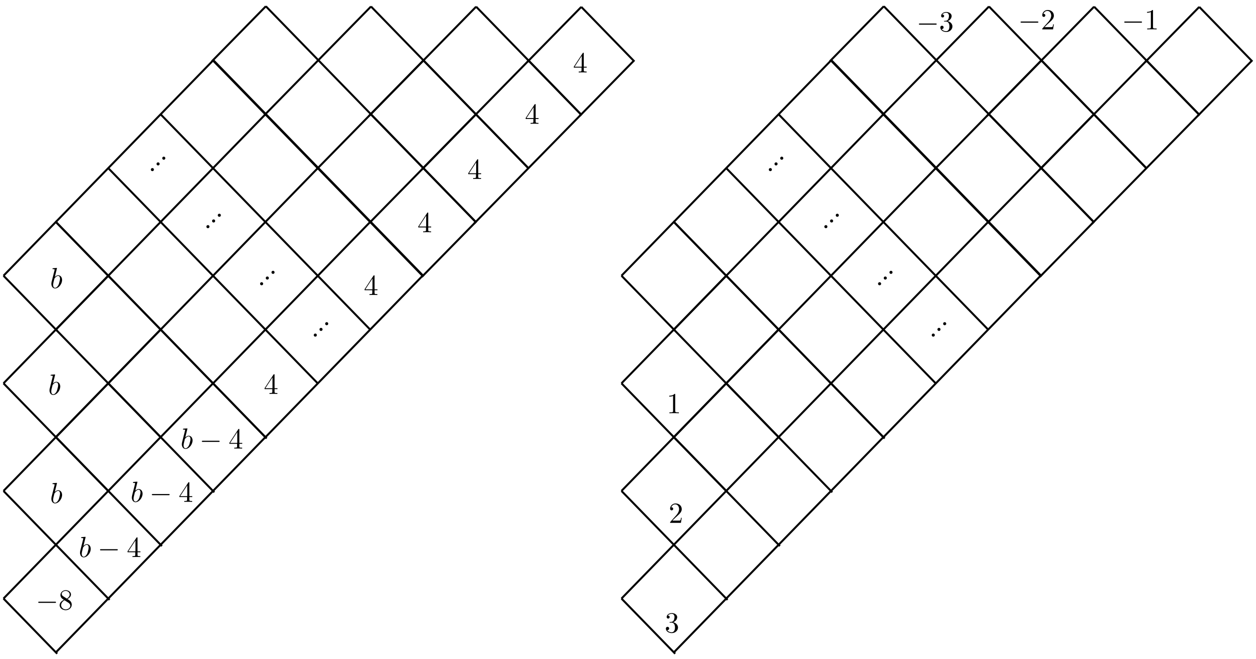
Combining our two equations for $\mathbb{E}[\mu, \varphi]$ yields the desired result. $\square$

A way to evaluate the error term, that is:

$$\frac{a-b}{a+b}\left(\sum_{i=0}^{a}(a-1-i)\mathbb{E}[\mu; \mathcal{T}^{+}_{(i,0)}] - \sum_{i=0}^{a-2} i\mathbb{E}[\mu; \mathcal{T}^{-}_{(i,\lambda_i),(i+1,\lambda_{i+1})}]\right)$$

would yield many results about the trapezoid. Down-degree is homomesic with respect to an invertible operator Φ, if for any orbit $\mathcal{O}$ of Φ, and the error term for the distribution $uni_{\mathcal{O}}$ is 0. The error term is 0 for the distribution defined by choosing a standard shifted Young Tableaux on $T_{a,b}$ uniformly at random and then choosing an ideal consisting of all numbers $\leq m$ for some m if and only if a conjecture of Reiner, Tenner and Yong ([RTY18], conjecture 2.24) is true. The error term is 0 for the distribution defined by choosing a plane partition of height m uniformly at random and then choosing one of the m ideals from the plane partition uniformly at random if and only if a conjecture of Hopkins ([Hop19] Conjecture 3.10) is true.

Figure 5: Rook arrangement defined by φ for $T_{4,b}$ and error term scaled down by $(a+b)/(a-b)$



In many general cases, we did not find a way to evaluate this error term. For rowmotion orbits, there is a way to trace out rowmotion orbits and consider a finite number of cases to show $\mathbb{E}[uni_{\mathcal{O}}; \text{ddeg}] = ab/(a+b)$ by locally (i.e. over some fixed number of rowmotion orbits) showing the error term is 0.

Theorem 3.7. *The action of rowmotion on the trapezoid exhibits For a rowmotion orbit $\mathcal{O}$ on $T_{3,n}$ or $T_{4,n}$,*

$$\mathbb{E}[uni_{\mathcal{O}}; \text{ddeg}] = \frac{ab}{a+b}.$$

For $T_{3,n}$, there were 3 cases we needed to consider and for $T_{4,n}$ there were 9 cases. We will not include this casework since there is a more powerful theorem we can show for $T_{3,n}$ and $T_{4,n}$:

Theorem 3.8. *For any toggle on anitchains-symmetric distribution μ on $T_{3,n}$ and $T_{4,n}$,*

$$\mathbb{E}[\mu; \text{ddeg}] = \frac{ab}{a+b}$$

In [Hop19], Hopkins proves a rowmotion orbit of order ideals is toggle on antichains-symmetric. Thus our above theorem implies theorem 3.7.

3.2 Proof of Theorem 3.8

Although Theorem 3.8 is only concerned with the $T_{3,n}$ and $T_{4,n}$ cases, we will state many of our lemmas more generally in hopes that this may help future readers can generalize our theorem.

Lemma 3.9. *For any toggle-symmetric distribution μ on the trapezoid $T_{a,b}$*

$$\sum_{i=0}^{a-1}(a-1-i)\mathbb{E}[\mu;\mathcal{T}^-_{(i,0)}] - \sum_{\substack{i+j\le a-1\\ j>0}} \mathbb{E}[\mu;\mathcal{T}^-_{(i,j)}] = 0$$

Proof. Define

$$\begin{aligned} a_n &:= \sum_{j=1}^{a-1-i} \mathbb{E}[\mu;\mathcal{T}^-_{(i,j)}]] \\ &= \Pr[\mu; I \text{ has a maximal element in } \{(i,j)|i+j=n, j>0\}] \\ b_n &:= \mathbb{E}[\mu;\mathcal{T}^-_{(i,j)}]] \\ &= \Pr[\mu;(n,0) \text{ is a maximal element in } I] \end{aligned}$$

Importantly, notice that

$$\begin{aligned} a_n + b_n &= \Pr[\mu; I \text{ has a maximal element in } \{(i,j)|i+j=n\}] \\ &= \Pr[\mu; I^C \text{ has a minimal element in } \{(i,j)|i+j=n+1|j>0\}] \\ &= \Pr[\mu; I \text{ has a maximal element in } \{(i,j)|i+j=n+1|j>0\}] \qquad \text{[by toggle-symmetry]} \\ &= a_{n+1} \end{aligned}$$

where I^C is the complement of the ideal I. The second to last equality follows from if an ideal is cut out with a Λ-shaped nook on a non-maximal element, then it must be followed by a V-shaped nook and vice versa. We now compute

$$\begin{aligned} \sum_{i=0}^{a-1}(a-1-i)\mathbb{E}[\mu;\mathcal{T}^-_{(i,0)}] - \sum_{\substack{i+j\le a-1\\ j>0}} \mathbb{E}[\mu;\mathcal{T}^-_{(i,j)}] &= \sum_{i=0}^{a-1}(a-1-i)b_i - \sum_{i=1}^{a} a_i \\ &= \left(\sum_{i=0}^{a-1}(a-1-i)(a_{i+1}-a_i)\right) - \sum_{i=1}^{a} a_i \\ &= \sum_{i=1}^{a} a_i - \sum_{i=1}^{a} a_i = 0 \end{aligned}$$

□

In particular, the above lemma says that a toggle-symmetric distribution μ satisfies $\mathbb{E}[\mu;\mathrm{ddeg}] = ab/(a+b)$ if and only if our error term form lemma 3.6 is equal to

$$\sum_{i=0}^{a-1}(a-1-i)\mathbb{E}[\mu;\mathcal{T}^-_{(i,0)}] - \sum_{\substack{i+j\le a-1\\ j>0}} \mathbb{E}[\mu;\mathcal{T}^-_{(i,j)}] = 0.$$

Thus we have

Proposition 3.10. *A toggle-symmetric distribution μ satisfies $\mathbb{E}[\mu;\mathrm{ddeg}] = ab/(a+b)$ if and only if*

$$\sum_{\substack{i+j\le a-1\\ j>0}} \mathbb{E}[\mu;\mathcal{T}^-_{(i,j)}] = \sum_{i=0}^{a-2} i\mathbb{E}[\mu;\mathcal{T}^-_{(i,\lambda_i),(i+1,\lambda_{i+1})}].$$

To make progress towards proving this equality, we will first convert the LHS into a sum of toggles on antichains. Then we will try to use the toggle on antichains-symmetry property to move our toggles on antichains in the RHS from the maximal elements to toggles on antichains near the minimal elements of the trapezoid.

Lemma 3.11. *For a trapezoid $T_{a,b}$, there is an equality of functions on $J(T_{a,b})$:*

$$\sum_{\substack{(i,j)|i+j\leq a-1\\ j>0}} \mathcal{T}^+_{(i,j)} = \sum_{n\geq 2} \sum_{\substack{|A|=n\\ A\in\{(i,j)|i+j\leq a+1\}}} (-1)^n \mathcal{T}^-_A$$

Proof. Consider an ideal in the restricted shape $S = T_{a,b} \cap \{(i,j)|i+j \leq a, (i,j) \neq (a-1,1)\}$, with the same $<$ relations as in $T_{a,b}$. It is clear that the functions on $J(T_{a,b})$

$$\sum_{\substack{(i,j)|i+j\leq a-1\\ j>0}} \mathcal{T}^+_{(i,j)}, \sum_{n\geq 2} \sum_{\substack{|A|=n\\ A\in\{(i,j)|i+j\leq a+1\}}} (-1)^n \mathcal{T}^-_A$$

may be viewed as functions on restrictions of ideals in $J(T_{a,b})$ to $J(S)$. Any ideal in $J(S)$ corresponds to the lattice path consisting of up-left and down-left steps starting at the rightmost corner of our shape and ending with an up-left step at one of $a-1$ of the leftmost vertices of our shape, which cuts out the ideal i.e. the set of elements below our path is the ideal. $\sum_{\substack{(i,j)|i+j\leq a-2\\ j>0}} \mathcal{T}^+_{(i,j)}$ counts the number of Λ-shaped nooks with both edges on a square (i,j) with $i+j \leq a-2$ and $j > 0$ of such a lattice path. For each of these Λ shaped nooks, there must be a V-shaped nook after and before it. Conversely given any two V-shaped nooks, there is a there must be a Λ-shaped nook inbetween them, and the edges of this nook must be on a square (i,j) with (i,j) with $i+j \leq a-2$ and $j > 0$. Thus

$$\sum_{p\in S} \mathcal{T}^+_p = 1 + \sum_{\substack{(i,j)|i+j\leq a-1\\ j>0}} \mathcal{T}^+_{(i,j)}$$

and

$$\begin{aligned}
\sum_{n\geq 2} \sum_{\substack{|A|=n\\ A\in\{(i,j)|i+j\leq a+1\}}} \mathcal{T}^-_A &= \sum_{n\geq 2} (-1)^n \binom{1 + \sum_{\substack{(i,j)|i+j\leq a-1\\ j>0}} \mathcal{T}^+_{(i,j)}}{n} \\
&= \binom{1 + \sum_{\substack{(i,j)|i+j\leq a-1\\ j>0}} \mathcal{T}^+_{(i,j)}}{1} + \sum_{n\geq 1} (-1)^n \binom{1 + \sum_{\substack{(i,j)|i+j\leq a-1\\ j>0}} \mathcal{T}^+_{(i,j)}}{n} \\
&= 1 + \sum_{\substack{(i,j)|i+j\leq a-1\\ j>0}} \mathcal{T}^+_{(i,j)}
\end{aligned}$$

□

Lemma 3.12. *For a trapezoid $T_{a,b}$, and a toggle on antichains-symmetric distribution μ,*

$$\sum_{i=0}^{a-2} \mathbb{E}[\mu; \mathcal{T}^-_{(i,\lambda_i),(i+1,\lambda_{i+1})}] = \sum_{i=1}^{a-1} \sum_{j<i} \mathbb{E}[\mu; T^-_{(j,i-j+1),(i,0)}] = \sum_{i=0}^{a-2} \mathbb{E}[\mu; \mathcal{T}^-_{(i,1)}]$$

Proof. Call an antichain A with two elements *adjacent* if $A = \{(i,j),(i',j')\}$ with $i+j = i'+j'+1$. Define the subsets of adjacent antichains

$$\begin{aligned}
S^- &:= \{A = \{(i,j),(i',j')\} \in T_{a,b} | i+j = i'+j'+1, j \neq \lambda_i\},\\
S^+ &:= \{A = \{(i,j),(i',j')\} \in T_{a,b} | i+j = i'+j'+1, 0 < j, j'\}\\
S &:= S^+ \cap S^-.
\end{aligned}$$

To get the first equality, we consider the following equation on ideals

$$f := \sum_{A\in S^+} \mathcal{T}_A^+ - \sum_{A\in S^-} \mathcal{T}_A^-$$
$$= \sum_{A\in S} \mathcal{T}_A + \sum_{i=0}^{a-2} \mathcal{T}^+_{(i,\lambda_i),(i+1,\lambda_{i+1})} - \sum_{i=1}^{a-1}\sum_{j<i} \mu; T^-_{(j,i-j+1),(i,0)}$$

We compute $\mathbb{E}[\mu; f]$ in two ways. First notice that if you can toggle in an antichain in S^+, then you can toggle out an antichain in S^-, thus for all ideals I, $f(I) = 0$ and so $\mathbb{E}[\mu; f] = 0$. Next

$$\mathbb{E}[\mu; f] = \sum_{A\in S} \mathbb{E}[\mu; \mathcal{T}_A] + \sum_{i=0}^{a-2} \mathbb{E}[\mu; \mathcal{T}^+_{(i,\lambda_i),(i+1,\lambda_{i+1})}] - \sum_{i=1}^{a-1}\sum_{j<i} \mathbb{E}[\mu; T^-_{(j,i-j+1),(i,0)}]$$
$$= \sum_{i=0}^{a-2} \mathbb{E}[\mu; \mathcal{T}^+_{(i,\lambda_i),(i+1,\lambda_{i+1})}] - \sum_{i=1}^{a-1}\sum_{j<i} \mathbb{E}[\mu; T^-_{(j,i-j+1),(i,0)}]$$

We get the second equality from toggle on antichains-symmetry:

$$\sum_{i=0}^{a-2} \mathbb{E}[\mu; \mathcal{T}^-_{(i,1)}] = \sum_{i=1}^{a-1}\sum_{j<i} \mathbb{E}[\mu; T^+_{(j,i-j+1),(i,0)}]$$
$$= \sum_{i=1}^{a-1}\sum_{j<i} \mathbb{E}[\mu; T^-_{(j,i-j+1),(i,0)}]$$

□

Proof of Theorem 3.8. Case 1: $T_{3,n}$. Between proposition 3.10 and lemma 3.12, all that remains, is to show

$$\mathbb{E}[\mu; \mathcal{T}^-_{(0,2)}] = \mathbb{E}[\mu; \mathcal{T}^-_{(1,\lambda_1),(2,\lambda_2)}].$$

Define S to be the set of anitchains

$$S := \{A \in T_{3,n} | A = \{(2, j), (1, j + 2)(0, j + 4)\} \text{ or } A = \{(2, j), (1, j + 2)(0, j + 5)\} \text{ and } j < \lambda_2\}$$

Then define a function of ideals,

$$f := \sum_{A\in S} \mathcal{T}_A - \mathcal{T}^-_{(0,2)} + \mathcal{T}_{(1,1)(0,3)} + \mathcal{T}_{(1,0)(0,2)} + \mathcal{T}_{(1,1)(0,4)} + \mathcal{T}_{(1,0)(0,3)} + \mathcal{T}^+_{(1,\lambda_1),(2,\lambda_2)}$$

We compute $\mathbb{E}[\mu; f]$ in two ways. First I claim for any ideal I, $f(I) = 0$. This can be checked for any ideal with all maximal elements in $\{(i, j)|i + j > \lambda_2\}$. For smaller j, we can check by hand that for the restriction of the ideal to $T_{a,b} \cap \{(i, j)|2i + j < 5\}$

$$\mathcal{T}^-_{(2,0),(1,2),(0,5)} + \mathcal{T}^-_{(2,0),(1,2),(2,4)} + \mathcal{T}^+_{(0,2)} + \mathcal{T}_{(1,1)(0,3)} + \mathcal{T}_{(1,0)(0,2)} + \mathcal{T}_{(1,1)(0,4)} + \mathcal{T}_{(1,0)(0,3)} = 0.$$

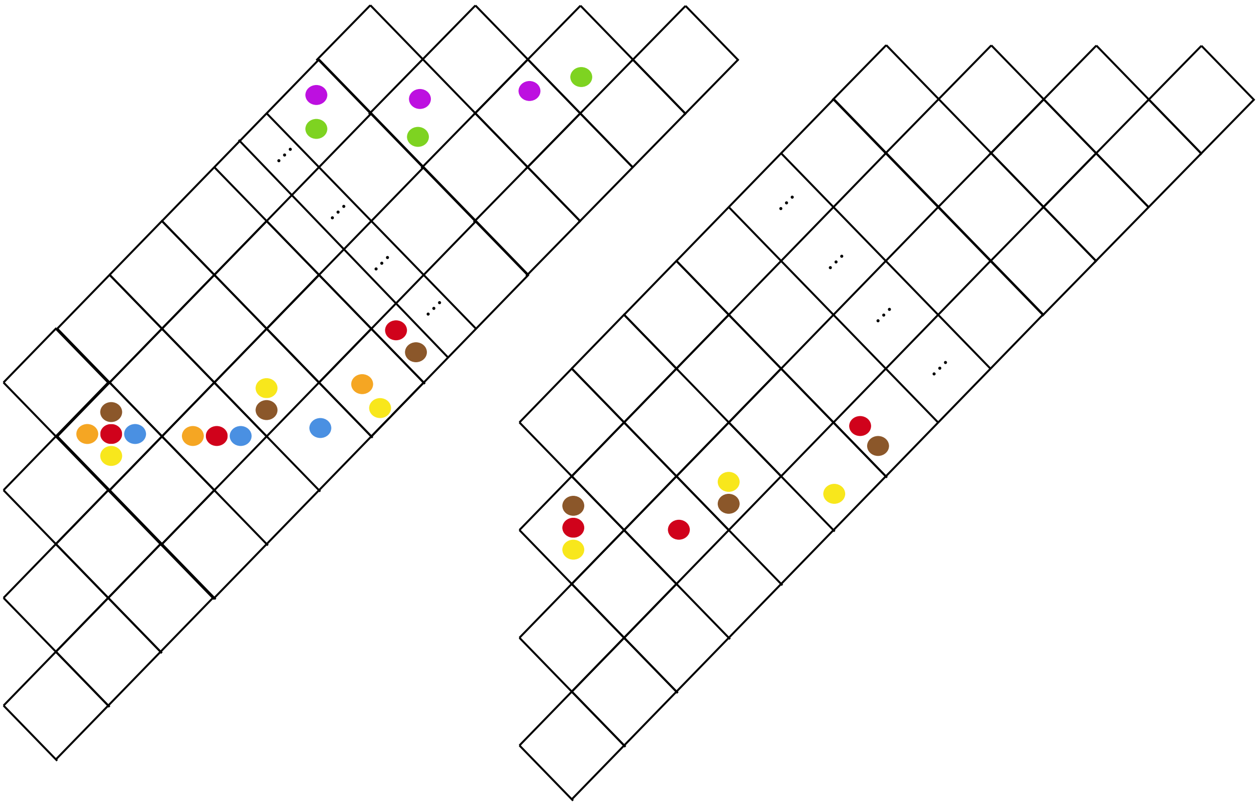
For other ideals, we notice that $\mathcal{T}^+_{(2,j),(1,j+2)(0,j+4)}(I) = 1$ for $j > 1$ if and only if $\mathcal{T}^-_{(2,j-1),(1,j+1)(0,j+3)}(I) = 1$.Thus $f(I) = 0$. On the other hand, from toggle on antichains symmetry,

$$\mathbb{E}[\mu; f] = \mathbb{E}[\mu; \mathcal{T}^-_{(0,2)}] - \mathbb{E}[\mu; \mathcal{T}^-_{(1,\lambda_1),(2,\lambda_2)}]$$

Thus we conclude the theorem for the $T_{3,n}$ case.

Case 2: $T_{4,n}$. We will do this in three computations:

Figure 6: The antichains of S' coded by color



1) Let S be the set of 4 element antichains of $T_{4,n}$ of the form $(3,j),(2,j+2),(3,j+4),(4,j+6+m)$ for $0 \le m \le 2$ or of the form $(3,j),(2,j+2),(3,j+5),(4,j+6+m)$ for $0 \le m \le 1$ and $j < \lambda_4 - 1$. Let S' be the set of color coded antichains in figure 6

Consider the function of ideals

$$f = \sum_{A \in S} \mathcal{T}_A + \sum_{A \in S'} \mathcal{T}_A + \mathcal{T}^{+}_{(2,\lambda_2),(3,\lambda_3)} - \mathcal{T}^{-}_{(1,1),(0,5)} - T^{-}_{(1,2),(0,5)} - T^{-}_{(1,2),(0,4)}$$

We compute $\mathbb{E}[\mu; f]$ in two ways. First I claim for any ideal I, $f(I) = 0$. This can be checked for any ideal with maximal elements (i,j) for $i+j > 8$ by noting if we can toggle in an antichain in S or S', then we can toggle our an antichain in S or S'. Moreover, in any ideal we can only toggle in at most 1 antichain in S or S' and in any ideal we can toggle out at most 1 antichain from S or S'. We can check for any ideal with a maximal element with $i+j < 8$, that these also satisfy $f(I) = 0$. On the other hand, from toggle on antichains symmetry,

$$\mathbb{E}[\mu; f] = \mathbb{E}[\mu; \mathcal{T}^{+}_{(2,\lambda_2),(3,\lambda_3)}] - \mathbb{E}[\mu; \mathcal{T}^{-}_{(1,1),(0,5)} + T^{-}_{(1,2),(0,5)} + T^{-}_{(1,2),(0,4)}]$$

2)Let S be the set of three element antichains of the form $(i_1,j_1),(i_2,j_2),(i_3,j_3)$ where $i_1 > i_2 > i_3$, $i_1+j_1+2 = i_2+j_2$, and $i_2+j_2+2 \le i_3+j_3 \le i_2+j_2+3$ with $i_1+j_1 \le 3+\lambda_3$ and $(i_2,j_2) \ne (2,\lambda_2)$. Let S' be the set of 4 element antichains of the form $(3,j),(2,j+2),(1,j+4),(0,j+6)$. Then define a function f on ideals

$$f = \sum_{A \in S} \mathcal{T}_A - \sum_{A \in S'} \mathcal{T}_A + \mathcal{T}^{+}_{(2,\lambda_2),(3,\lambda_3)} + \mathcal{T}^{+}_{(1,\lambda_1),(2,\lambda_2)} + \mathcal{T}^{-}_{(2,1),(1,3),(0,5)} - \sum_{i=1}^{2} \left(\sum_{0 \le x < i} T^{-}_{(i,1),(x,i+2-x)} + T^{-}_{(i,1),(x,i+3-x)} \right)$$

We compute $\mathbb{E}[\mu; f]$ in two ways. First, I claim for any ideal I, $f(I) = 0$. For ideals which contain maximal elements (i,j) for $i+j > \lambda_3 + 2$, this can be checked since there are few since there are few possible nonzero terms in f. For ideals which contain multiple maximal elements (i,j) with $i+j \leq 8$, this can also be checked since there are few possible nonzero terms in f. For other ideals I, by chasing V and Λ nooks, we see you can toggle in an element of S' if and only if you can toggle out an element of S', and that the number of elements of S which can by toggled in equals the number of elements of S which can be toggled out. We conclude that $f(I) = 0$. On the other hand, from toggle on antichains symmetry,

$$\mathbb{E}[\mu; f] = \mathbb{E}\left[\mu; \mathcal{T}^+_{(2,\lambda_2),(3,\lambda_3)} + \mathcal{T}^+_{(1,\lambda_1),(2,\lambda_2)} + \mathcal{T}^-_{(2,1),(1,3),(0,5)} - \sum_{i=1}^{2}\left(\sum_{0\leq x<i} T^-_{(i,1),(x,i+2-x)} + T^-_{(i,1),(x,i+3-x)}\right)\right]$$

3) By proposition 3.10, lemma 3.12 and adding our calculations in (1) and (2), showing

$$\mathcal{T}^-_{(1,1),(0,5)} + T^-_{(1,2),(0,5)} + T^-_{(1,2),(0,4)} + \sum_{i=1}^{2}\left(\sum_{0\leq x<i} T^-_{(i,1),(x,i+2-x)} + T^-_{(i,1),(x,i+3-x)}\right)$$
$$-\mathcal{T}^-_{(2,1),(1,3),(0,5)} + \sum_{i=1}^{a-1}\sum_{j<i}\mathbb{E}[\mu; T^-_{(j,i-j+1),(i,0)}] = \sum_{k\geq 2}\sum_{\substack{|A|=k \\ A\in\{(i,j)|i+j\leq a+1\}}} (-1)^k \mathcal{T}^-_A$$

will imply our theorem for $T_{4,n}$. This can be checked for any ideal by restricting the ideal to $T_{4,n} \cap \{(i,j)|i+j \leq 8\}$ and checking all ideals in this shape. □

Conjecture 3.13 ([Hop19, Conjecture 4.30])**.** *There is a size-preserving bijection φ between the rowmotion orbits of* $\mathrm{PP}^\ell(R_{a,b})$ *and* $\mathrm{PP}^\ell(T_{a,a+b})$*. Moreover, for all rowmotion orbits $\mathcal{O}$ for* $\mathrm{PP}^\ell(R_{a,b})$*, we have:*

$$\sum_{T\in\mathcal{O}} \operatorname{ddeg} T = \sum_{T\in\varphi(\mathcal{O})} \operatorname{ddeg} T.$$

Conjecure 2.7 gives a candidate for the bijection φ for the case $\ell = 1$. Together with homomesy of rowmotion orbits of the trapezoid, this approach would be enough to prove the conjecture for the case $\ell = 1$.

4 Acknowledgements

This research was carried out as part of the 2019 REU program at the School of Mathematics of the University of Minnesota, Twin Cities. The authors are grateful for the support of NSF RTG grant DMS-1148634 for the REU program. The authors would like to thank Sam Hopkins, Andy Hardt and Vic Reiner for their mentorship and guidance.

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