REU REPORT

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Abstract. We look at a generalized version of degree sequence of hypergraphs, where we consider linear combinations of (hyper)-edges with rational, integer or positive integer coefficients, and try to describe the algebraic objects thus defined ($\mathbb{Q}\mathrm{E}(G)$, $\mathbb{Z}\mathrm{E}(G)$, $\mathbb{Z}_{+}\mathrm{E}(G)$) in terms of properties of the hypergraph.

We find a restrictive definition of connectivity for hypergraphs which ensures a description of $\mathbb{Z}\mathrm{E}(G)$ in terms of partitions of k into positive integers.

We also look at the integral closure of the $\mathbb{Z}_{+}\mathrm{E}(G)$, which has been fully described for graphs, and give varied examples of why generalizations to higher dimensions encounter problems.

Contents

1. Introduction and Case $k = 2$

We first define k-hypergraphs, with and without degenerate edges. Let $\mathbb{N} := \{1, 2, 3...\}$ and $[n] := \{1, 2, 3, ..., n\}$. For a set A we denote by $\binom{A}{k}$ the family of k-element subsets of A.

Definition 1.1. A k-hypergraph G is a pair (V, E), where $V \subset \mathbb{N}$ and $E \subset \binom{V}{k}$. The elements of E are called the *edges* of G.

Now consider multisets, where the same element is allowed to appear a finite number of times; the cardinality of a multiset will count each element with its

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multiplicity, and from now on a sum over the elements of a multiset will take each element with the corresponding multiplicity, unless otherwise noted. Given a set A, denote $\left(\!\!\binom{A}{k}\!\!\right) := \{B \text{ multiset } : |B| = k \text{ and if } e \in B \text{ then } e \in A\}$.

Definition 1.2. A generalized *k-hypergraph* G, or for short a *k-hygraph*, is a pair (V, E), with $V \subset \mathbb{N}$ and $E \subset \left(\!\!\binom{V}{k}\!\!\right)$. A multiset $e \in E$ is called an *edge*, and *degenerate edge* if at least one element in e has multiplicity greater than one.

Notice that even for a k-hygraph, E is defined as a set, not a multiset, thus each edge is counted at most once. On a different note, further definitions for generalized k-hypergraphs will restrict naturally to k-hypergraphs.

Definition 1.3. (characteristic vectors) Given a multiset $A \in \left(\!\!\binom{[n]}{m}\!\!\right)$, for some $n,\ m \in \mathbb{N}$, define the characteristic vector $\chi_A \in \mathbb{R}^n$ of the multiset A by $\chi_A := \sum_{i \in A} e_i$, where e_i, for $i = 1, 2, ..., n$, are the standard basis vectors of $\mathbb{R}^n$.

The degree sequence $d(G) \in \mathbb{R}^n$ of a k-hygraph $G = ([n], E)$ can be defined as $d(G) := \sum_{e \in E} \chi_e$. A classical question is to determine a condition for a vector in

$$\mathbb{Z}^n_{\equiv 0(k)} := \{v \in \mathbb{Z}^n : \sum_{j=1}^{n} v_j \equiv 0 (\text{mod } k)\}$$

to be a degree sequence of a k-hygraph, for k and n fixed, $k \leq n$.

Branching off from this question, one can look at a generalized form of degree sequences, where we allow coefficients in front of the χ_e's.

Definition 1.4. Given an additive semigroup $\mathcal{R} \subset \mathbb{R}$ and a finite subset M of $\mathbb{R}^n$, define the semigroup $\mathcal{R}\mathrm{M} \subset \mathbb{R}^n$ as

$$\mathcal{R}\mathrm{M} = \left\{ \sum_{v \in M} \alpha_v v : \ \alpha_v \in \mathcal{R} \right\}$$

Given a k-hygraph $G = (V, E)$ define $\mathcal{R}\mathrm{E(G)} = \mathcal{R}\{\chi_e : e \in E\}$.
We will usually use:

$\mathcal{R} = \mathbb{Z}_+ := \{v \in \mathbb{Z} : v \geq 0\}$
$\mathcal{R} = \mathbb{Q}_+ := \{v \in \mathbb{Q} : v \geq 0\}$
$\mathcal{R} = \mathbb{R}$
$\mathcal{R} = \mathbb{Q}$
$\mathcal{R} = \mathbb{Z}$.

Notice that $\mathbb{Z}_+\mathrm{E}(G)$ allows non-negative coefficients for each edge, $\mathbb{Z}\mathrm{E}(G)$ considers all integer coefficients, while $\mathbb{Q}_+\mathrm{E}(G)$ describes the rational points included in the cone $\mathbb{R}_+\mathrm{E}(G)$.

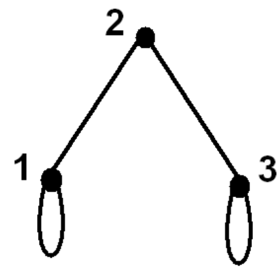
Remark 1.5. The study of $\mathbb{R}\mathrm{E}(G)$ is essentially equivalent to the study of $\mathbb{Q}\mathrm{E}(G)$, in the sense that $\mathbb{R}\mathrm{E}(G) = \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}\mathrm{E}(G)$, because we have $\chi_e \in \mathbb{Q}^n$ for any $e \in E$, and $\mathbb{R}/\mathbb{Q}$ is a field extension. A geometric property that now follows is $\mathbb{Q}\mathrm{E}(G) = \mathbb{R}\mathrm{E}(G) \cap \mathbb{Q}^n$.

A similar remark applies for $\mathbb{R}_+\mathrm{E}(G)$ and $\mathbb{Q}_+\mathrm{E}(G)$. From now on, we will focus on describing $\mathbb{Q}\mathrm{E}(G)$, and $\mathbb{Q}_+\mathrm{E}(G)$.

There is one more object that we will study:

Definition 1.6. (*saturation* or *integral closure*) For a k-hygraph $G = ([n], E)$ define $\mathbb{Z}_+^{\mathrm{sat}}\mathrm{E}(G) := \mathbb{Q}_+\mathrm{E}(G) \cap \mathbb{Z}\mathrm{E}(G)$.

Let's consider an example to illustrate these notions:



Example 1.7. Let $G = ([3], E)$ with $E = \{\{1,1\}, \{1,2\}, \{2,3\}, \{3,3\}\}$. Then:

- $\chi_{\{1,1\}} = (2,0,0)$, $\chi_{\{1,2\}} = (1,1,0)$ etc.
- it is easy to see that $\mathbb{Z}\mathrm{E}(G) = \mathbb{Z}^3_{\equiv 0(2)}$.
- $\mathbb{Z}_+\mathrm{E}(G) \subsetneqq \mathbb{Z}_+^{\mathrm{sat}}\mathrm{E}(G)$, because

  $(1,0,1) = \frac{1}{2}(2,0,0) + \frac{1}{2}(0,0,2) = \frac{1}{2}\chi_{\{1,1\}} + \frac{1}{2}\chi_{\{3,3\}} \in \mathbb{Q}_+\mathrm{E}(G)$ and

  $(1,0,1) = \chi_{\{2,3\}} - \chi_{\{1,2\}} + \chi_{\{1,1\}} \in \mathbb{Z}\mathrm{E}(G)$

  so $(1,0,1) \in \mathbb{Z}_+^{\mathrm{sat}}\mathrm{E}(G)$ but $(1,0,1) \notin \mathbb{Z}_+\mathrm{E}(G)$. Moreover, we will see that $(1,0,1)$ is unique in the sense that together with $E(G)$ it generates $\mathbb{Z}_+^{\mathrm{sat}}\mathrm{E}(G)$.

We will focus on describing $\mathbb{Q}\mathrm{E}(G)$, $\mathbb{Z}\mathrm{E}(G)$, $\mathbb{Z}_+^{\mathrm{sat}}\mathrm{E}(G)$ in terms of graphical properties of G. The case $k = 2$ yields complete characterizations of all these objects.

Remark 1.8. We allow *loops*, i.e. degenerate edges $\{i, i\}$, and we consider a *loop* to be an odd cycle.

We start by describing $\mathbb{Z}\mathrm{E}(G)$ and $\mathbb{Q}\mathrm{E}(G)$:

Proposition 1.9. *($\mathbb{Z}E(G)$ for graphs) Assume $G = ([n], E)$ is a graph (a 2-hygraph) which is connected, then:*

- *if G does not contain any odd cycles (is bipartite), with $V_1 \sqcup V_2 = [n]$ such that for any for $e \in E$, $e \cap V_i \neq \varnothing$, $i = 1, 2$ (that is, every edge in E has one end in V_1 and the other in V_2), define*

$$\mathbb{H} = \left\{ v \in \mathbb{R}^n : \sum_{i \in V_1} v_i = \sum_{j \in V_2} v_j \right\}$$

.
Then $\mathbb{Z}E(G) = \mathbb{Z}^k_{\equiv 0(2)} \cap \mathbb{H} = \mathbb{Z}^n \cap \mathbb{H}$ *and* $\mathbb{Q}E(G) = \mathbb{Q}^n \cap \mathbb{H}$*;*

- *if* G *contains an odd cycle (is not bipartite), then* $\mathbb{Z}E(G) = \mathbb{Z}^n_{\equiv 0(2)}$ *and* $\mathbb{Q}E(G) = \mathbb{Q}^n$*.*

Another nice property for $k = 2$ is that $\mathbb{Z}\mathrm{E}(G)$ is the set of all integer points with even sum of coordinates in the cone $\mathbb{Q}\mathrm{E}(G)$. It is obvious that $\mathbb{Z}\mathrm{E}(G) \subseteq \mathbb{Q}\mathrm{E}(G) \cap \mathbb{Z}^n_{\equiv 0(2)}$, but the equality is proven by:

Proposition 1.10. *For a graph* $G = ([n], E)$ *we have* $\mathbb{Z}E(G) = \mathbb{Q}E(G) \cap \mathbb{Z}^n_{\equiv 0(2)}$ *hence* $\mathbb{Z}^{sat}_+ E(G) = \mathbb{Q}_+ E(G) \cap \mathbb{Z}^n_{\equiv 0(2)}$*.*

Proof. We distinguish two cases:

- If G contains no odd cycles, then from Proposition 1.9 we deduce $\mathbb{Z}\mathrm{E}(G) = \mathbb{Z}^n_{\equiv 0(2)} \cap \mathbb{H}$. We have that for any $e \in E$, $\chi_e \in \mathbb{H}$, so $\mathbb{Q}\mathrm{E}(G) \subset \mathbb{H}$, thus

$$\mathbb{Z}\mathrm{E}(G) = \mathbb{Q}\mathrm{E}(G) \cap (\mathbb{Z}^n_{\equiv 0(2)} \cap \mathbb{H}) = (\mathbb{Q}\mathrm{E}(G) \cap \mathbb{H}) \cap \mathbb{Z}^n_{\equiv 0(2)} = \mathbb{Q}\mathrm{E}(G) \cap \mathbb{Z}^n_{\equiv 0(2)}$$
.

- If G contains at least one odd cycle, then by Proposition 1.9 we already have $\mathbb{Z}\mathrm{E}(G) = \mathbb{Z}^n_{\equiv 0(2)} \subset \mathbb{Q}\mathrm{E}(G)$.

□

The most important result for $k = 2$ is the description of $\mathbb{Z}^{\mathrm{sat}}_+\mathrm{E}(G)$ in the case of a connected graph G. We will sketch a proof drawn from [5, Th 8.7.9] and especially from [2]; the idea is to clearly separate the steps of the proof, with the purpose of adapting them for the later study of $\mathbb{Z}^{\mathrm{sat}}_+\mathrm{E}(G)$ for $k > 2$.

Definition 1.11. A vector $v \in \mathbb{Z}^n_+$ is called a *hole* of $\mathbb{Z}^{\mathrm{sat}}_+\mathrm{E}(G)$ if $v \in \mathbb{Z}^{\mathrm{sat}}_+\mathrm{E}(G) \setminus \mathbb{Z}_+\mathrm{E}(G)$.

Theorem 1.12. *For a connected graph* $G = ([n], E)$*, denote*

$$W = \{\chi_w : w \subset E,\ w = \mathcal{O}_1 \sqcup \mathcal{O}_2,\ \textit{where } \mathcal{O}_1 \textit{ and } \mathcal{O}_2 \textit{ are disjoint}$$
$$\textit{chordless odd cycles not connected by one edge } \}$$

Then

$$\mathbb{Z}^{sat}_+ E(G) = \mathbb{Z}_+(E(G) \cup W).$$

Proof. ***(Step 1)*** Take a hole $v \in \mathbb{Q}_+\mathrm{E}(G) \cap \mathbb{Z}\mathrm{E}(G) \setminus \mathbb{Z}_+\mathrm{E}(G)$. We can write $v = \sum_{e \in E_0} r_e \chi_e$, with $E_0 \subseteq E$ and $r_e \in \mathbb{Q}_+$ for all $e \in E_0$.

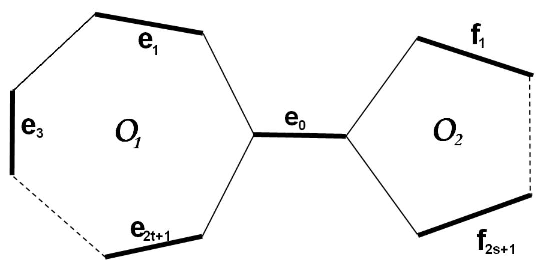
A crucial step is that we can assume w.l.og. that the vectors in $\chi_{E_0} := \{\chi_e : e \in E_0\}$ are linearly independent over $\mathbb{R}$ by Caratheodory's theorem.

Now consider $v' = \sum_{e \in E_0} (r_e - \lfloor r_e \rfloor)\chi_e$ and we will write $v' = \sum_{e \in E_1} q_e \chi_e$, with $q_e \in (0,1) \cap \mathbb{Q}$ for all $e \in E_1 \subset E_0$ and χ_{E_1} linearly independent. Notice that v' is also a hole because $v = v' + \sum_{e \in E_0} a_e \chi_e$ with $a_e \in \mathbb{Z}_+$ for all $e \in E_0$. This ensures $E_1 \neq \varnothing$. We proceed by proving that $v' \in \mathbb{Z}_+(E(G) \cup W)$.

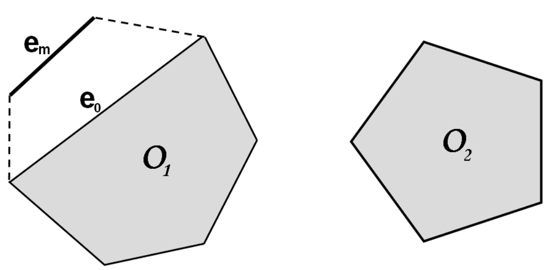
(Step 2) We look at the graph $G' = (V, E_1)$ formed by the edges in E_1 and their vertices $V = \{i \in [n] : i \in e \text{ for some } e \in E_1\}$:

(1) every vertex in G' has degree at least 2, because $v'_i = \sum_{i \in e} q_e$ and $0 < q_e < 1$ for each $e \in E_1$; thus any connected component of G' contains cycles;
(2) any even cycle gives a linear dependency (the alternate sum of edges is equal to 0), so G' contains only odd cycles;
(3) For a connected component T of G' we cannot have more than one odd cycle. Assume we have at least two odd cycles and a path that joins them (if the cycles intersect pick the empty path); this gives rise to a degenerated even cycle, which gives a linear dependency. It follows that T is exactly an odd cycle, because all degrees are at least 2. Thus G' is a union of disjoint odd cycles.
(4) We find that for each cycle, the weights q_e of the edges have to be exactly $\frac{1}{2}$, hence v''s entries are only 0 and 1, and each odd cycle contributes an odd number of 1's. It follows that G' has an even number of odd cycles, because $v' \in \mathbb{Z}^k_{\equiv 0(2)}$;

(Step 3) It is enough to prove that $v' \in \mathbb{Z}_+(E(G) \cup W)$ when G' has exactly two odd cycles, $\mathcal{O}_1$ and $\mathcal{O}_2$. We can prove this statement by induction on the sum of lengths of the two cycles.



Firstly, notice that if there is an edge e_0 connecting the two cycles, suppose that the cycles $\mathcal{O}_1$ and $\mathcal{O}_2$ excluding the edges ending in the vertices of e_0 are $\{e_1, e_2, ..., e_{2t+1}\}$ and $\{f_1, f_2, ..., f_{2s+1}\}$ respectively, where the edges are taken in the order they appear in the cycles. Then already $v' \in \mathbb{Z}_+\mathrm{E}(G)$, because $v' = \chi_{e_0} + (\chi_{e_1} + \chi_{e_3} + ... + \chi_{e_{2t+1}}) + (\chi_{f_1} + \chi_{f_3} + ... + \chi_{f_{2s+1}})$. So suppose there is no edge connecting $\mathcal{O}_1$ and $\mathcal{O}_2$.



If both cycles are chordless, by definition $v' \in W$. If cycle $\mathcal{O}_1$ has a chord e_0, we can write $\mathcal{O}_1 = \mathcal{O}_1' \sqcup \{e_1, ..., e_{2t+1}\}$ where $\mathcal{O}_1'$ is an odd cycle and the edges e_1, ... e_{2t+1} are part of the original cycle $\mathcal{O}_1$, in this order. Then by induction $\chi_{\mathcal{O}_1'} + \chi_{\mathcal{O}_2} \in \mathbb{Z}_+(E(G) \cup W)$, and $v' = \chi_{\mathcal{O}_1'} + \chi_{\mathcal{O}_2} + \chi_{v_1} + \chi_{v_3} + ... + \chi_{v_{2k+1}}$.

□

2. The integer lattice $\mathbb{Z}\mathrm{E}(G)$

2.1. **Strongly connected k-hygraphs.** We will now try to find characterizations of $\mathbb{Q}\mathrm{E}(G)$ and $\mathbb{Z}\mathrm{E}(G)$ for the more general case $k \geq 2$. We need two concepts: a connectivity condition to be imposed on the k-hygraphs, and a generalized version of the restrictions imposed by the bipartite-ness in the case $k = 2$ in Proposition 1.9.

Definition 2.1. A k-hygraph $G = ([n], E)$ is called *strongly connected* if for any two edges $a,\ b \in E$ there exist $m \in \mathbb{N} \setminus \{0\}$ and edges $a = e_1,\ e_2,\ \ldots,\ e_m = b$ such that $|e_i \cap e_{i+1}| \geq k - 1$ for any $i = 1, 2, ..., m-1$.

It is easy to see that this definition reduces to the usual connectivity in the case of graphs. We present an example to underline the multiset properties used in the case of degenerate edges:

Example 2.2. The 4-hygraph $G = ([4], \{\{1,2,2,3\}, \{2,2,3,4\}\})$ is strongly connected because $|\{1,2,2,3\} \cap \{2,2,3,4\}| = |\{2,2,3\}| = 3$.

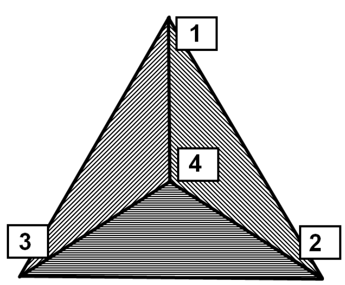
Nevertheless, the 4-hygraph $G = ([4], \{\{1,2,2,3\}, \{2,3,3,4\}\})$ is *not* strongly connected because $|\{1,2,2,3\} \cap \{2,3,3,4\}| = |\{2,3\}| = 2$.

The condition in Proposition 1.9 that a graph has no odd cycles is equivalent to the existence of a coloring of the vertices of the graph with two colors such that every edge has an endpoint of each color. We can generalize this concept in the following way:

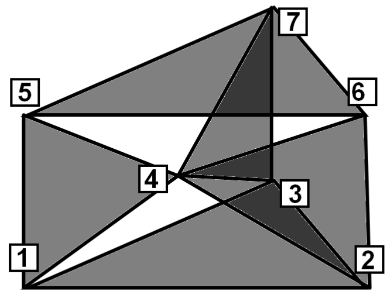
Definition 2.3. Given a k-hygraph $G = ([n], E)$ and a partition λ of k as sum of positive integers $k = \sum_{i=1}^{p} k_i$, the k-hygraph G is called λ-partite if there is a partition $[n] = \bigsqcup_{i=1}^{p} A_p$ such that for any edge $e \in E$ and $i \in \{1, 2, ..., p\}$ the sum of e's multiplicities over all elements of A_i is k_i.

Thus a k-hygraph is λ-partite if we can color its vertices with colors 1 though p and then every edge has exactly k_i vertices (counted with multiplicities) colored with color i. For $k = 2$, a bipartite graph is λ-partite with λ the partition $2 = 1 + 1$.

Focusing on $k = 3$, we will give an example where a 3-hygraph G is only $(3 = 2 + 1)$-partite and one where G is only $(3 = 3)$-partite.



Example 2.4. (3=2+1) Consider $G = ([4], \{\{1,2,4\},\{2,4,3\},\{3,4,1\}\})$. It is clear that G is not $(3 = 1+1+1)$-partite, but G is $(3 = 2+1)$-partite with $B_1 = \{1,2,3\}$ and $B_2 = \{4\}$.



Example 2.5. (3=3) Consider $G = ([7], E)$ with

$$E = \{\{1,2,3\},\{5,6,7\},\{1,4,5\},\{2,4,6\},\{3,4,7\}\}$$

It is clear that G does not accept any partition other than the trivial one.

Notice that if a k-hygraph is λ-partite for some λ, we obtain a constraint on $\mathbb{Q}\mathrm{E}(G)$ given by a linear relation: with the notations in Definition 2.3, for any $v \in \mathbb{Q}\mathrm{E}(G)$ we have

$$\frac{1}{k_1}\sum_{j\in A_1} v_j = \frac{1}{k_2}\sum_{j\in A_2} v_j = ... = \frac{1}{k_p}\sum_{j\in A_p} v_j$$

Indeed, the above relation is true by definition when $v = \chi_e$ and e is an edge, and $\mathbb{Q}\mathrm{E}(G)$ is generated by $\{\chi_e : e \in E\}$. The main result says that in the case of *strongly connected* k-hygraphs, this is the only restriction for $\mathbb{Q}\mathrm{E}(G)$, and the only restriction for $\mathbb{Z}\mathrm{E}(G)$ beyond $\mathbb{Z}\mathrm{E}(G) \subset \mathbb{Z}^n_{\equiv 0(k)}$.

Definition 2.6. Given a k-hygraph $G = ([n], E)$, call vertices $i_0,\ i_1 \in [n]$ *equivalent*, and denote $i_0 \sim i_1$, if $e_{i_0} - e_{i_1} \in \mathbb{Z}\mathrm{E}(G)$, where e_j are the standard basis for $\mathbb{R}^n$, for $j = 1, 2, ..., n$.

Then "$\sim$" is an equivalence relation on $[n]$, whom it partitions into equivalence classes, $[n] = \bigsqcup_{j=1}^p B_j$.

Now the main result:

Theorem 2.7. *Given a strongly connected k-hygraph $G = ([n], E)$ which has p equivalence classes B_1,...,B_p given by the "$\sim$" relation, we have that:*

i) *if $p = 1$ then $\mathbb{Z}E(G) = \mathbb{Z}^n_{\equiv 0(k)}$;*
ii) *if $p > 1$ then there exists a unique partition $\lambda = \{k_1, k_2, ..., k_p\}$ such that G is λ-partite, and the partition of $[n]$ is given exactly by $[n] = \bigsqcup_{j=1}^p B_j$, and*

$$\mathbb{Z}E(G) = \left\{ v \in \mathbb{Z}^n_{\equiv 0(k)} : \frac{1}{k_1}\sum_{j\in B_1} v_j = \frac{1}{k_2}\sum_{j\in B_2} v_j = ... = \frac{1}{k_p}\sum_{j\in B_p} v_j \right\}$$

We will use the following lemma:

Lemma 2.8. *Suppose $w \in \mathbb{Z}^n$, and for any $l = 1, 2, ..., p$ we have $\sum_{j \in B_l} w_j = 0$. Then $w \in \mathbb{Z}E(G)$.*

Proof. Assume $B_1 = \{i_1, i_2, ..., i_b\}$, then we can find integers α_1, α_2,..., α_{b-1} such that if $w_1 = \alpha_1(e_{i_1} - e_{i_2}) + \alpha_2(e_{i_2} - e_{i_3}) + ... + \alpha_{b-1}(e_{i_b-1} - e_{i_b})$ then w and w_1 have the same entries at indices in B_1.

Now consider $w - w_1$ and repeat the procedure for B_2 and so on. At the end we get $w = w_1 + ... + w_p$, and all $w_j \in \mathbb{Z}\mathrm{E}(G)$ (because $e_{i_0} - e_{i_1} \in \mathbb{Z}\mathrm{E}(G)$ when $i_0 \sim i_1$), so $w \in \mathbb{Z}\mathrm{E}(G)$. □

Proof. (of Theorem 2.7)

Consider a graph $G'([p], E')$. Consider a map $f : [n] \to [p]$ such that when $i \in B_l$ we have $f(i) = l$. By an extension of notation we write $f(e) = \{f(a_1), f(a_2), ..., f(a_k)\}$ when $e = \{a_1, a_2, ..., a_k\}$. Denote

$$E' = \left\{ f(e) \in \left(\binom{[p]}{k} \right) : \ e \in E \right\}$$

.

Notice that G' is also strongly connected: consider $a,\ b \in E'$, then take $a_1, b_1 \in E$ such that $f(a_1) = a_0$ and $f(b_1) = b_0$. Because G is strongly connected we have $a = e_1,\ e_2, ..., e_m = b \in E$ such that $|e_i \cap e_{i+1}| \geq k - 1$ for all $i = 1, 2, ...m - 1$, so $a_1 = f(e_1), f(e_2), ..., f(e_m) = b_1 \in E'$ have the property that $|f(e_i) \cap f(e_{i+1})| \geq k-1$ for all $i = 1, 2, ..., m - 1$.

Suppose $|E'| > 1$, then we can find $e_0,\ e_1 \in E'$, $|e_0 \cap e_1| = k - 1$, so say $e_0 \setminus e_1 = \{i\}$ and $e_1 \setminus e_0 = \{j\}$, obviously with $i \neq j$. It follows that $i \sim j$ in G' because $e_i - e_j = \chi_{e_0} - \chi_{e_1} \in \mathbb{Z}E(G')$. We will prove that this forces two equivalence classes in G to be the same, which is a contradiction.

Take any edges $d_0, d_1 \in E$ such that $f(d_0) = e_0$ and $f(d_1) = e_1$, and $i_1 \in B_i$ and $j_1 \in B_j$. Then the vector $(e_{i_1} - e_{j_1}) - (\chi_{d_0} - \chi_{d_1})$ satisfies the condition in Lemma 2.8, which means that $e_{i_1} - e_{j_1} \in \mathbb{Z}\mathrm{E}(G)$ so we get that equivalence classes $B_i \cap B_j \neq \varnothing$, contradiction.

Now we know that G' has exactly one edge e_0, and assume any $l = 1, 2, ..., p$ appears with multiplicity k_l in the multiset e_0. It follows that for any edge $e \in E$ and any $l = 1, 2, ..., p$ we have

$$\left| e \cap \left(\binom{B_l}{k} \right) \right| = k_l,$$

so G is λ-partite with λ the partition of k as $k = \sum_{l=1}^{p} k_l$.

Now take any $w \in \mathbb{Z}^n_{\equiv 0(k)}$ such that

$$\frac{1}{k_1} \sum_{j \in B_1} w_j = \frac{1}{k_2} \sum_{j \in B_2} w_j = ... = \frac{1}{k_p} \sum_{j \in B_p} w_j = \alpha$$

We can prove $\alpha \in \mathbb{Z}$ because $w \in \mathbb{Z}^n_{\equiv 0(k)}$ and

$$\sum_{i=1}^{n} w_i = \sum_{l=1}^{p} \sum_{i \in B_l} w_i = \sum_{l=1}^{p} \alpha \cdot k_l = k \cdot \alpha,$$

but then choose any edge $e \in E$ so $(w - \alpha \cdot e) \in \mathbb{Z}\mathrm{E}(G)$ by Lemma 2.8, so $w \in \mathbb{Z}\mathrm{E}(G)$. This completes the proof, because it is clear that any $w \in \mathbb{Z}\mathrm{E}(G)$ has to be of the above form.

□

Corollary 2.9. *In the condition of Theorem 2.7, we have:*

i) *if* $p = 1$ *then* $\mathbb{Q}E(G) = \mathbb{Q}^n$*;*

ii) *if* $p > 1$ *then*

$$\mathbb{Q}E(G) = \left\{ v \in \mathbb{Q}^n : \frac{1}{k_1} \sum_{j \in B_1} v_j = \frac{1}{k_2} \sum_{j \in B_2} v_j = ... = \frac{1}{k_p} \sum_{j \in B_p} v_j \right\}$$

As soon as we drop the hypothesis of *strongly connected*, we find examples where the coloring given by partitions of k do not encapsulate all information about $\mathbb{Q}\mathrm{E}(G)$.

Example 2.10. Consider the 3-hygraph G in example 2.5. We have seen that G is λ-partite only if λ is the trivial partition $3 = 3$, which imposes no restriction on $\mathbb{Q}\mathrm{E}(G)$. Nevertheless, $\mathbb{Q}\mathrm{E}(G) \neq \mathbb{Q}^n$, more specifically:

$$\mathbb{Q}\mathrm{E}(G) = \{v \in \mathbb{Q}^n : v_1 + v_6 = v_5 + v_2 \text{ and } v_2 + v_7 = v_6 + v_3\}$$

This encourages the search for a more general idea of "coloring" than the partitions of k. We have obtained the following conjecture:

Conjecture 2.11. *Let* $G_k = ([k], E_k)$ *be the complete* k*-hygraph on* $[k]$*.*

Given a k*-hygraph* $G([n], E)$ *, (easily connected?), consider any function* $f : [n] \to [k]$ *and extend it naturally to* $f : E \to E_k$*, and define* $G_f := ([k], f(E))$*. Define further* $\phi : \mathbb{Q}^n \to \mathbb{Q}^k$ *by* $\phi(e_i) = e_{f(i)}$*. Then:*

$$\mathbb{Q}E(G) = \bigcap_{f:[n]\to[k]} \phi^{-1}(\mathbb{Q}E(G_f))$$

In plain words, the only restrictions on $\mathbb{Q}\mathrm{E}(G)$ are given by colorings where we use at most k colors, but a certain set of possible edges is permitted (instead of just one edge, as previously).

Note that only some of these colorings impose restrictions, and some are equivalent to others in terms of equations imposed.

If the conjecture is true, it would mean that we only need to study the $\mathbb{Q}\mathrm{E}(G)$ for k-hygraphs on at most k vertices to understand the $\mathbb{Q}\mathrm{E}(G)$ in the general case.

2.2. **Reduction algorithms and loosely connected k-hygraphs.** Is there any way to use the ideas in Theorem 2.7 for a k-hygraph G which is not strongly connected? Consider the following algorithm:

Consider a subgraph $G' \subset G$, $G = (V, E)$ and $G' = (V', E')$ such that $V' \subset V$, $E' \subset E$ and G' strongly connected, and uniquely partition $V' = \bigsqcup_{l=1}^{p} B_l$ as in Theorem 2.7.

Now create hygraph $G_1 = ((V \setminus V') \cup V_1, E_1)$, where $V_1 = \{b_1, b_2, ..., b_l\}$ from G in the following way: for each $1 \leq l \leq p$ replace all vertices in B_l with vertex b_l and replace all edges containing vertices in B_l with edges containing b_l with the respective multiplicity.

Theorem 2.7 guarantees that if $v \in \mathbb{Z}E(G_1)$ then for any vector

$$w \in \mathbb{Z}^{|V|} \text{ with } \sum_{j \in B_l} w_j = v_{b_l}, \text{ and } w_i = v_i \text{ when } i \in (V \setminus V'),$$

we have $w \in \mathbb{Z}\mathrm{E}(G)$.

We can repeat the process until there are no more strongly connected subgraphs (one step reduces the number of vertices), which leads us to the following definition, for the special case $k = 3$:

Definition 2.12. Consider a 3-hygraph G, and suppose that given any two distinct edges $a, b \in E$ we have

- $|a \cap b| \leq 1$;
- there exist edges $a = e_1,\ e_2, ...,\ e_m = b$ such that $|e_i \cap e_{i+1}| = 1$

Then we call G *loosely connected*.
If G satisfies only the latter condition, we call it *weakly connected*.

Notice that this is the same as saying that any two multisets that represent edges are either disjoint or intersect at a multiset with cardinality 1. The previous procedure says that we only need to understand the $\mathbb{Q}\mathrm{E}(G)$ and $\mathbb{Z}\mathrm{E}(G)$ for loosely connected graphs.

2.3. A bound on the dimension of $\mathbb{Q}\mathbf{E}(G)$ for loosely connected 3-hygraphs.

It is natural to ask whether a loosely connected 3-hygraph can be drawn on a certain two-dimensional surface (sphere, torus, etc).

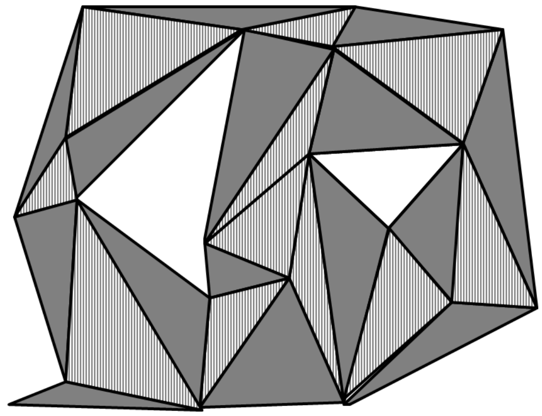
Definition 2.13. Let's think of a 3-hygraph as n points on a two-dimensional surface σ, where *edges* are subsurfaces of σ which contain exactly three vertices, and are homeomorphic to solid triangles (moreover, we can take the pre-images of the vertices of the solid triangle to be the three vertices united by the edge). Most importantly, we impose that two different edges are either disjoint, intersect at a vertex, or along a simple curve uniting two vertices. Then we say that G is embedded on the surface σ.

Notice that a 3-hygraph embedded on σ partitions the surface into edges and other subsurfaces homeomorphic to solid polygons (again, the pre-images of vertices of the polygon are exactly the vertices of G contained in the subsurface). We call such a subsurface which is not an edge an *empty cycle* (because it contains no edges inside).

We first analyze the case of a plane (or equivalently, a sphere).

Proposition 2.14. *Suppose G, a loosely connected 3-hygraph, is embedded on the sphere, and we have $m > 0$ pairwise disjoint empty cycles of lengths k_1, k_2,..., and k_m, then:*

$$\left\lfloor \frac{n}{2} \right\rfloor \leq |E| \leq \left\lfloor n - \frac{\sum_{i=1}^{m} k_i}{3} + m + 2 \right\rfloor$$



Proof. We can suppose w.l.o.g. G is embedded in the plane such that the edges are solid triangles, and the exterior polygon represents one of the m empty cycles on the sphere, with k_1 sides. Triangulate all empty cycles excluding the selected $m-1$, such that we have:

- t is the number of triangles representing edges of G;
- T counts all triangles, including the ones used to triangulate some empty cycles, but not counting any of the $m-1$ empty cycles themselves;
- n is the number of the vertices of G;
- e is the number of segments used in the configuration;

For example, in the image above the edges are colored dark gray, while the other triangles are dashed gray; the $m-1$ empty cycles are left white. We have:

- $m=3$, $k_1=9$, $k_2=3$ and $k_3=4$;
- $t=15$ and $T=26$;
- $n=20$, $e=37$;

We calculate T by the equation giving the sum of angles around all vertices confined to the inside of the exterior polygon:

$$\pi \cdot T + \sum_{i=2}^{m} \pi \cdot (k_i - 2) = \pi \cdot (k_1 - 2) + 2\pi(n - k_1) \quad \Rightarrow \quad T = 2n - \sum_{i=1}^{m}(k_i - 2) + 4$$

Every triangle contains three segments, and all segments are counted twice, except those on an empty cycle, so:

$$3T = 2e - \sum_{i=1}^{m} k_i$$

Now if a certain vertex v has d_v incident segments, the number of incident edges is at most $\frac{d}{2}$, because G is *loosely connected*. The sum of $\frac{d_v}{2}$'s over all vertices is e, so we get:

$$3t \le e = \frac{1}{2}\left(3T + \sum_{i=1}^{m} k_i\right) = 3n - \sum_{i=1}^{m} k_i + 3m + 6$$

The proof for the left inequality is an easy exercise using the fact that G is connected. Asymptotical equality is achieved in this case, by the graph:

$$G = ([2n-1], \{1, 2, n+1\}, \{2, 3, n+2\}, ..., \{n-1, n, n+n-1\})$$

□

In the case of a general case of a surface σ of genus g, if σ is partitioned into the t edges and M other surfaces, a simple application of Euler's formula gives:

$$t = \frac{n}{2} + 2g - 2 + M$$

The effect on $\mathbb{Q}\mathrm{E}(G)$ given by these propositions follows from

$$\dim \mathbb{Q}\mathrm{E}(G) \leq |E|.$$

2.4. **Another class of particular 3-hygraphs.** We also looked at 3-hygraphs with only degenerated edges (see definition 1.2):

Definition 2.15. If $e \in E$ and $e = \{a, b, b\}$ with $a \neq b$ we call e an *arrow* from a to b and denote $e = [a, b\}$. If $e = \{a, a, a\}$ we call e a *simple loop.*

Proposition 2.16. *The $\mathbb{R}E(G)$ of a connected 3-hygraph with only degenerated edges is a hyperplane if it either contains no cycles or simple loops, or all the cycles (of arrows) are even and have an equal amount of arrows going clockwise as counterclockwise. Otherwise $\mathbb{R}E(G) = \mathbb{R}^n$.*

Proof. :

It is easy to see that if the hygraph is connected and has no cycles, then there are exactly $n-1$ edges and that the corresponding vectors are linearly independent. If there exists a simple loop i , then we will look at $([n], E\backslash\{i\})$ which is still connected. If the remaining graph contains a simple loop, we will continue the same procedure until we get an irreducible graph G'. If $G' = ([n], E')$ contains cycles, we will fix one of edges (j) in one of the cycles look a the gpaph $([n], E'\backslash\{j\})$ and continue doing this until again we reach an irreducible graph G'' without cycles or simple loops. The dimension of its real lattice is $n-1$. It is easy to see that any vector corresponding to the simple loop that we took out of the graph is linearly independent with all the edges of A by first numbering the vertices in such a way that all the diagonal entries of the edge matrix are nonzero and then looking at the determinant of the vectors in G'' and the vector of the simple loop. The determinant equals to the product of the diagonal entries and therefore is not equal to zero.

If there are no simple loops in the graph then look at any cycle. We will number the vertices of the cycle according to their cyclic order. If we look at the matrix of the obtained cycle (It has nonzero entries only on the main diagonal and the diagonal on top of it and the bottom left corner, also each row has one 1 and one 2 and the rest are zeros)

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & . & 0 & 0 \\ 0 & a_{2,1} & a_{2,2} & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & a_{n-1,1} & a_{n-1,2} \\ a_{n,2} & 0 & 0 & . & 0 & a_{n,1} \end{pmatrix}$$

then it is not hard to see that its determinant is zero iff the cycle is even and there is an equal amount of arrows going in both directions since $det(A) = \prod_{1=1}^{n} a_{1,i} + (-1)^{n-1} \cdot \prod_{1=1}^{n} a_{2,i}$. Therefore the the dimension of the real lattice is $n-1$ only if there are no simple loops, odd cycles or even cycles with an equal amount of arrows going each direction, otherwise it is n.

□

3. The quotient group for $\mathbb{Z}\mathrm{E}(G)$

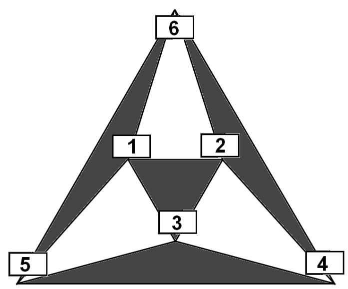
The study of $\mathbb{Z}\mathrm{E}(G)$ in general conditions poses additional problems, because in general we do **not** have $\mathbb{Z}\mathrm{E}(G) = \mathbb{Q}\mathrm{E}(G) \cap \mathbb{Z}^n_{\equiv 0(k)}$.

Definition 3.1. When referring to a k-hygraph $G = ([n], E)$, we define the *quotient group* to be the group $(\mathbb{Q}\mathrm{E}(G) \cap \mathbb{Z}^n_{\equiv 0(k)})/\mathbb{Z}\mathrm{E}(G)$.

By Theorem 2.7 and Corollary 2.9, we have that:

Proposition 3.2. *For a strongly connected k-hygraph G the* quotient group *is trivial, so $\mathbb{Z}E(G) = \mathbb{Q}E(G) \cap \mathbb{Z}^n_{\equiv 0(k)}$.*

Nevertheless, the case when G is not strongly connected yields a counter-example:



Example 3.3. Take $G = ([6], E)$ where $E = \{\{1,2,3\},\{1,5,6\},\{2,4,6\},\{3,4,5\}\}$. Then $v = (1,1,1,1,1,1)$ is a nonzero element of the quotient group, because $2 \cdot v = \sum_{e \in E} \chi_e$, but it can be seen that $v \notin \mathbb{Z}\mathrm{E}(G)$.

A few questions to ask are:

- Can the order of the quotient group be arbitrarily large for fixed k?
- Is there any way to describe the quotient group, for k-hygraphs with special restrictions?

Example 3.4. The quotient group of

$$G = ([n], \{\{1,2,2\},\{2,3,3\},\{3,4,4\},...,\{n-1,n,n\},\{n,1,1\}\})$$

is isomorphic to $\mathbb{Z}_{|1+(-2)^n|}$, which can be easily seen by looking at the Smith normal form of the matrix whose rows are the characteristic vectors of the edges in E. A simple series of row and column operations gives:

$$\begin{pmatrix} 1 & 2 & . & 0 & 0 \\ 0 & 1 & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & 1 & 2 \\ 2 & 0 & . & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & . & 0 & 0 \\ 0 & 1 & . & 0 & 0 \\ . & . & . & . & . \\ -4 & 0 & . & 1 & 0 \\ 0 & 0 & . & 0 & 1 \end{pmatrix} \sim ...$$

$$
... \sim \begin{pmatrix} 1 & 2 & . & 0 & 0 \\ (-1)^n \cdot 2^{n-1} & 1 & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & 1 & 0 \\ 0 & 0 & . & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1+(-2)^n & 0 & . & 0 & 0 \\ 0 & 1 & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & 1 & 0 \\ 0 & 0 & . & 0 & 1 \end{pmatrix}
$$

We also look at the quotient group for 3-hygraphs with all edges arrows (See Definition 2.15). Moreover, let's restrict our attention to *trees*, i.e. the simple graph obtained by transforming all arrows into simple edges is a tree. Clearly any tree has $(n-1)$ edges-arrows.

Proposition 3.5. *For any 3-hygraph which is a tree of edge-arrows we have*

$$
(\mathbb{Q}E(G) \cap \mathbb{Z}_3^n)/\mathbb{Z}E(G) \simeq \bigoplus_{i=1}^{k} \mathbb{Z}_{2^{l_i}}
$$

where $k \leq n-1$ *and* $l_i \in \mathbb{Z}_+$.

Proof. If we look at the $n \times (n-1)$ matrix generated by the vectors that represent the edges of the tree and find its smith normal form, then the rows will give us the basis vectors of $\mathbb{Z}E(G)$. This shows that $k \leq n-1$.

The elements in the quotient group can also be described as any element $\overline{x} \notin \mathbb{Z}E(G)$ such that there exists a number $p \in \mathbb{N} \setminus \{0\} : \ p \cdot \overline{x} \in \mathbb{Z}E(G)$. Proving the proposition now would be the same as proving that the order of any element is a power of 2. Suppose that there is a prime $q \neq 2$ such that an element $\overline{a}$ is of order q. This means that $q \cdot \overline{a} = \sum_{e \in E} a_e \chi_e$. Let's look a one of the leaves e_j of the graph: it is the only edge that touches some vertex and therefore in the linear combination $\sum_{e \in E} a_e \chi_e$, a_{e_j} is divisible by q, because the entries of χ_e can only be 0,1 or 2 for any $e \in E$. Now let's look at $q \cdot \overline{a} - a_{e_j}\chi_j$. All its coordinates are still divisible by q therefore we can look at any leaf of the graph $([n], E \setminus e_j)$ and repeat the procedure. This way we can see that for any e, a_e is divisible by q. Therefore $\sum_{e \in E} \frac{a_e}{q} \chi_e = \overline{a} \in \mathbb{Z}\mathrm{E}(G)$. This contradiction proves that any element has order power of two, so by the classification theorem for finite abelian groups we get that the quotient group is of the desired form.

□

We can generalize this proposition in the following way. If we consider k-hygraphs with edges that have $n-1$ of element A and one element B, where $A \neq B$ then for these kinds of hygraphs the following is true:

Proposition 3.6. $(\mathbb{Q}E(G) \cap \mathbb{Z}_k^n)/\mathbb{Z}E(G) \simeq \bigoplus_{i=1}^{k} (\mathbb{Z}_{2^{l_i 2}} \oplus \mathbb{Z}_{3^{l_i 3}} \oplus ... \oplus \mathbb{Z}_{s^{l_i s}})$ *where* $k \leq n-1$, $l_i \in \mathbb{Z}_+$ *and* s *is the greatest prime such that* $s < k$.

It can be proved using the exact same method as the 3-hygraph case.

One might ask whether there are any examples where we get orders of elements which are higher powers of 2, that is, if l_i can be at least 2. Here is one example:

Example 3.7. Consider the following graph:

$$G = ([9], \{\{1, 2, 2\}, \{2, 3, 3\}, \{3, 4, 4\}, \{4, 5, 5\}, \{3, 6, 6\}, \{6, 7, 7\}, \{6, 8, 8\}, \{9, 8, 8\}\})$$

.

The Smith Normal Form of the 9×8 matrix generated by the vectors that correspond to the edges is a matrix with entries $1, 1, 1, 1, 1, 1, 2, 4$ on the main diagonal, which means that the quotient group is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_4$.

4. Holes of $\mathbb{Z}_+^{\mathrm{SAT}}\mathrm{E}(G)$

The study of the saturation $\mathbb{Z}_+^{\mathrm{sat}}\mathrm{E}(G)$ implies the study of the vectors v called *holes* defined as $v \in \mathbb{Z}_+^{\mathrm{sat}}\mathrm{E}(G) \setminus \mathbb{Z}_+\mathrm{E}(G)$. We will exhibit a few counter-examples to conjectures of "desirable" properties of the holes. First, drawing from the proof of Theorem 1.12 we define:

Definition 4.1. Consider a k-hygraph $G = ([n], E)$, and a vector $v \in \mathbb{Z}_+^{\mathrm{sat}}\mathrm{E}(G)$. Suppose there exists $E_0 \subseteq E$ and numbers $q_e \in (0, 1) \cap \mathbb{Q}$ for all $e \in E_0$ such that:

- the set χ_{E_0} is linearly independent over $\mathbb{Z}$ (thus over $\mathbb{R}$);
- $v = \sum_{e \in E_0} q_e \chi_e$.

Then we call v an *irreducible hole* of the saturation $\mathbb{Z}_+^{\mathrm{sat}}\mathrm{E}(G)$.

We need a few more definitions before we proceed:

Definition 4.2. ([3, Def 1.6])
Consider two sets $x, y \subset \mathbb{N}$ with $|x| = |y| = k$ and $x = \{x_1 < x_2 < ... < x_k\}$ and $y = \{y_1 < y_2 < ... < y_k\}$. We define the *Gale order* on such sets by setting $x \leq y$ if $x_j \leq y_j$ for all $j = 1, 2, ..., k$.

Definition 4.3. We call a k-hypergraph $G = ([n], E)$ *shifted* if its edges form an order ideal in the Gale order, or, equivalently, if the following statement holds:
for any $e \in E$ and $f \subset \mathbb{N}$, $|f| = k$, with $f \leq e$, then $e \in E$.
Moreover, we call an edge e in a shifted graph *maximal* if it there is no edge $f \in E$ with $f > e$ in the Gale order.

We also introduce a refinement of the definition of *strongly connected*:

Definition 4.4. A k-hygraph $G = ([n], E)$ is called *diameter* k if for any two edges $a,\ b \in E$ there exist edges $a = e_0,\ e_1,\ \dots,\ e_k = b$ such that $|e_i \cap e_{i+1}| \geq k - 1$ for any $i = 0, 1, ..., k - 1$.

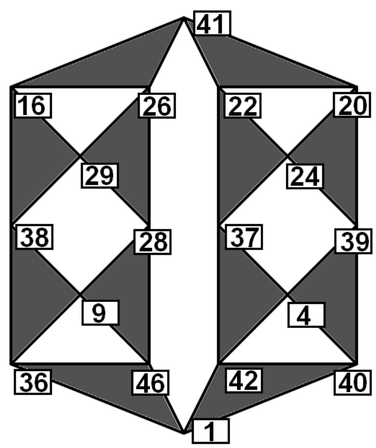
4.1. **Four false conjectures.** Here are four candidates for tempting, yet false, conjectures:

(1) Given an irreducible hole v (see Definition 4.1) for a k-hygraph G, then $v_i < k$ for each $i = 1, 2, ..., n$. (True for $k = 2$)

(2) Given a strongly connected k-hygraph G which is λ-partite with λ the partition $k = 1 + 1 + ... + 1$ (thus $\mathbb{Z}\mathrm{E}(G) = \mathbb{Z}^n_{\equiv 0(k)}$), then $\mathbb{Z}_+^{\mathrm{sat}}\mathrm{E}(G) = \mathbb{Z}_+\mathrm{E}(G)$, i.e. there are no holes. (True for $k = 2$)
(3) Same as previous, replacing *strongly connected* by *diameter* k.
(4) Given a shifted k-hygraph G, $\mathbb{Z}_+^{\mathrm{sat}}\mathrm{E}(G) = \mathbb{Z}_+\mathrm{E}(G)$. (True for $k = 2$)

Conjecture (3) and the definition of diameter k are motivated by the following fact: White proved in [6] that $\mathbb{Z}_+^{\mathrm{sat}}\mathrm{E}(G) = \mathbb{Z}_+\mathrm{E}(G)$ for a certain class of k-hygraphs, namely *matroids*, which happen to be diameter k, thus strongly connected. De Negri also proved this result for k-hygraphs which are shifted and with exactly one maximal edge in [4, Cor. 3.5]; these conditions actually imply that the k-hygraph is a matroid. It is thus natural to ask if either *diameter* k or *shifted*, together with other conditions, force $\mathbb{Z}_+^{\mathrm{sat}}\mathrm{E}(G) = \mathbb{Z}_+\mathrm{E}(G)$.

Xun Dong already gave a counter-example to conjecture (4) for $k = 4$ in [1]. We will start by giving a counter-example with $k = 3$, followed by a counter-example to (3) for $k = 3$, which implies (2) is also false for $k = 3$.



Example 4.5. Consider a 3-hypergraph $G = ([46], E)$.

Denote $E_0 = \{\{1, 36, 46\}, \{9, 28, 46\}, \{9, 36, 38\}, \{26, 28, 29\}, \{16, 29, 38\}, \{16, 26, 41\}, \{1, 40, 42\}, \{4, 37, 42\}, \{4, 39, 40\}, \{22, 24, 37\}, \{20, 24, 39\}, \{20, 22, 41\}\}$.

Let $E = E_0 \bigsqcup \{\{a, b, c\} \subset [46] : a < b < c \text{ and } a + b + c \leq 82\}$.

Now consider the vector $w \in \mathbb{Z}^{46}$, $w_i = \begin{cases} 1, & i \in e, \text{ for some } e \in E_0; \\ 0, & \text{otherwise.} \end{cases}$

Then G is shifted, and w is a hole in the saturation, $w \in \mathbb{Z}_+^{\mathrm{sat}}\mathrm{E}(G) \setminus \mathbb{Z}_+\mathrm{E}(G)$.

Proof. For $e \in E$, $e = \{a_1, a_2, a_3\}$, consider $e' \in [46]$, $e' = \{b_1, b_2, b_3\}$ with $e' < e$, then $b_1 + b_2 + b_3 < a_1 + a_2 + a_3 \leq 83$ thus $e' \in E$. Thus G is shifted.

We start by proving that $w \in \mathbb{Z}_+^{\mathrm{sat}}\mathrm{E}(G) = \mathbb{Q}_+\mathrm{E}(G) \cap \mathbb{Z}\mathrm{E}(G)$. We can first write $w = \sum_{e \in E_0} \frac{1}{2}\chi_e \in \mathbb{Q}_+\mathrm{E}(G)$, then notice that G is strongly connected and it contains the edges $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$ and $\{2, 3, 4\}$. This forces $1 \sim 4 \sim 2 \sim 3$, thus G has only one equivalence class given by "$\sim$", thus by Theorem 2.7 we have $\mathbb{Z}\mathrm{E}(G) = \mathbb{Z}^{46}_{\equiv 0(3)}$. It follows that $w \in \mathbb{Z}\mathrm{E}(G)$, because $\sum_{i=1}^{46} w_i = 18$.

Consider the set $A = \{j \in [46] : j \in e$ for some $e \in E_0\}$. Because for each $j \in A$ we have $|\{e \in E_0 : j \in e\}| = 2$ we can write

$$\sum_{i=1}^{46} w_i \cdot i = \sum_{j \in A} j = \sum_{e \in E_0} \frac{1}{2} \sum_{j \in e} j = 12 \cdot \frac{1}{2} \cdot 83 = 6 \cdot 83 \tag{4.1}$$

Now assume by contradiction that $w \in \mathbb{Z}_+\mathrm{E}(G)$, then there is a set $E_1 \in E$ and we can write $w = \sum_{e \in E_1} a_e \chi_e$ with $a_e \in \mathbb{Z}_+ \setminus \{0\}$ for all $e \in E_1$. Because $\sum_{i=1}^{46} w_i = 18$ we obtain $\sum_{e \in E_1} a_e = 6$, and by evaluating the sum in (4.1) again we obtain:

$$\sum_{i=1}^{46} w_i \cdot i = \sum_{e \in E_1} a_e \sum_{j \in e} j \leq \sum_{e \in E_1} a_e \cdot 83 = 6 \cdot 83$$

But we know equality holds, which happens only if $\sum_{j \in e} j = 83$ for each $e \in E_1$, thus $E_1 \subseteq E_0$. This leads to a contradiction: we know $w_9 = w_{36} = w_{46} = 1$, and the only edges in E_0 incident to vertices 9, 36 and 46 are $\{1, 36, 46\}$, $\{9, 36, 38\}$ and $\{9, 28, 46\}$. Either two edges cannot both be in E_1, and one edge does not satisfy all three vertices.

□

To construct the counter-example to (3) we will modify the base set of edges E_0 in example 4.5:

Consider the (3=1+1+1)-partition given by $B_1 = \{1, 4, 9, 24, 29, 41\}$, $B_2 = \{16, 22, 28, 36, 39, 42\}$ and $B_3 = \{20, 26, 37, 38, 40, 46\}$. Then:

then:

- if $m \in B_1$, replace the number m with $3m$
- if $m \in B_2$, replace the number m with $3m + 1$
- if $m \in B_3$, replace the number m with $3m + 2$

We obtain the following (3=1+1+1)-partite 3-hypergraph, where the partitions are given by the residue classes, modulo 3, of the vertices:

Example 4.6. Consider a 3-hypergraph $G = ([140], E)$.

Denote $E_0 = \{\{3, 109, 140\}, \{27, 85, 140\}, \{27, 109, 116\}, \{80, 85, 87\}, \{49, 87, 116\}, \{49, 80, 123\}, \{1, 122, 127\}, \{12, 113, 127\}, \{12, 118, 122\}, \{67, 72, 113\}, \{62, 72, 118\}, \{62, 67, 123\}\}$.

Let $E = E_0 \bigsqcup \{\{a, b, c\} \subset [140] : a \not\equiv b \not\equiv c \not\equiv a (\mathrm{mod}\ 3)$ and $a + b + c \leq 251\}$.

Now consider the vector $w \in \mathbb{Z}^{140}$, $w_i = \begin{cases} 1, & i \in e, \text{ for some } e \in E_0; \\ 0, & \text{otherwise.} \end{cases}$

Then G is $(3 = 1{+}1{+}1)$-partite, G is diameter 3 and w is a hole in the saturation, $w \in \mathbb{Z}_+^{\mathrm{sat}}\mathrm{E}(G) \setminus \mathbb{Z}_+\mathrm{E}(G)$.

Proof. Because "diameter 3" implies "strongly connected", a similar argument as in proof of example 4.5 shows that v is a hole. Thus we only need to show that G is diameter 3.

W.l.o.g. we only have to study a generic situation:

Consider two distinct, disjoint edges $e_0 = \{a, b, c\}$ and $e_3 = \{A, B, C\}$ such that $a + b + c \le A + B + C \le 252$, $a \equiv A(\mathrm{mod}\ 3)$, $b \equiv B(\mathrm{mod}\ 3)$ and $c \equiv C(\mathrm{mod}\ 3)$. We can also assume w.l.o.g. that $a < A$, then we have two cases:

- we have $b \le B$ or $c \le C$. Again assume w.l.o.g. the former takes place, then take $e_1 = \{a, b, C\}$ and $e_2 = \{a, B, C\}$ then we see that
$$a + b + C \le a + B + C < A + B + C \le 252$$
so $e_1,\ e_2 \in E$ .Thus $e_0,\ e_1,\ e_2,\ e_3$ satisfy the condition in Definition 4.4;

- we have $b \ge B$ and $c \ge C$. Then take $e_1 = \{a, B, c\}$ and $e_2 = \{a, B, C\}$, so for both $j = 1, 2$ we have that either $e_j = e_0$ or
$$\sum_{i \in e_j} i < a + b + c \le 252,$$
so we have that $e_1,\ e_2 \in E$. Thus $e_0,\ e_1,\ e_2,\ e_3$ satisfy the condition in Definition 4.4.

$\square$

To give a counter-example to (1) we need a proposition regarding the existence of holes. The problem arises from the fact that just the condition $v \in \mathbb{Q}_+\mathrm{E}(G) \setminus \mathbb{Z}_+\mathrm{E}(G)$ and $v \in \mathbb{Z}^n_{\equiv 0(k)}$ does not assure that v is a hole. Naturally, the *missing ingredient* is the condition $v \in \mathbb{Z}\mathrm{E}(G)$! Indeed, together with this hypothesis we have that $v \in \mathbb{Z}^{\mathrm{sat}}_+\mathrm{E}(G) \setminus \mathbb{Z}_+\mathrm{E}(G)$; moreover, we have a counter-example for when $v \notin \mathbb{Z}\mathrm{E}(G)$: the vector v in Example 3.3 is not a hole in the saturation, because $v \notin \mathbb{Z}\mathrm{E}(G)$. Nevertheless, we are assured by the following proposition that $\mathbb{Z}\mathrm{E}(G)$ does not pose an insurmountable obstruction, and that $v \in (\mathbb{Q}_+\mathrm{E}(G) \setminus \mathbb{Z}_+\mathrm{E}(G)) \cap \mathbb{Z}^n_{\equiv 0(k)}$ is essentially all we need in order to engineer a hole:

Proposition 4.7. *Consider a k-hygraph $G = ([n], E)$ and a vector $v \in \mathbb{Z}^n_{\equiv 0(k)}$ such that $v \in \mathbb{Q}_+E(G) \setminus \mathbb{Z}_+E(G)$ and we can write $v = \sum_{e \in E_0} q_e \chi_e$ with E_0 linearly independent over $\mathbb{Z}$ and $q_e \in (0, 1) \cap \mathbb{Q}$ for all $e \in E_0$. Moreover, assume $v \notin \mathbb{Z}E(G)$.*

Then there exists a k-hygraph $G' = ([n'], E')$ with the following properties:

- *$n < n'$ and $E \subset E'$;*
- *for any $e \in E' \setminus E$ we have that $e \not\subseteq [n]$;*
- *G' is strongly connected;*

In these condition, the vector $v' \in \mathbb{Z}^{n'}_{\equiv 0(k)}$ given by $v' = \sum_{e \in E_0} q_e \chi_e$ is an irreducible hole for $\mathbb{Z}^{sat}_+ E(G')$.

Proof. After proving the existence of G', we only need to prove that $v' \notin \mathbb{Z}_+E(G')$ and $v' \in \mathbb{Z}\mathrm{E}(G')$.

Suppose we can write
$$v' = \sum_{e \in E'} \alpha_e \chi_e$$

with α_e non-negative integers. It follows that $\alpha_e = 0$ if $e \in E' \setminus E$, but this leads to $v \in \mathbb{Z}_+\mathrm{E}(G)$, contradiction.

The second statement follows from Proposition 3.2, because if M is the $l.c.m$ of the denominators of the q_e's, then $M \cdot v' \in \mathbb{Z}\mathrm{E}(G)$, thus $v' \in \mathbb{Z}\mathrm{E}(G)$ because G', being strongly connected, has trivial quotient group.

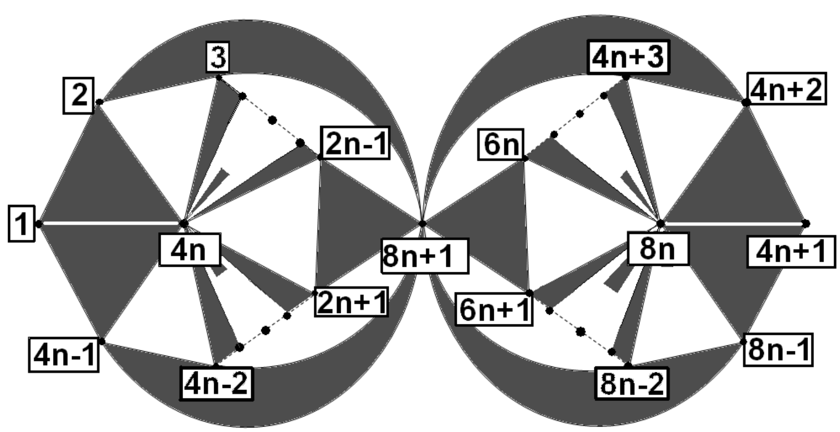
To construct G' we will add vertices and edges to strongly connect the k-hygraph. It is enough to exhibit the procedure for two edges $a, b \in E$ such that $0 < |a \cap b| < k-1$; assume $a \cap b = \{c_1, c_2, ...c_l\}$ and that $a \setminus (a \cap b) = \{a_1, a_2, ..., a_{k-l}\}$, $b \setminus (a \cap b) = \{b_1, b_2, ..., b_{k-l}\}$, then consider graph $G_1 = ([n+1], E \cup E')$, where

$$E' = \{\{c_1, ..., c_l, a_1, ..., a_j, n+1, b_{j+2}, ..., b_{k-l}\} : \text{ for } j = 0, 1, ..., (k-l-1)\}$$

The edges in E' strongly connect the edges a and b, thus the number of strongly connected components (partition of E) decreases. By repeating the procedure a finite number of times we obtain G'.

□

And now the counter-example to property (1), which will actually exhibit irreducible holes in k-hypergraphs having arbitrarily large coordinates:



Example 4.8. Let's look at a 3-hypergraph $([8n+1], E)$, where $E = E_1 \cup E_2 \cup E_3 \cup E_4$ is given by:

$$E_1 = \{\{1, 2, 4n\}, \{3, 4, 4n\}, ...\{4n-3, 4n-2, 4n\}, \{4n-1, 1, 4n\}\};$$

$$E_2 = \{\{2, 3, 8n+1\}, \{4, 5, 8n+1\}, ..., \{4n-2, 4n-1, 8n+1\}\};$$

$$E_3 = \{\{4n+1, 4n+2, 8n\}, \{4n+3, 4n+4, 8n\}, ...\{8n-3, 8n-2, 8n\}, \{8n-1, 4n+1, 8n\}\};$$

$$E_4 = \{\{4n+2, 4n+3, 8n+1\}, \{4n+4, 4n+5, 8n+1\}, ..., \{8n-2, 8n-1, 8n+1\}\}$$

We will show that G and the vector

$$v = (\underbrace{1, 1, ..., 1}_{4n-1}, n, \underbrace{1, 1..., 1}_{4n-1}, n, 2n-1)$$

satisfies the hypothesis of Proposition 4.7. We can write

$$v = \sum_{e \in E} \frac{1}{2}\chi_e \in \mathbb{Q}_+ E(G)$$

We also need to show that $v \notin \mathbb{Z}_+\mathrm{E}(G)$; given the above way of writing v, it is enough to show that χ_E is linearly independent. Assume the contrary: consider a linear dependency

$$\sum_{e \in E} a_e \cdot \chi_e = 0 \tag{4.2}$$

Each one of the vertices $1, 2, ..., 4n-1$ is included in exactly two edges. If the coefficient $a_{\{1,2,4n\}} = a$ then for relation 4.2 to hold we need $a_{\{2,3,8n+1\}} = -a$, $a_{\{3,4,4n\}} = a$ and in general $a_{\{2k+1,2k+2,4n\}} = -a_{\{2k+2,2k+3,8n+1\}} = a$ for any $k = \overline{1...2n-2}$. This means that $a_{\{4n-1,1,4n\}} = a$ but then the first coordinate of the sum in 4.2 will be $2a$. This means that $a = 0$.

The same procedure can be done with $E_3 \cup E_4$. This shows that all the coefficients in 4.2 are zero, so χ_E is linearly independent, thus by Proposition 4.7 we have produced irreducible holes with arbitrarily large entries.

The next question that can be asked is:

Question 4.9. Given a shifted k-hygraph G, with $k > 2$, what is the minimum integer $m(k)$ of maximal elements in E such that $\mathbb{Z}_+^{\mathrm{sat}}\mathrm{E}(G) = \mathbb{Z}_+\mathrm{E}(G)$ does not necessarily take place?

To recapitulate, we know that:

- for $k = 2$ any shifted graph has no holes (the simple proof is based on the fact that *any* two odd cycles will be connected by one edge);
- for $k > 2$ and $m(k) = 1$ there are no holes ([6], [4]);

The search for a hole in the case $k = 3$, $m(3) = 2$ has been, to date, inconclusive, and we have not found a proof that $m(3) = 2$ guarantees no holes, either.

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