Support Equalities of Ribbon Schur Functions

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October 2, 2017

Abstract

In [6], McNamara proved that two skew shapes can have the same Schur support only if they have the same number of $k \times \ell$ rectangles as subdiagrams. This implies that two connected ribbons can have the same Schur support only if one is obtained by permuting row lengths of the other. We present substantial progress towards classifying when a permutation $\pi \in S_m$ of row lengths of a connected ribbon α produces a ribbon α_π with the same Schur support as α; when this occurs for all $\pi \in S_m$, we say that α has *full equivalence class*. Our main results include a sufficient condition for a connected ribbon α to have full equivalence class. Additionally, we prove a separate necessary condition, which we conjecture to be sufficient. Finally, we show that our separate necessary and sufficient conditions fully classify when ribbons with at most 4 rows have full equivalence class.

Introduction

The question of when two skew diagrams yield equal skew Schur functions has been studied in detail; for instance, see [1], [8], and [9]. However, the related question of when two skew diagrams have the same Schur support (see Definition 0.2) has received less attention, with the most substantial progress occurring in [6] (2007) and in [7] (2011). Similarly, the question of when two skew diagrams have equal monomial support (see Definition 0.4) has not been addressed to our knowledge.

In [6], P. R. W. McNamara proves that any two skew diagrams with the same Schur support necessarily contain the same number of $k \times \ell$ rectangles, for every $k,\ \ell \geq 1$. In [7], P. R. W. McNamara and S. van Willigenburg explicitly determine the Schur support for a special class of skew shapes called *equitable ribbons*.

In this paper, we expand on the results presented in [6] and [7] by working to classify which connected ribbons have the same Schur support under all permutations of their row lengths; we say these ribbons have *full equivalence class* (see Definition 0.8).

In the next section, we provide preliminary information to aid the understanding of the rest of the paper. In Section 1 (resp. Section 2), we provide a sufficient (resp. necessary) condition for a connected ribbon to have full equivalence class. Then in Section 3 we show that the sufficient condition from Section 1 and the necessary condition from Section 2 completely classify which ribbons with 3 or 4 rows have full equivalence class.

In the Appendix, we present miscellaneous results pertaining to the equality of skew Schur support, classify the equality of monomial support among skew shapes, and include sample code used to test Conjecture 2.4.

Preliminaries

We begin by establishing some preliminary information regarding Schur functions, ribbons, Yamanouchi words, Littlewood-Richardson fillings, and R-matrices. In addition, we will present preliminary results regarding certain edge cases for which we can easily classify which ribbons have full equivalence class.

Schur Functions

The *Young diagram* corresponding to a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of an integer n is a collection of boxes arranged in left-aligned rows, where the i-th row from the top has λ_i boxes. A filling of a Young diagram with integers is called *semistandard* if the integers increase weakly across rows and strictly down columns. Such a filled-in Young diagram is called a *semistandard Young tableau (SSYT)*.

Example 0.1. The following is an example of a semistandard Young tableau:

1	1	1	1	1	2	2	4
2	2	3	3	4	5	5	
3	4	5					

We use *weight* or *content* to refer to multi-set of integers in the filling of a tableau. The weight or content is denoted as a tuple $\nu = (\nu_1, \nu_2, \ldots, \nu_k)$, where ν_i is the number of i's in the filling of the tableau. For example, the content of the tableau from Example 0.1 is $\nu = (5, 4, 3, 3, 3, 3)$.

Schur functions are often considered to be the most important basis for the ring of symmetric functions. Schur functions are indexed by integer partitions, where the Schur function s_λ corresponding to a partition λ is defined as

$$s_\lambda(x_1, x_2, x_3, \ldots) = \sum_{\substack{T\,:\,\text{SSYT of}\\ \text{shape } \lambda}} x^T = \sum_{\substack{T\,:\,\text{SSYT of}\\ \text{shape } \lambda}} x_1^{t_1} x_2^{t_2} x_3^{t_3} \cdots \tag{1}$$

where t_i is the number of occurrences of i in T. The Schur function corresponding to λ can also be written as a linear combination of monomial symmetric functions:

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu.$$

Here, $K_{\lambda\mu}$ denotes a *Kostka number*, which is equal to the number of semistandard Young tableau of shape λ and weight μ, and m_μ is the sum of all monomials $x^{\mu'}$, where μ' ranges over all distinct permutations of the parts of μ.

We can generalize this notion of Schur functions to apply to skew shapes, which are obtained by removing the Young diagram corresponding to the partition μ from the top-left corner of a larger Young diagram corresponding to the partition λ. Here, we require that the diagram for μ is contained in the diagram for λ, and we write the resulting skew shape as λ/μ. When μ is the empty partition, we call λ/μ "straight." Skew Schur functions have an analogous definition to that of straight Schur functions, where the sum in Equation 1 is instead over semistandard Young tableau of shape λ/μ.

Skew Schur functions have the nice property that they are Schur-positive, meaning that for any skew shape λ/μ,

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu,\nu} s_\nu$$

where ν denotes a straight partition, and where all coefficients $c^\lambda_{\mu,\nu} \geq 0$. The coefficients $c^\lambda_{\mu,\nu}$ are called *Littlewood-Richardson coefficients*, and will play an important role in the Littlewood-Richardson rule (which we introduce in Theorem 0.10). This relationship between skew Schur functions and straight Schur functions motivates the following definition:

Definition 0.2. The *Schur support* of a skew shape λ/μ, denoted $[\lambda/\mu]$, is defined as

$$[\lambda/\mu] = \{\nu :\ c^\lambda_{\mu,\nu} > 0\}.$$

In other words, the support of a skew shape is the set of straight shapes ν such that s_ν appears with nonzero coefficient in the expansion of $s_{\lambda/\mu}$ into a linear combination of straight Schur functions.

Remark 0.3. It is well known [10, Exer. 7.56(a)] that $[\alpha^\circ] = [\alpha]$, where α° is the antipodal (180°) rotation of a ribbon α.

Since each skew Schur function can be written as a linear combination of straight Schur functions, and since each straight Schur function can be written as a linear combination of monomial symmetric functions, it follows that each skew Schur function can be written as a linear combination of monomial symmetric functions.

Definition 0.4. The *monomial support* of a skew shape λ/μ, denoted $\mathrm{mSupp}(\lambda/\mu)$, is the set of straight shapes ν such that m_ν appears with positive coefficient in the expansion of $s_{\lambda/\mu}$ into a linear combination of monomial symmetric functions.

Until the Appendix, when we say "support," we mean Schur support.

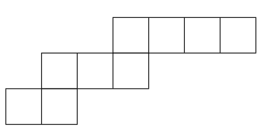
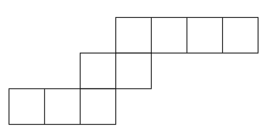
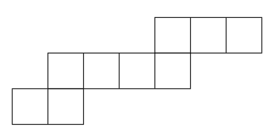
Ribbons

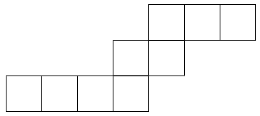
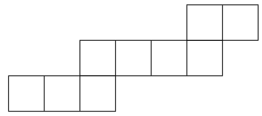
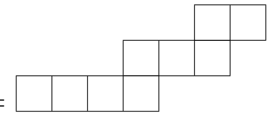
A *ribbon* is a skew shape which does not contain a 2×2 block as a subdiagram. A skew shape is said to be *connected* if there exists a path between any two boxes of the diagram using only north, east, south, and west steps such that the path is contained in the diagram.

In this paper, we consider *connected ribbons* (i.e. skew shapes in which each pair of consecutive rows overlaps in exactly one column). As such, any composition α of an integer n determines a unique connected ribbon. We will use the notation $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ to denote a connected ribbon with m rows, where row 1 is at the top of the ribbon, where row i has length α_i, and where $\alpha_i \neq 0$ for all $1 \leq i \leq m$. For the remainder of the paper, when we say "ribbon," we mean "connected ribbon."

Definition 0.5. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $\alpha_\pi = (\alpha_{\pi^{-1}(1)}, \alpha_{\pi^{-1}(2)}, \ldots, \alpha_{\pi^{-1}(m)})$ be connected ribbons, where $\pi \in S_m$ is a permutation written in cycle notation. We say α_π is a *permutation* of α.

Example 0.6. Below are all the permutations of the ribbon $\alpha = (4, 3, 2)$:

$\alpha =$  $\alpha_{(2\ 3)} =$  $\alpha_{(1\ 2)} =$ 

$\alpha_{(1\,3\,2)} =$  $\alpha_{(1\,2\,3)} =$  $\alpha_{(1\,3)} =$ 

In [6], McNamara proves that two skew shapes have the same Schur support only if they have the same number of $k \times \ell$ rectangles for all $k, \ell \geq 1$. This result has the following implication for connected ribbons:

Proposition 0.7. *Let α and β be connected ribbons such that $[\alpha] = [\beta]$. Then $\beta = \alpha_\pi$ for some permutation $\pi \in S_m$.*

Proof. By [6], α and β contain the same number of 2×1 rectangles as subdiagrams, and therefore have the same number of rows — let's say m rows. Label the row lengths of α, indexing so that the row lengths weakly decrease as the index increases (i.e. $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$). Label the row lengths of β in the same way. It suffices to show that $\alpha_i = \beta_i$ for all $1 \leq i \leq m$.

Suppose for the sake of contradiction that there exists an $i \in \{1, 2, \ldots, m\}$ for which $\alpha_i \neq \beta_i$. Choose the minimal such i and assume, without loss of generality, that $\alpha_i > \beta_i$. It follows that α has more $1 \times \alpha_i$ rectangles than β, contradicting McNamara's necessary condition for Schur support equality. Therefore, $\alpha_i = \beta_i$ for all $1 \leq i \leq m$, completing the proof. □

We define a relation $\sim$ between connected ribbons α and β, where $\alpha \sim \beta$ when $[\alpha] = [\beta]$. It is clear that $\sim$ is an equivalence relation. Then by Proposition 0.7, $\alpha \sim \beta$ only if β is a permutation of α.

Definition 0.8. A ribbon $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is said to have *full equivalence class* if for any permutation $\pi \in S_m$, we have $\alpha \sim \alpha_\pi$.

For instance, the ribbon $\alpha = (4, 3, 2)$ from Example 0.6 has full equivalence class, since all of its permutations have support $\{(5, 3, 1), (5, 4), (4, 4, 1), (5, 2, 2), (4, 3, 2), (6, 3), (7, 1, 1), (6, 2, 1), (7, 2)\}$.

Yamanouchi Words and Tableaux

We now introduce the concepts of *Yamanouchi words* and *Yamanouchi tableaux*, which will be essential for using and defining our main tool for proving equality of support — the Littlewood-Richardson rule.

A *Yamanouchi word* is a word with the property that all of its initial subwords contain no more $i+1$'s than i's, for all integers i. For our purposes, we are concerned with the *reverse reading word* of a tableau, which reads right-to-left across rows and top-to-bottom from one row to the next. A *Yamanouchi tableau* is a tableau whose reverse reading word is Yamanouchi.

Example 0.9. The following tableau is Yamanouchi because at any point along its reverse reading word, 112213321, the number of 2's is never greater than the number of 1's, and the number of 3's is never greater than the number of 2's.

					1	1
			1	2	2	
1	2	3	3			

Littlewood-Richardson Fillings

Littlewood-Richardson fillings (which we often abbreviate as *LR-fillings*) are fillings which are both semistandard and Yamanouchi. These fillings play an important role in the Littlewood-Richardson rule, which we are now ready to state.

Theorem 0.10. *(Littlewood-Richardson rule) [5] If*

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu,\nu} s_{\nu},$$

then $c^{\lambda}_{\mu,\nu}$ is the number of Littlewood-Richardson fillings of λ/μ of content ν.

The following corollary follows immediately from Theorem 0.10 and the definition of Schur support (Definition 0.2), and will be more directly applicable to the proofs in the remainder of the paper.

Corollary 0.11. *For any straight shape ν and skew shape λ/μ, $\nu \in [\lambda/\mu]$ if and only if there exists a Littlewood-Richardson filling of λ/μ with content ν.*

Example 0.9 continued. Notice that the filling in this example is semistandard and has content $\nu = (4, 3, 2)$. It follows from Theorem 0.11 that the straight shape $(4, 3, 2)$ is in the support of the ribbon with row lengths $(2, 3, 4)$.

In proofs throughout the remainder of the paper, we frequently utilize Corollary 0.11 by constructing Littlewood-Richardson fillings of tableaux. In proving sufficient conditions for same support, the approach will be to show that if μ is in the support of a ribbon α, then we can find a LR filling of α' with content μ given an LR filling of α with the same content. To prove necessary conditions for equal support, we will show that if α satisfies the conditions $(*)$ and α' doesn't, then there exists a content for an LR filling of α (or α') that cannot be a content for a filling of α' (or α respectively).

Notation. We represent the filling of a row by the numbers that appear from left to right in that row. We denote a string of k i's in a row by i^k. We will use the notation $i^{\text{rem.}}$ to mean we use all remaining i's in that location, while i^{fill} will mean we write exactly the number of i's which will fill the row (after having accounted for what else is in the row). Finally, we may want to fill a row with i's, but then finish filling the row with j's in the event that the i's run out before the row is full. We denote this by writing i^{fill} and (j^{fill}).

R-Matrices

We will now introduce an algorithm which will be instrumental in proving our sufficient condition for a connected ribbon to have full equivalence class. The *R-matrix algorithm*, described in [3, Section 2.2.3], provides a way to swap two consecutive row lengths in an arbitrary ribbon with a semistandard filling so that the filling within the two rows remains semistandard and has the same content as before. Note, however, that semistandardness of the ribbon as a whole is not necessarily preserved.

Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a connected ribbon with filling $\mathcal{F}$, and let α_j and α_{j+1} be the two row lengths we wish to swap. Observe that we can assume $\alpha_j > \alpha_{j+1}$: if $\alpha_j = \alpha_{j+1}$, we're done; if $\alpha_j < \alpha_{j+1}$, we can consider the antipodal rotation of α (by Remark 0.3). The R-matrix algorithm will utilize this fact and assume that $\alpha_j > \alpha_{j+1}$. The algorithm proceeds as follows:

1. Convert the filling of rows j and $j+1$ to a box-ball system with the boxes corresponding to the j^{th} row on the left and the boxes corresponding to the $(j+1)^{st}$ row on the right.

2. For each ball on the right (in an arbitrary order), connect it to the unconnected ball on the left above it which is lowest. If there is no such ball on the left, connect it to the lowest unconnected ball on the left.

3. Shift all unconnected balls on the left horizontally to the right.

4. Convert this box-ball system back into rows of a ribbon.

Example 0.12. [3, Sect. 2.2.3] Suppose we have a ribbon whose j^{th} and $(j+1)^{st}$ rows are as follows.

		1	3	3	4	7
1	3	5				

Figure 1: Rows before swapping

Steps 1-3 of the R-matrix algorithm as applied to this partial tableau are depicted below. Notice that the only ball movement is two balls in the third box from the top shifting from the left to the right, as these were the only two unconnected balls on the left.

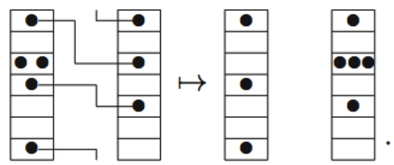


Figure 2: Application of R-matrix algorithm

After applying step 4 of the R-matrix algorithm, we obtain the following partial tableau:

				1	4	7
1	3	3	3	5		

Figure 3: Rows after swapping

Notice that the row lengths have swapped, while the content and semistandardness of the filling has been preserved, as promised.

Edge Cases

Recall that we wish to determine when a connected ribbon has full equivalence class. In certain edge cases, the answer to this question is straightforward.

Proposition 0.13. *Any connected ribbon α with fewer than three rows has full equivalence class.*

Proof. In the case that α has only one row, the proposition follows trivially. Assume that α has two rows. Then for any $\pi \in S_2$, either $\alpha_\pi = \alpha$ or $\alpha_\pi = \alpha^\circ$. In this case, the proposition follows from Remark 0.3. □

As previously mentioned, we will frequently construct Littlewood-Richardson fillings as a method of proving that two connected ribbons have equal or unequal Schur support. Fortunately, it is often possible to make these fillings quite simple.

Definition 0.14. We call a filling of a tableau which uses only 1's and 2's a *1-2 filling.*

Proposition 0.15. *A ribbon $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ has a 1-2 Littlewood-Richardson filling if and only if $\alpha_i \geq 2$ for $2 \leq i \leq m-1$.*

Proof. Observe that if row i has length 1 for some $2 \leq i \leq m-1$, then any semistandard filling of α contains a 3. On the other hand, if $\alpha_i \geq 2$ for $2 \leq i \leq m-1$, we can fill the first row with 1^{α_1}, the last row with 1^{α_m}, and all other rows with $1^{\text{fill}}2$ to obtain a 1-2 LR-filling. □

With the help of Proposition 0.15, we can classify when any connected ribbon with at least one row of length 1 has full equivalence class.

Proposition 0.16. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a ribbon with $k > 0$ rows of length 1. Then α has full equivalence class if and only if $k = m$.*

Proof. We will split this proof into three cases.

Case 1: $k = 1$. Consider any permutation α_π of α such that the the first row has length 1, and any permutation α_σ of α such that neither the first nor last row has length 1. Then by Proposition 0.15, α_π has a 1-2 LR-filling and α_σ does not. Therefore, by the Littlewood-Richardson rule (specifically Corollary 0.11), there exists a straight shape of the form $\lambda = (\lambda_1, \lambda_2)$ in the support of α_π that is not in the support of α_σ.

Case 2: $2 \leq k \leq m-1$. We will split this case into two subcases:
Subcase 1: $k = 2$. Consider the ribbon $\alpha = (1, 1, \alpha_3, \alpha_4, \ldots, \alpha_m)$ and the permutation $\alpha_{(2\ m)} = (1, \alpha_m, \alpha_3, \alpha_4, \ldots, \alpha_{m-1}, 1)$ of α, where $\alpha_i \geq 2$ for $i \in \{3, 4, \ldots, m\}$. Then by Proposition 0.15, α does not have a 1-2 LR-filling, while $\alpha_{(2\ m)}$ does. The result follows from the Littlewood-Richardson rule.
Subcase 2: $3 \leq k \leq m-1$. Consider the ribbon

$$\alpha = (1, \alpha_2, \underbrace{1, 1, \ldots, 1, 1}_{k-1}, \alpha_{m-(k+1)}, \alpha_{m-k}, \ldots, \alpha_m),$$

and the permutation

$$\alpha_{(1\ 2)} = (\alpha_2, \underbrace{1, 1, \ldots, 1, 1}_{k}, \alpha_{m-(k+1)}, \alpha_{m-k}, \ldots, \alpha_m)$$

of α. In the event that $k = m - 1$, omit $\alpha_{m-(k+1)}, \alpha_{m-k}, \ldots, \alpha_m$ from the ribbons. Then any content of a semistandard filling of $\alpha_{(1\ 2)}$ contains a k, while it is easy to see that there exists a content of an LR-filling of α that does not contain a k, so the result again follows from the Littlewood-Richardson rule.

Case 3: $k = m$**.** In this case, $\alpha_\pi = \alpha$ for any $\pi \in S_m$. It follows trivially that α has full equivalence class. □

Remark 0.17. Having covered ribbons with fewer than three rows (Proposition 0.13) and with any number of rows of length 1 (Proposition 0.16), as we proceed we only consider connected ribbons with at least three rows and no rows of length 1.

Now that we have established the necessary preliminary information and have excluded edge cases from our consideration, we are ready to present our main results.

1 A Sufficient Condition

In this section, we prove a sufficient condition for a ribbon to have full equivalence class (Corollary 1.5).

We begin with two lemmas, each of which establishes a property about the R-matrix algorithm (introduced in the Preliminaries), which will be essential for the proof of Theorem 1.3 (which implies Corollary 1.5). In Lemma 1.1, we prove that the R-matrix algorithm preserves the Yamanouchi property of a ribbon with a LR-filling. In Lemma 1.2, we prove that the bottom-left entry in the $(j+1)^{st}$ row of a ribbon with a LR-filling is not increased by the R-matrix algorithm — a step towards showing that for certain ribbons, two rows can be swapped while preserving semistandness of the filling.

We use the results of these two lemmas, in addition to some casework, to show in Theorem 1.3 that under a certain condition on three adjacent row lengths of a ribbon with an LR-filling, the bottom two of the three adjacent row lengths can be swapped while preserving the Yamanouchi property and semistandardness. By imposing this condition on the entire ribbon, we get as a corollary a sufficient condition for a ribbon to have full equivalence class.

Lemma 1.1. *Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a ribbon. Let $i \in \{1, 2, \ldots, m-1\}$. For a given LR-filling $\mathcal{F}$ of α, applying the R-matrix operation to rows i and $i+1$ of α produces a filling $\mathcal{F}'$ of $\alpha_{(i\, i+1)}$ which is Yamanouchi (but not necessarily semistandard).*

Proof. Since $\mathcal{F}$ is Yamanouchi, we only need to show that the initial reverse reading words up to the i^{th} and $(i+1)^{st}$ rows of $\mathcal{F}'$ are Yamanouchi. Denote the filling of the i^{th} and $(i+1)^{st}$ rows of $\mathcal{F}$ by $\mathcal{R}$ and the filling of the i^{th} and $(i+1)^{st}$ rows of $\mathcal{F}'$ by $\mathcal{R}'$. Fix any j and assume that j and $j+1$ appear in $\mathcal{R}$ as follows:

$$\cdots j^a (j+1)^b \cdots$$

$$\cdots j^c (j+1)^d \cdots$$

Let η_j and η_{j+1} denote the number of j's and $(j+1)$'s, respectively, in the reverse reading word of $\mathcal{F}'$ by the end of the $(i-1)^{st}$ row. Let $M = \eta_j - \eta_{j+1}$. Since $\mathcal{F}$ is assumed to be Yamanouchi, we have that $M \geq b$ and $M \geq b + d - a$.

Let x be the number of connected j's on the left when executing the R-matrix algorithm. Similarly, let y be the number of connected $(j+1)$'s on the left. Notice that $x \geq \min(a, d)$.

Following the R-matrix algorithm, j and $j+1$ occur in $\mathcal{R}'$ as:

$$\cdots j^x (j+1)^y \cdots$$

$$\cdots j^{a+c-x} (j+1)^{b+d-y} \cdots$$

Define the function $r(n)$ to be the number of $(j+1)$'s minus the number of j's which have occurred within the first n elements of the initial reverse reading word of $\mathcal{R}'$. (For instance, $r(y) = y$ since the reverse reading word of $\mathcal{R}'$ begins with $(j+1)^y$.) Clearly r is maximal after a string of $(j+1)$'s, so either after the length y string of $(j+1)$'s or after $b + d + x$ elements have been seen. We only have left to show that the function r never exceeds M.

Notice that $r(y) = y$ and $r(b + d + x) = (y + (b + d - y)) - x = b + d - x$. Since $y \leq b \leq M$, we have that $r(y) \leq M$. For $r(b + d + x)$, we consider two cases. If $x \geq d$, then $r(b + d + x) = b + d - x \leq b \leq M$, as desired. On the other hand, if $x < d$, then since $x \geq \min(a, d)$ (as noted above), we have $x \geq a$. Then $r(b + d + x) = b + d - x \leq b + d - a \leq M$. This completes the proof. □

We have just shown that the R-matrix algorithm preserves the Yamanouchi property of a ribbon with an LR-filling. Recall from the Preliminaries that the R-matrix operation as applied to a ribbon with an LR-filling preserves semistandardness of the filling within the two rows that are swapped; however, semistandardness of the filling of the entire ribbon is not necessarily preserved. With the next lemma, we prove another property of the R-matrix algorithm so as to work towards establishing how we might use the R-matrix algorithm while preserving the semistandardness of the entire ribbon.

Lemma 1.2. *Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a ribbon. Let $i \in \{1, 2, \ldots, m-1\}$. For a given LR-filling $\mathcal{F}$ of α, applying the R-matrix algorithm to rows i and $i+1$ to obtain the filling $\mathcal{F}^{'}$ does not increase the leftmost entry in the $(i+1)^{st}$ row from the top.*

Proof. Suppose x is the entry of the leftmost box in the $(i+1)^{st}$ row of $\mathcal{F}$. Then by the R-matrix algorithm, x is also in the $(i+1)^{st}$ row of $\mathcal{F}^{'}$. Now the result follows from the fact that the R-matrix operation preserves semistandardness within the two rows. ☐

The remaining way in which $\alpha_{(i\ i+1)}$ with filling $\mathcal{F}^{'}$ may not be semistandard is for the number in the rightmost box of the i^{th} row of $\alpha_{(i\ i+1)}$ to be less than or equal to the leftmost box in the $(i-1)^{st}$ row (where α and $\mathcal{F}^{'}$ are as in Lemma 1.2). This is the main focus of the following proof.

Theorem 1.3. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a ribbon. Let $\alpha_i < \alpha_{i-1} + \alpha_{i+1}$ for some $1 \le i \le (m-1)$, where $\alpha_0 = \infty$. Then $[\alpha] \subseteq [\alpha_{(i,i+1)}]$.*

Proof. Let $\mathcal{F}$ be an LR-filling of α with content μ. Since antipodal rotation preserves Schur support (Remark 0.3), we can assume without loss of generality that $\alpha_i > \alpha_{i+1}$. Perform the R-matrix algorithm on rows i and $i+1$ of α to obtain the ribbon $\alpha_{(i\ i+1)}$ with filling $\mathcal{F}^{'}$. We will refer to the labeling of boxes of α as shown in Figure 4, where the top row shown in the diagram is the $(i-1)^{st}$ row of α.

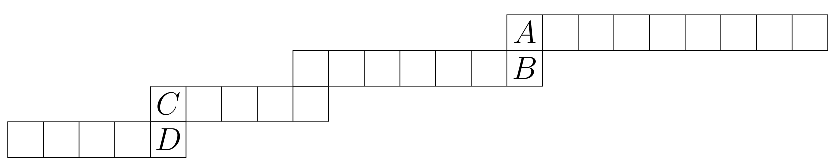


Figure 4: Labeled boxes of α

We denote the corresponding boxes of $\alpha_{(i\ i+1)}$ as A', B', C', and D'. By Lemma 1.1, $\mathcal{F}^{'}$ satisfies the Yamanouchi property. By Lemma 1.2, if $i = 1$ (the top row is swapped), then we have achieved an LR-filling and we are done. So, assume $i > 1$ and we will now ensure semistandardness in the filling of boxes A' and B' of $\alpha_{(i\ i+1)}$. Let the integers in boxes A' and B' be a and b, respectively. If $a < b$, we are done.

If $b < a$, simply swap a and b to obtain an LR-filling of $\alpha_{(i\,i+1)}$, and we are done. Now, assume $a = b$.

Notice that if there is an entry $w > b$ in the $(i-1)^{st}$ row which is not rightmost, then we may select the minimal such w and swap the b in box B' with the leftmost occurrence of w in the $(i-1)^{st}$ row to obtain an LR-filling with content μ. Thus we may assume that all entries except the rightmost entry of the $(i-1)^{st}$ row are b. Since $\mathcal{F}$ was semistandard, the element in box B of $\mathcal{F}$ must have been strictly greater than b. Then since that entry is clearly not in the i^{th} row of $\mathcal{F}'$, it must have been taken to the $(i+1)^{st}$ row by the R-matrix algorithm. Hence, there must be some entry of $\mathcal{F}'$ in the $(i+1)^{st}$ row of $\alpha_{(i\,i+1)}$ which is greater than b.

Now, if any entry x (besides the leftmost) of the $(i+1)^{st}$ row is less than b, we can select the maximal such x and swap the b in box A' with the rightmost occurrence of x to obtain a valid LR-filling. Therefore we can assume that all entries except the leftmost of the $(i+1)^{st}$ row are greater than or equal to b.

Additionally, if any entry (besides the leftmost) of the i^{th} row is less than b, we can choose the rightmost such entry and exchange it with the b in box A' to obtain a valid LR-filling. Thus we can assume that all entries except the leftmost entry are equal to b.

Now, let y denote the leftmost entry of the i^{th} row, and consider the case where $y \neq b$. In this case, $y < b$ by the preservation of semistandardness within the two rows swapped. It follows that we can swap y with the b in box A' and be done, since we know that the rightmost entry in the $(i+1)^{st}$ row is strictly greater than b. Therefore we may assume that $y = b$.

By the above arguments, we may assume the entries shown in Figure 5, where q and z are unknown. (In case the notation is unclear, in the $(i+1)^{st}$ row, we are attempting to convey that all but possibly the leftmost entry is at least b; additionally, recall that the rightmost entry is strictly greater than b.)

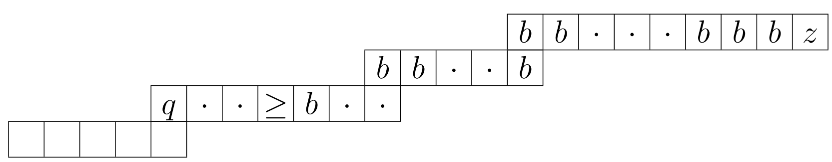


Figure 5: Assumed entries of $\mathcal{F}'$

We finish the proof by looking at two cases. In order to distinguish between the two cases, we need to set up some additional notation. Let s be value of the leftmost entry in the $(i+1)^{st}$ row which is greater than b. Let $\mathcal{R}$ be the filling of the subdiagram of $\mathcal{F}'$ consisting exactly of the $(i-1)^{st}$, i^{th} and $(i+1)^{st}$ rows, except excluding the rightmost box of the $(i-1)^{st}$ row and all boxes to the left of the leftmost entry with values less than or equal to b in the $(i+1)^{st}$ row.

Let $\mathcal{V}_z$ be the initial partial tableau of $\mathcal{L}$ which ends immediately after the z. For every integer n, let M_n be the number of n's which occur in the initial reverse reading word of $\mathcal{V}_z$.

Case 1: $M_b > M_{b+1}$**.** In this case, we argue we can swap the b in box B' with an appropriate entry from the row below.

Consider the leftmost box in the $(i+1)^{st}$ row with entry s. If this is the rightmost box of row $i+1$, then the i^{th} and $(i+1)^{st}$ rows together contain at most two entries not equal to b: q and s, where $q < b < s$. Since s is the lone entry greater than b in rows i and $i+1$ of $\mathcal{F}^{'}$, in order for $\mathcal{F}$ to have been semistandard in rows $i-1$ and i, the s must have been in box B of $\mathcal{F}$. Now, looking at the 2×1 box overlap between rows i and $i+1$, we see q must have been the leftmost entry of the i^{th} row of $\mathcal{F}$.

However, the R-matrix algorithm would have kept q in the i^{th} row; more specifically, having no other numbers smaller than b on the left, a b on the right would be connected to q on the left. This contradicts the filling $\mathcal{F}^{'}$. We therefore conclude that the leftmost s cannot be in the rightmost box of the $(i+1)^{st}$ row of $\mathcal{F}^{'}$. It follows that swapping the leftmost s with the b in box B' of $\mathcal{F}^{'}$ will produce a semistandard Young tableau. We will call the filling after this swap $\mathcal{L}$.

We now argue that $\mathcal{L}$ will also be Yamanouchi. If $s \neq b+1$, then the filling clearly retains its Yamanouchi property — this would mean that there are no occurrences of either $b+1$ or $s-1$ in the i^{th} or $(i+1)^{st}$ rows and so making s appear earlier and b appear later in this segment will not violate the Yamanouchi property.

Thus we may assume that $s = b+1$ (but we will still write s for formatting purposes). Then $\mathcal{L}$ has the entries shown in Figure 6, where we indicate that the $(i+1)^{st}$ row, to the right of its leftmost entry, has a string of b's, followed by a string of numbers at least s.

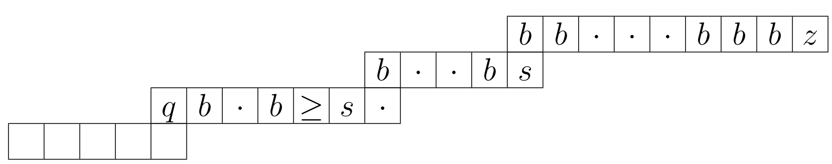


Figure 6: Assumed entries of $\mathcal{L}$

Let $r(n)$ be the number of b's that occur in the first n elements of the initial reverse reading word of $\mathcal{R}$, minus the number of s's that occur in the same string. Since $M_b > M_s$ by assumption, in order to show that $\mathcal{L}$ is Yamanouchi, it will be sufficient to show that $r(n)$ is never less than -1.

The reverse reading word of $\mathcal{R}$ is $b^{\alpha_{i-1}-1}sb^{\alpha_{i+1}-1} * \cdots * s$, where the $*$'s are all at least s. Clearly the function r is minimal when all the $*$'s are equal to s, so we may assume that we have $b^{\alpha_{i-1}-1}sb^{\alpha_{i+1}-1}s^k$, where $k \leq \alpha_i - 1$. To see that $r(n)$ is at least -1 everywhere, we want to show that $\alpha_{i-1} - 1 \geq 1$ (which is clear) and that $(\alpha_{i-1} - 1 + \alpha_{i+1} - 1) - (k+1) \geq -1$. It will be helpful to notice that $\alpha_i < \alpha_{i-1} + \alpha_{i+1}$ can be rewritten as $\alpha_i + 1 \leq \alpha_{i-1} + \alpha_{i+1}$. We then get that

$$k \leq \alpha_i - 1 = (\alpha_i + 1) - 2 \leq \alpha_{i-1} + \alpha_{i+1} - 2.$$

Finally, $k \leq \alpha_{i-1} + \alpha_{i+1} - 2$ implies that $(\alpha_{i-1} - 1 + \alpha_{i+1} - 1) - (k+1) \geq -1$. This completes Case 1.

Case 2: $M_b = M_{b+1}$**.** Let $u = b + 1$ for notational simplicity. First we argue that $z \neq b$ by proving the following claim.

<u>Claim 1:</u> In the reverse reading word of $\mathcal{V}_z$, the last u must occur after the last b.

Proof of Claim 1. Suppose for the sake of contradiction that the last u occurs before the last b in the reverse reading word of $\mathcal{V}_z$. Then the initial reverse reading word of $\mathcal{V}_z$ ending immediately before the last b contains M_u u's and $M_b - 1 = M_u - 1$ b's, meaning this initial reverse reading word is not Yamanouchi. This contradiction completes the proof. □

In particular, this implies that $z \neq b$. Now, let t denote the entry immediately above z in $\mathcal{F}'$. If $t \neq b$, then the b in box B' can be swapped with one of z or t to obtain a valid LR-filling. We can therefore assume $t = b$. Then by Claim 1, we must have $z = u$. So, we can update our assumed entries of $\mathcal{F}'$.

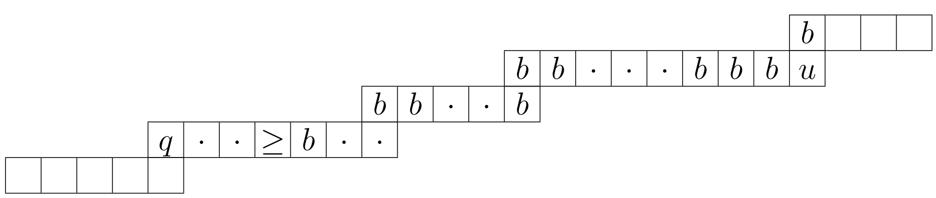


Figure 7: Updated assumed entries of $\mathcal{F}'$

Now scan the tableau corresponding to $\mathcal{F}'$ from the $(i-2)^{nd}$ row upwards. First suppose we reach a row which contains multiple entries greater than b. Then we can easily swap the leftmost such entry with the b in box B' to obtain an LR-filling. As a result, we may assume that we do not encounter a row with more than one entry greater than b. In particular, we may assume that any u appearing above the $(i-1)^{st}$ row must be in the rightmost box of its row, by the semistandardness of $\mathcal{F}'$. We will use the following claim to complete the proof of this theorem.

Claim 2: There exists a u in the rightmost box of some row above the $(i-1)^{st}$ such that the entry immediately above the u is less than b.

Proof of Claim 2. First we argue that there exists a u in a row above the $(i-1)^{st}$ row. Suppose that the u in the $(i-1)^{st}$ row is the topmost u in the tableau. Consider what is immediately to the right of the b in the leftmost box of the $(i-2)^{nd}$ row. This entry cannot be b by the assumption that $M_b = M_u$; however, it also cannot be greater than u since the tableau is Yamanouchi. Thus we conclude that there exists a u in a row above the $(i-1)^{st}$ row. Additionally, by the paragraph above the statement of Claim 2, any such u must be in the rightmost box of its row.

Notice that by the Yamanouchi condition, no u can appear in the first row (since $u > b$), so each u necessarily has a box immediately above it. Assume for the sake of contradiction that every u above the $(i-1)^{st}$ row has a b immediately above it. Then since $M_b = M_u$, we have that all of the b's in the first $(i-2)$ rows are immediately above a u. Now, consider the topmost b. Since it is immediately above a u, it must be the leftmost box in its row. Since all rows are at least 2 boxes in length (Remark 0.17), there is an entry immediately to the right of this topmost b. This entry must be at least b by semistandardness. However, if it is b, it contradicts the assumption that $M_b = M_u$; if it is greater than b, it contradicts the fact that our tableau is Yamanouchi. In either case, we reach a contradiction, so we conclude that there exists a u in the first $(i-2)$ rows with an entry not equal to b immediately above it. By semistandardness, this entry must be less than b. □

It is easy to see that swapping the u given by Claim 2 with the b in box B' produces an LR-filling. This completes the proof of Theorem 1.3.

□

Since adjacent transpositions generate the symmetric group, the above theorem gives the following sufficient condition for a connected ribbon to have full equivalence class. Before stating this corollary, we give a quick definition.

Definition 1.4. We say integers $x \le y \le z$ *satisfy the strict triangle inequality* if $z < x + y$. In this case, we may also say that $\{x, y, z\}$ satisfies the strict triangle inequality.

Corollary 1.5. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a ribbon. If all 3-subsets of $\{\alpha_i\}_{i=1}^m$ satisfy the strict triangle inequality, then α has full equivalence class.*

Proof. Let $i \in \{1, 2, \ldots, m-1\}$ be arbitrary. As noted above, it will be sufficient to show that $[\alpha] = [\alpha_{(i,i+1)}]$. If $\alpha_i = \alpha_{i+1}$, this result follows trivially. Then by Remark 0.3, we can assume without loss of generality that $\alpha_i > \alpha_{i+1}$. By assumption, $\alpha_i < \alpha_{i-1} + \alpha_{i+1}$, so Theorem 1.3 implies that $[\alpha] \subseteq [\alpha_{(i,i+1)}]$.

To show containment the other way, consider the antipodal rotation $\alpha^\circ = (\alpha_m, \alpha_{m-1}, \ldots, \alpha_1)$ of α, with $\alpha^\circ_{m+1} = 0$ for convenience. Let $j = m + 1 - i$, so that $\alpha^\circ_j = \alpha_i$ and $\alpha^\circ_{j-1} = \alpha_{i+1}$. Note that $j \geq 2$ since $i < m$. Thus we want to show that $[\alpha^\circ_{(j-1,j)}] \subseteq [\alpha^\circ]$.

Since $\alpha_i > \alpha_{i+1}$, we have that $\alpha^\circ_j > \alpha^\circ_{j-1}$ and by assumption, $\alpha^\circ_j < \alpha^\circ_{j-1} + \alpha^\circ_{j-2}$. Since swapping α°_j and α°_{j-1} in $\alpha^\circ_{(j-1,j)}$ gives us α° back, Theorem 1.3 gives us that $[\alpha^\circ_{(j-1,j)}] \subseteq [\alpha^\circ]$, completing the proof. □

Having proven a sufficient condition for a ribbon to have full equivalence class, we now prove a separate necessary condition.

2 A Necessary Condition

Theorem 2.1. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a ribbon, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$. If α has full equivalence class, then $N_j < \sum_{i=j+1}^m \alpha_i - (m-j-2)$ for all $1 \leq j \leq m-2$, where*

$$N_j = \max\{k | \sum_{i \leq j:\, \alpha_i < k} (k - \alpha_i) \leq m - j - 2\}.$$

Although this condition may appear a bit convoluted, the following remark and lemma may help motivate it.

Remark 2.2. We will always have $\alpha_j \leq N_j \leq \alpha_j + m - j - 2$. In particular, $N_j = \alpha_j$ whenever $j = m-2$, while $N_j = \alpha_j + m - j - 2$ if and only if $\alpha_j \leq \alpha_{j-1} - (m-j-2)$.

Lemma 2.3. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a ribbon, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$. If $N_j \geq \sum_{i=j+1}^m \alpha_i - (m-j-2)$ for some $j \in \{1, 2, \ldots, m-2\}$, then $\alpha_j > \alpha_{j+1}$.*

Proof. Assume that $N_j \geq \sum_{i=j+1}^m \alpha_i - (m-j-2)$ for some $j \in \{1, 2, \ldots, m-2\}$. It follows that

$$\begin{aligned}\alpha_{j+1} &\le N_j + (m-j-2) - \sum_{i=j+2}^{m} \alpha_i \\ &\le \alpha_j + 2(m-j-2) - \sum_{i=j+2}^{m} \alpha_i \\ &< \alpha_j,\end{aligned}$$

where the second inequality follows from the upper bound on N_j given in Remark 2.2 and the third inequality follows from Remark 0.17. □

Additionally, our proof of Theorem 2.1 should make the condition more intuitive.

Proof of Theorem 2.1. We prove by contrapositive. Fix $j \in \{1, 2, \ldots, m-2\}$ such that $N_j \ge \sum_{i=j+1}^{m} \alpha_i - (m-j-2)$. We will give an LR-filling of $\alpha_{(j\ j+1)}$ of content μ such that α has no LR-filling of content μ.

Fill $\alpha_{(j\ j+1)}$ as follows (we'll call this filling $\mathcal{F}$). Fill the i^{th} row entirely with i's for $i \le j$. Put α_{j+1} $(j+1)$'s in the rightmost boxes of the $(j+1)^{st}$ row and fill the remaining boxes in this row with j's. Note that by Lemma 2.3, the leftmost entry of the $(j+1)^{st}$ row in this filling is a j (meaning this row is longer than α_{j+1} in length).

We now fill the remaining $m-j-1$ rows with as many $(j+1)$'s as possible; put $(j+1)$'s in all but the leftmost box of the next $m-j-2$ rows, as well as in every box in the last row. Now the only empty boxes are the leftmost boxes in rows $j+2, \ldots, m-1$. We will call these remaining boxes *critical boxes.* Fill the critical boxes from top to bottom according to the following algorithm: in each box, put the largest integer $\le j$ such that the initial reverse reading word through that box remains Yamanouchi. In practice, we will use exclusively j's until the number of j's in the tableau equals the number of $(j-1)$'s. Then, we will alternate between $(j-1)$'s and j's until the number of $(j-1)$'s equals the number of $(j-2)$'s. At this point, we rotate between placing j's, $(j-1)$'s, and $(j-2)$'s until the number of $(j-2)$'s equals the number of $(j-3)$'s. We continue in this manner until all boxes have been filled. In order to prove this algorithm gives an LR-filling, we will show that this filling has exactly N_j j's.

First we define a "round". Consider the sequence of numbers $c_1, c_2, \ldots, c_{m-j-2}$ written into the critical boxes from top to bottom. Let $J = \{c_j : c_j = j\}$. Now partition $c_1, \ldots, c_{m-j-2}$ into *rounds*, where each round is a consecutive subsequence of $c_1, \ldots, c_{m-j-2}$ whose last element is in J but with no other elements in J (i.e. a round ends if and only if a j is encountered).

Claim: If r rounds can be completed before reaching the bottom, then at the end of the r^{th} round, we have filled exactly

$$\sum_{i\leq j:\ \alpha_i<\alpha_j+r} (\alpha_j + r - \alpha_i)$$

critical boxes. In particular, each number $i \leq j$ such that $\alpha_i \leq \alpha_j + r$ has occurred in exactly $\alpha_j + r - \alpha_i$ critical boxes.

Proof of Claim. We will use induction on r. The claim trivially holds when $r = 0$. Now consider an arbitrary $r > 0$ (such that r rounds can be completed before reaching the bottom) and assume the claim holds for $r-1$. In the r^{th} round, we will write every number that was used in the $(r-1)^{st}$ round one more time, as well as any number ℓ satisfying $\alpha_\ell = \alpha_j + r - 1$. Therefore, the latter numbers will each fill exactly 1 critical box after r rounds, as appropriate since, by choice of ℓ, $\alpha_j + r - \alpha_\ell = 1$. All numbers which appeared in the $(r-1)^{st}$ round have now occurred in a critical box one more time than before. For a fixed number i, by the induction hypothesis, this is $\alpha_j + (r-1) - \alpha_i + 1 = \alpha_j + r - \alpha_i$ times. This completes the proof of the Claim. □

Clearly the number of j's in $\mathcal{F}$ is α_j plus the number of rounds executed before running out of critical boxes. That is,

$$\begin{aligned} \mu_j &= \alpha_j + \max\{r \mid \sum_{i\leq j:\alpha_i<\alpha_j+r} \alpha_j + r - \alpha_i \leq m - j - 2\} \\ &= \max\{k : \sum_{i\leq j:\alpha_i<k} (k - \alpha_i) \leq m - j - 2\} = N_j. \end{aligned}$$

In particular, $\mu_j = N_j$.

By construction, $\mu_j \leq \mu_{j-1} \leq \cdots \leq \mu_1$. Semistandardness is also clear by construction, so we all that is left to check is that $\mu_j \geq \mu_{j+1}$. Indeed, $\mu_{j+1} = \sum_{i=j+1}^m \alpha_i - (m-j-2)$, so this condition follows by our assumption that $N_j \geq \sum_{i=j+1}^m \alpha_i - (m-j-2)$, which gives that $\mu_j = N_j \geq \sum_{i=j+1}^m \alpha_i - (m-j-2) = \mu_{j+1}$.

We now show that α does not have an LR-filling of content μ. By the Yamanouchi property and semistandardness, there cannot be any $(j+1)$'s above the $(j+1)^{st}$ row. It follows that the maximum number of $(j+1)$'s is $\sum_{i=j+1}^m \alpha_i - (m-j-1) < \mu_{j+1}$. Therefore there is no LR-filling of α with content μ and by Corollary 0.11, μ is in the support of $\alpha_{(j\ j+1)}$ but not of α. □

Conjecture 2.4. *Let $\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_m)$ be a ribbon with each $\alpha_i \geq 2$ and $m \geq 3$. Then, α has full equivalence class if and only if $N_j < \sum_{i=j+1}^m \alpha_i - (m-j-2)$ for all $1 \leq j \leq m-2$.*

We have proven that the condition in Theorem 2.1 is necessary for a ribbon to have full equivalence class, and have just conjectured it to be sufficient. We now show that this condition is in fact sufficient for $m = 3$ and $m = 4$. We have also verified by computation that this condition is sufficient for $m = 5$, $m = 6$, and $m = 7$ for certain n (see Appendix for sample code).

3 General Results Applied to Small Cases

In this section, we support Conjecture 2.4 by proving that the condition it mentions is both necessary and sufficient for $m = 3$ and $m = 4$.

3.1 Ribbons With 3 Rows

Let us first consider what the condition from Conjecture 2.4 is in the $m = 3$ case. This condition requires that $1 \leq j \leq m - 2 = 1$, meaning that we only need to consider $j = 1$. Then by Remark 2.2, $N_j = \alpha_j = \alpha_1$. The conjectured necessary and sufficient condition from Conjecture 2.4 therefore amounts to

$$N_1 = \alpha_1 < \sum_{i=2}^{3} \alpha_i - (m - j - 2) = \alpha_2 + \alpha_3 - (3 - 1 - 2) = \alpha_2 + \alpha_3$$

,

As we see, Conjecture 2.4 would imply that when $m = 3$, having row lengths which satisfy the strict triangle inequality is both necessary and sufficient for having full equivalence class. Before proving in Theorem 3.4 that this is in fact the case, we state the lemmas that we need for the proof of Theorem 3.4 and a definition needed for the lemmas. Since the proofs of these lemmas are long and technical, we will defer their proofs until later.

Definition 3.1. In a ribbon $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, a row j is considered to be *long* if $1 \leq j < m$ and $\alpha_j > \sum_{i=1}^{j-1} \alpha_i - 2(j-2)$, or if $j = m$ and $\alpha_j \geq \sum_{i=1}^{j-1} \alpha_i - 2(j-2)$.

Lemma 3.2. *Let α_j be the longest length of a row in the connected ribbon α. Then no long rows occur beneath the first occurrence of a row of length α_j.*

Lemma 3.3. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a ribbon. Let j be the largest row index such that row j is long; if α has no long rows, let $j = 1$. Let $\ell = \max(\alpha_j + m - 2, \lceil n/2 \rceil)$ if $j = 1$ or $j = m$, and let $\ell = \max(\alpha_j + m - 3, \lceil n/2 \rceil)$ if $1 < j < m$. Then there exists a 1-2 LR-filling with p 1's if and only if*

$$\ell \leq p \leq n - (m - 1).$$

Now we can state and prove the theorem we wanted:

Theorem 3.4. *Any ribbon of the form $\alpha = (\alpha_1, \alpha_2, a_3)$ where $\alpha_1 \geq a_2 \geq a_3$, has full equivalence class if and only if $\alpha_1, \alpha_2, \alpha_3$ satisfy the strict triangle inequality.*

Proof. If $\alpha_1, \alpha_2, \alpha_3$ satisfy the strict triangle inequality, we have by Corollary 1.5 that α has full equivalence class. Conversely, assume that $\alpha_1 \geq \alpha_2 + \alpha_3$. It follows that $\alpha_1 + m - 3 = \alpha_1 \geq \lceil n/2 \rceil$ and that $\alpha_1 > \alpha_2$. Therefore Lemmas 3.2 and 3.3 imply that there exists a 1-2 LR-filling of $\alpha_{(1,2)} = (\alpha_2, \alpha_1, \alpha_3)$. However, the same lemmas also imply that any 1-2 LR-filling of α requires at least $\alpha_1 + m - 2 = \alpha_1 + 1$ 1's. So $(\alpha_1, \alpha_2 + \alpha_3) \in [\alpha_{(1,2)}]$, but $(\alpha_1, \alpha_2 + \alpha_3) \notin [\alpha]$. □

3.2 Ribbons With 4 Rows

This subsection will proceed analogously to Subsection 3.1, but will also include some additional general results (such as Corollary 3.7) and results specific to $m = 4$ (such as parts of Theorem 3.5), which are not strictly necessary for proving that Conjecture 2.4 holds for $m = 4$.

As in the previous subsection, let us first consider what Conjecture 2.4 implies in the case of $m = 4$. Since $1 \leq j \leq m - 2 = 4 - 2 = 2$ in the statement of Conjecture 2.4, we need only consider $j = 1$ and $j = 2$. In the case of $j = 1$, we get that $N_1 = \max\{k | \ k - \alpha_1 \leq 4 - 1 - 2 = 1\}$, which implies that $N_1 = \alpha_1 + 1$. Therefore, we get the condition

$$\begin{aligned} N_1 = \alpha_1 + 1 <& \alpha_2 + \alpha_3 + \alpha_4 - (m - j - 2) = \alpha_2 + \alpha_3 + \alpha_4 - 1 \\ \implies \alpha_1 <& \alpha_2 + \alpha_3 + \alpha_4 - 2 \end{aligned}$$

In the case that $j = 2$, we have from Remark 2.2 that $N_2 = \alpha_2$, which implies the condition

$$N_2 = \alpha_2 < \alpha_3 + \alpha_4 - (m - j - 2) = \alpha_3 + \alpha_4.$$

We will soon prove, in Theorem 3.6, that these conditions classify exactly when ribbons with 4 rows have full equivalence class. Before we do that, we state Theorem 3.5, which will be helpful in proving Theorem 3.6. Similar to Subsection 3.1, we will defer the long and technical proof of Theorem 3.5 until after we have proved the more significant result of Theorem 3.6.

Hypothesis 1. *Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be a ribbon where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = n$ and the following inequalities are satisfied:*

(a) $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4$

(b) $\alpha_1 \geq \alpha_2 + \alpha_3$

(c) $\alpha_2 < \alpha_3 + \alpha_4$

(d) $\alpha_1 < \alpha_2 + \alpha_3 + \alpha_4$

Theorem 3.5. *Suppose we have a ribbon $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ satisfying the conditions of Hypothesis 1. Then the following hold:*

1. *α has maximal support equivalence class if and only if $\alpha_1 < \lceil n/2 \rceil - 1$.*
2. *For n even, if $\alpha_1 = n/2-1$, then $(n/2, n/2) \notin [(\alpha_1, \alpha_2, \alpha_3, \alpha_4)]$, but $(n/2, n/2) \in [(\alpha_2, \alpha_1, \alpha_3, \alpha_4)]$.*
3. *For n odd, if $\alpha_1 = \lceil n/2 \rceil - 1$, then $(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor, 1)$ and $(\lceil n/2 \rceil, \lfloor n/2 \rfloor)$ are in the support of $(\alpha_2, \alpha_1, \alpha_3, \alpha_4)$, but not in the support of $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.*

Theorem 3.6. *Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be a ribbon where $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4$. Then α has full equivalence class if and only if*

(i) $\alpha_1 < \alpha_2 + \alpha_3 + \alpha_4 - 2$, *and*

(ii) $\alpha_2 < \alpha_3 + \alpha_4$.

Proof. As explained at the beginning of the section, conditions (i) and (ii) are necessary for a connected ribbon α to have full equivalence class (by Theorem 2.1).

Conversely, if $\alpha_1 < \alpha_2 + \alpha_3$, then it follows from Corollary 1.5 that α has full equivalence class. So, assume that $\alpha_1 \geq \alpha_2 + \alpha_3$. Notice that by (i), $2\alpha_1 < n - 2$, meaning $\alpha_1 < \lceil n/2 \rceil - 1$. In this case, we have from Theorem 3.5 (1) that α has full equivalence class. □

We now turn our attention to proving the auxiliary results we used above: Lemmas 3.2, 3.3, and Theorem 3.5. We are ready to prove the two lemmas, but Theorem 3.5 takes more work to prove; we finally do so at the end of this section.

Lemma 3.2. *Let α_j be the longest length of a row in the connected ribbon α. Then no long rows occur beneath the first occurrence of a row of length α_j.*

Proof. Assume, for the sake of contradiction, that there exists a long row, k, occurring beneath a row of length α_j. Then

$$\alpha_k > \sum_{i=1}^{k-1} \alpha_i - 2(k-2) = \alpha_j + \Big[\sum_{i=1}^{j-1} \alpha_i + \sum_{i=j+1}^{k-1} \alpha_i - 2(k-2) \Big] \geq \alpha_j,$$

where we have used our assumption from Remark 0.17. However, this is a contradiction, since $\alpha_k \leq \alpha_j$ by assumption. □

We will now prove Lemma 3.3, which classifies exactly which contents are possible for 1-2 LR-fillings of a connected ribbon with four rows. This classification, as well as being helpful for proving Theorems 3.4 and 3.6 above, will be an essential tool in proofs throughout the remainder of the subsection.

Lemma 3.3. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a ribbon. Let j be the largest row index such that row j is long; if α has no long rows, let $j = 1$. Let $\ell = \max(\alpha_j + m - 2, \lceil n/2 \rceil)$ if $j = 1$ or $j = m$, and let $\ell = \max(\alpha_j + m - 3, \lceil n/2 \rceil)$ if $1 < j < m$. Then there exists a 1-2 LR-filling with p 1's if and only if*

$$\ell \leq p \leq n - (m - 1).$$

Proof. For any value of j, we can obtain a 1-2 filling with $n - (m-1)$ 1's by filling all boxes with 1's except the boxes which are the bottom box of a 2×1 rectangle, which we will fill with 2's. Since each row has length at least 2 (by our assumption stated in Remark 0.17), we obtain a LR-filling.

To obtain a LR-filling with exactly ℓ 1's, we will split the proof into three cases: $j = 1$, $j = m$, and $1 < j < m$. Once we have obtained such a filling, we can replace some 2's with 1's until we reach $n - (m-1)$ 1's to obtain a filling with $\ell \leq p \leq n - (m - 1)$ 1's.

Case 1 ($j = 1$): We show that in both subcases (i.e. $\ell = \alpha_1 + m - 2$ and $\ell = \lceil n/2 \rceil$), there exists a 1-2 LR-filling with ℓ 1's.

<u>Subcase 1 ($\lceil n/2 \rceil \leq \alpha_1 + m - 2$)</u>: Fill the first row of α with 1's. Proceed to put 1's in the top box of the remaining 2×1 rectangles (of which there are $m - 2$). Thus we have used $(\alpha_1 + m - 2)$ 1's. Fill the remaining boxes with 2's. Since $\lceil n/2 \rceil \leq \alpha_1 + m - 2$, we know that this filling has more 1's than 2's. Assume, for the sake of contradiction, that we have violated the Yamanouchi property. Then there exists some row k in which the number of 2's surpasses the number of 1's for the first time. We have by assumption (Remark 0.17) that every row has length at least 2. Therefore, since each remaining row has at most one 1 (and since row m is all 2's), the number of 2's that will be added below row k is greater than the number of 1's to be added below row k. This contradicts the fact that there are ultimately more 1's than 2's in the filling. Therefore, our filling is a LR-filling.

<u>Subcase 2 ($\lceil n/2 \rceil > \alpha_1 + m - 2$)</u>: Begin with the filling described in Subcase 1 (which now has fewer 1's than 2's). Starting in the second row and reading left to right across rows, replace every 2 that is not part of a 2×1 rectangle with a 1 until there are $\lceil n/2 \rceil$ 1's. Suppose this process terminates at row k. It is clear that we do not violate the Yamanouchi property in any row higher than k. Assume, for the sake of

contradiction, that we violate the Yamanouchi property in the row k. If there are x 2's in this row, we get

$$x + (k-2) > \sum_{i=1}^{k-1} \alpha_i - (k-2).$$

This implies that $x > \sum_{i=1}^{k-1} \alpha_i - 2(k-2)$, contrary to the assumption that $j = 1$ (i.e. that no row past row 1 is long). It is clear that in the rows after we have placed $\lceil n/2 \rceil$ 1's, we do not violate the Yamanouchi property.

Case 2 ($j = m$): We have that $n = \sum_{i=1}^{j} \alpha_i$ and that $\alpha_m \geq \sum_{i=1}^{m-1} \alpha_i - 2(m-2)$. It follows that

$$\lceil n/2 \rceil = \left\lceil \frac{\alpha_m + \sum_{i=1}^{m-1} \alpha_i}{2} \right\rceil \leq \left\lceil \frac{\alpha_m + (\alpha_m + 2(m-2))}{2} \right\rceil = \alpha_m + m - 2$$

Therefore, $\ell = \alpha_m + m - 2$. Begin by filling the first row of α with 1's. Put 2's in the rightmost box of each of the rows 2 through $m-1$. Fill the remaining boxes of rows 2 through $m-1$ with 1's. The number of 1's used in the filling so far is $\sum_{i=1}^{m-1} \alpha_i - (m-2)$, while the number of 2's used so far is $m-2$. Now, fill the rightmost portion of row m with $\sum_{i=1}^{m-1} \alpha_i - 2(m-2)$ 2's (bringing the total number of 2's to $\sum_{i=1}^{m-1} \alpha_i - (m-2)$). Next, fill the rest of row m with $\alpha_m - (\sum_{i=1}^{m-1} \alpha_i - 2(m-2))$ 1's, bringing the total number of 1's to

$$\alpha_m - \sum_{i=1}^{m-1} \alpha_i + 2(m-2) + (\sum_{i=1}^{m-1} \alpha_i - (m-2)) = \alpha_m + (m-2).$$

It is clear that we have not violated the Yamanouchi property.

Case 3 ($1 < j < m$): We will consider both the subcase where $\lceil n/2 \rceil \leq \alpha_j + m - 3$ and the subcase where $\lceil n/2 \rceil > \alpha_j + m - 3$.

<u>Subcase 1 ($\lceil n/2 \rceil \leq \alpha_j + m - 3$):</u> Fill the first j rows with $\alpha_j + (j-2)$ 1's and $\sum_{i=1}^{j-1} \alpha_i - (j-2)$ 2's, as in Case 2. Since $j < m$, we have $\alpha_j > \sum_{i=1}^{j-1} \alpha_i - 2(j-2)$ (the number of 2's in row j), meaning there is at least one 1 in row j. To fill the remaining $m-j$ rows, place a 1 in the top box of each remaining 2×1 rectangles (of which there are $m-j-1$); this brings the total number of 1's to $\alpha_j + (j-2) + (m-j-1) = \alpha_j + m - 3$. Finally, we will use $\sum_{i=j+1}^{m} \alpha_i - (m-j-1)$ 2's to fill the remaining boxes.

Assume, for the sake of contradiction, that we have violated the Yamanouchi property. It is clear that we do not violate the Yamanouchi property above or in row

j. Then there exists some row $k > j$ in which the number of 2's surpasses the number of 1's for the first time. However, by the same argument as in Case 1: Subcase 1, this would contradict the fact that there are ultimately more 1's than 2's in the filling. Therefore, our filling is a LR-filling.

Subcase 2 ($\lceil n/2 \rceil > \alpha_j + m - 3$): Start with the filling described in Subcase 1, which now, by assumption, has strictly fewer than $\lceil n/2 \rceil$ 1's. Starting from the j^{th} row and reading left-to-right across rows, change all 2's that are not part of a 2×1 rectangle to 1's until there are $\lceil n/2 \rceil$ 1's. It is clear that we do not violate the Yamanouchi property in all of the rows above where we stop changing 2's to 1's (since each such row has at most one 2). If we stop changing 2's to 1's in the j^{th} row, it is again clear that we do not violate the Yamanouchi property (since we have by the construction in Case 2 that first j rows comply with the Yamanouchi property). Assume, for the sake of contradiction, that we violate the Yamanouchi property in a row $k > j$ where we place the $(\lceil n/2 \rceil)^{nd}$ 1. Then the number of 1's in the first $k-1$ rows is

$$\sum_{i=1}^{k-1} \alpha_i - (k-2) \tag{2}$$

and the number of 2's in the first k rows is

$$(k-2) + [\alpha_k - (\lceil n/2 \rceil - \sum_{i=1}^{k-1} \alpha_i + (k-2))]. \tag{3}$$

Since, by assumption, we violate the Yamanouchi property, it must be the case that (3) > (2), which gives

$$\alpha_k > \lceil n/2 \rceil - (k-2).$$

Additionally, we know that there are fewer than $\lceil n/2 \rceil$ 1's in the first $k-1$ rows, meaning

$$\lceil n/2 \rceil > \sum_{i=1}^{k-1} \alpha_i - (k-2).$$

Combining these inequalities yields

$$\alpha_k > \sum_{i=1}^{k-1} \alpha_i - 2(k-2), \tag{4}$$

but we let j be the last row such that α_j satisfies (4). Therefore, we have a contradiction, meaning we do not violate the Yamanouchi property in the row we stop placing 1's. Below this row, we clearly do not violate the rule, since we already have $\lceil n/2 \rceil$ 1's.

Converse: We now show that if $p < \ell$ or $p > n-(m-1)$, then there does not exist a 1-2 LR-filling with p 1's. Clearly there is no way to fill with more than $n-(m-1)$ 1's because that would require some 2×1 rectangle to contain two 1's, which violates semistandardness. Additionally, regardless of j, if $\ell = \lceil n/2 \rceil$, there is clearly no 1-2 LR-filling with $p < \ell$ 1's. Since $\lceil n/2 \rceil = \alpha_m + m - 2$ in Case 2, the converse of Case 2 has been shown, and we only have Case 1: Subcase 1 and Case 3: Subcase 1 left to consider.

It is clear by construction that the filling in Case 1: Subcase 1 uses the minimum number of 1's, since using fewer 1's necessarily violates either the Yamanouchi property or semistandardness. In Case 3: Subcase 1 (where $\ell = \alpha_j + m - 3$), note that the number of 2's in the first j rows is at most the number of 1's in the first $j-1$ rows. Moreover, the number of 1's in the first $j-1$ rows is at most $\sum_{i=1}^{j-1} \alpha_i - (j-2)$. Thus the number of 1's in the first j rows is at least

$$\sum_{i=1}^{j} \alpha_i - (\sum_{i=1}^{j-1} \alpha_i - j + 2) = \alpha_j + j - 2.$$

Since there are $m-j$ rows below the j-th row, there are $m-j-1$ further occurrences of 2×1 rectangles, so the number of total 1's is at least $\alpha_j + j - 2 + (m-j-1) = \alpha_j + m - 3 = \ell$. □

We now present a corollary of Lemma 3.3, which is another necessary condition for a ribbon to have full equivalence class. Although the necessary condition in this corollary is strictly weaker than the necessary condition from Theorem 2.1, it is much more easily understood.

Corollary 3.7. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a ribbon of size n with a full equivalence class. Then there exists an m-gon with side lengths being the row lengths of α.*

Proof. To prove the contrapositive, we assume that there does not exist an m-gon with side lengths equal to the row lengths of α. Then there exists some $\alpha_j \geq \sum_{i=1}^{j-1} \alpha_i + \sum_{i=j+1}^{m} \alpha_i$.

Consider a permutation α_π of α where $\pi(j) = 1$. Let us show that α_π has no long rows beneath $\alpha_{\pi^{-1}(1)} = \alpha_j$. Assume, for the sake of contradiction, that $\alpha_{\pi^{-1}(k)}$ is long for some $k > 1$. Then we have

$$\alpha_{\pi^{-1}(k)} \geq \sum_{i=1}^{k-1} \alpha_{\pi^{-1}(i)} - 2(k-2) = \alpha_{\pi^{-1}(1)} + (\sum_{i=2}^{k-1} \alpha_{\pi^{-1}(i)} - 2(k-2)) \geq \alpha_{\pi^{-1}(1)} \geq \sum_{i=2}^{m} \alpha_{\pi^{-1}(i)} > \alpha_{\pi^{-1}(k)} \ ,$$

a contradiction; here, we have used our assumptions from Remark 0.17.

Notice that since $m \geq 3$ and $\alpha_j \geq \lceil n/2 \rceil$, we have that $\alpha_j + m - 2 > \alpha_j + m - 3 \geq \lceil n/2 \rceil$. Therefore, we have from the $j = 1$ case of Lemma 3.3 that there does not exist a 1-2 LR-filling of α_π with fewer than $\alpha_j + m - 2$ 1's.

Now, consider a permutation α_σ of α where $\sigma(j) = 2$. By a similar argument as before, there does not exist an $\alpha_{\sigma^{-1}(k)}$ with $k \neq 2$ such that $\alpha_{\sigma^{-1}(k)} \geq \sum_{i=1}^{k-1} \alpha_{\sigma^{-1}(i)} - 2(k-2)$. Therefore, applying the $1 < j < m$ case of Lemma 3.3, we get that there exists a 1-2 LR-filling with $\alpha_j + m - 3$ 1's.

Therefore, $(\alpha_j + m - 3, n - (\alpha_j + m - 3)) \in [\alpha_\sigma]$, but $(\alpha_j + m - 3, n - (\alpha_j + m - 3)) \notin [\alpha_\pi]$, meaning α does not have full support equivalence class. □

Now that we have established Lemma 3.3 (our main tool for the remaining proofs in this section), we establish a set of conditions that will help us classify the ribbons $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ which have full equivalence class but do not satisfy the condition of Corollary 1.5 (the strict triangle inequality). (These ribbons are the subject of Theorem 3.5.)

Remark 3.8. Let $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$ be a ribbon such that $\beta_1 \geq \beta_2 \geq \beta_3 \geq \beta_4$ and such that every size 3 subset of $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ satisfies the strict triangle inequality. Notice that β then satisfies conditions (a), (c), and (d) of Hypothesis 1. However, for the proofs in this subsection, it is often convenient to assume condition (b) of Hypothesis 1. As a result, we use Corollary 1.5 to classify ribbons of the same form as β, and will not include such ribbons in Hypothesis 1.

We now establish two lemmas which are also used to prove the main result of this subsection, Theorem 3.5.

Lemma 3.9. *Let $\alpha = (\alpha_1, a_2, a_3, a_4)$ be a ribbon which satisfies the conditions of Hypothesis 1. Let A be the set of permutations α_π of α, where $\pi(1) = 1$. Then for any $\alpha_\gamma, \alpha_\delta \in A$, $[\alpha_\gamma] = [\alpha_\delta]$.*

Proof. Let $\alpha_\gamma, \alpha_\delta \in A$. Let γ_1, γ_2, γ_3, and γ_4 be the lengths of the first, second, third and fourth rows of α_γ, respectively, and define δ_i analogously. Thus, we have $\gamma_1 = \delta_1 = \alpha_1$ and $\{\gamma_2, \gamma_3, \gamma_4\} = \{\delta_2, \delta_3, \delta_4\} = \{\alpha_2, \alpha_3, \alpha_4\}$. Let $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$, where μ_i indicates the number of i's in the content. In the following cases, we show that for any content μ, there is an LR-filling of α_γ with content μ if and only if there is an LR-filling of α_δ with content μ. More specifically, for each possible type of μ, we give an algorithm for constructing an LR-filling of α_γ with content μ; since this algorithm does not depend on the order of the last three rows of α_γ, this functions to show that $[\alpha_\gamma] = [\alpha_\delta]$.

Case 1: $\mu_3 = 0$. By Lemma 3.2, the second, third, and fourth rows of both α_γ and α_δ are not long. It follows from Lemma 3.3 that there is an LR-filling of α_γ with content μ if and only if there is an LR-filling of α_δ with the same content.

Case 2: $\mu_4 > 0$. Within this case, we examine two subcases.

Subcase 2a: $\mu_2 \geq \alpha_1$.

Row	Filling
1	1^{α_1}
2	$1^{\text{fill}}(2^{\text{fill}})2^{\mu_3}$
3	$1^{\text{rem.}}2^{\text{fill}}3^{\mu_3}$
4	$2^{\text{rem.}}4^{\mu_4}$

We fill the first row entirely with 1's. The rightmost portion of the second row is filled with as many 2's as there are 3's in the whole filling; the leftmost portion of the second row is then filled with 1's, and is finished with 2's if we run out of 1's. (Notice we can't run out of both 1's and 2's in the second row because there are at least α_1 of each and there are only $\alpha_1 + \gamma_2$ boxes in the first two rows.) We next fill the rightmost portion of the third row with all of the 3's. Then we fill the leftmost portion with the remaining 1's, using 2's if we run out of 1's. The last row has all of the 4's, in addition to the remaining 2's. To show that this algorithm gives a valid LR-filling of α_γ, we check that (i) we have a SSYT, (ii) the number of 1's, 2's, 3's, and 4's is consistent with the case we're examining and with the shape of the tableau, and (iii) the number of $x+1$'s never overtakes the number of x's in the reverse reading word, for $x \in \{1, 2, 3\}$.

(i) We clearly have a SSYT since the last number in each row increases every time, and since we never decrease across rows.

(ii) By construction, we have the correct number of 3's and 4's in this filling (μ_3 and μ_4 respectively). Moreover, the 3's and 4's each occupy less than one row since we have at least $2\alpha_1$ boxes with a 1 or a 2, leaving $n - 2\alpha_1 \leq n - \alpha_1 - (\alpha_2 + \alpha_3) = \alpha_4$ boxes containing a 3 or a 4. We now argue that the 1's must expire within the first three rows. Suppose for the sake of contradiction that we have 1's remaining after the third row. Then the second row must contain exactly μ_3 2's, meaning the third row would contain no 2's. However, the fourth row must then have $\mu_2 - \mu_3 > \alpha_1 - \alpha_4 > \alpha_2$ 2's, which clearly cannot fit in the fourth row. It follows by contradiction that the 1's fit in the first three rows. Since the 2's are simply used as row fillers when there are unfilled boxes, the 2's must fit by the assumption that $|\mu| = n$.

(iii) Clearly the 4's never overtake the 3's and the 3's never overtake the 2's. Finally, the 2's cannot overtake the 1's before the last row since there are α_1 1's in the first row, which is more boxes than there are in the second and third rows combined. Since all of the 1's appear before the last row, this implies that the 2's never overtake the 1's.

For the remaining cases and subcases, we will not justify the algorithms in detail, having now demonstrated what needs to be checked.

Subcase 2b: $\mu_2 < \alpha_1$.

Row	Filling
1	1^{α_1}
2	$(1^{\text{fill}})2^{\text{fill}}$
3	$(1^{\text{fill}})2^{\text{fill}}3^{\mu_4}$
4	$1^{\text{rem.}}2^{\text{rem.}}3^{\text{rem.}}4^{\mu_4}$

Case 3: $\mu_3 > 0$**;** $\mu_4 = 0$**.** Within this case, there are three subcases:

Row \ Subcase	3a: $\mu_2 \geq \alpha_1$	3b: $\mu_2 < \alpha_1$; $\mu_3 \leq n - 2\alpha_1$	3c: $\mu_2 < \alpha_1$; $\mu_3 > n - 2\alpha_1 \geq 1$
1	1^{α_1}	1^{α_1}	1^{α_1}
2	$1^{\text{fill}}(2^{\text{fill}})2^{\mu_3}$	$1^{\text{fill}}(2^{\text{fill}})2$	$(1^{\text{fill}})2^{\text{fill}}$
3	$1^{\text{rem.}}2^{\text{fill}}$	$(1^{\text{fill}})2^{\text{fill}}$	$(1^{\text{fill}})2^{\text{fill}}3^{\max(1,\mu_3-\delta_4)}$
4	$2^{\text{rem.}}3^{\mu_3}$	$1^{\text{rem.}}2^{\text{rem.}}3^{\mu_3}$	$1^{\text{rem.}}2^{\text{rem.}}3^{\text{rem.}}$

□

Corollary 3.10. *Let* $\alpha = (\alpha_1, a_2, a_3, a_4)$ *be a ribbon which satisfies the conditions of Hypothesis 1. Let* α_π *be a permutation of* α*, where* $\pi(1) = 4$*. Then* $[\alpha_\pi] = [\alpha]$*.*

Proof. This corollary follows immediately from the proof of Lemma 3.9 and the fact that rotating a skew shape antipodally preserves its support (Remark 0.3). □

The next lemma and corollary demonstrate an analogous result for the cases where $\pi(1) = 2$ or $\pi(1) = 3$.

Lemma 3.11. *Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be a ribbon which satisfies the conditions of Hypothesis 1. Let A be the set permutations α_π of α, where $\pi(1) = 2$. Then for any $\alpha_\gamma, \alpha_\delta \in A$, $[\alpha_\gamma] = [\alpha_\delta]$.*

Proof. We proceed as in the proof of Lemma 3.9 and use the variables as defined in this proof.

Case 1: $\mu_3 = 0$.

Row	Filling
1	1^{γ_1}
2	$1^{\alpha_1-\gamma_1}2^{\gamma_1}$
3	$1^{\text{rem.}}2^{\text{fill}}$
4	$2^{\text{rem.}}$

Case 2: $\mu_4 > 0$.

Row \ Subcase	2a: $\mu_2 \geq \alpha_1$	2b: $\mu_2 < \alpha_1$
1	1^{γ_1}	1^{γ_1}
2	$1^{\alpha_1-\gamma_1}2^{\gamma_1}$	$1^{\text{fill}}2^{\min(\mu_2,\gamma_1)}$
3	$1^{\text{rem.}}2^{\text{fill}}3^{\mu_4}$	$(1^{\text{fill}})2^{\text{fill}}3^{\mu_4}$
4	$2^{\text{rem.}}3^{\text{rem.}}4^{\mu_4}$	$1^{\text{rem.}}2^{\text{rem.}}3^{\text{rem.}}4^{\mu_4}$

Notice that if $\gamma_1 < \mu_4$, then there is in fact no permutation of the first, third and fourth rows for which there is an LR-filling of content μ.

Case 3: $\mu_3 > 0; \mu_4 = 0$.

Row \ Subcase	3a: $\mu_2 \geq \alpha_1$	3b: $\mu_2 < \alpha_1$; $\mu_3 \leq \alpha_4$	3c: $\mu_2 < \alpha_1$; $\mu_3 > \alpha_4$
1	1^{γ_1}	1^{γ_1}	1^{γ_1}
2	$1^{\alpha_1-\gamma_1}2^{\gamma_1}$	$1^{\text{fill}}2^{\min(\mu_2-1,\gamma_1)}$	$1^{\text{fill}}2^{\min(\mu_2,\gamma_1)}$
3	$1^{\text{rem.}}2^{\text{fill}}$	$1^{\text{fill}}(2^{\text{fill}})2$	$(1^{\text{fill}})2^{\text{fill}}3^{\max(1,\mu_3-\gamma_4)}$
4	$2^{\text{rem.}}3^{\mu_3}$	$1^{\text{rem.}}2^{\text{rem.}}3^{\mu_3}$	$1^{\text{rem.}}2^{\text{rem.}}3^{\text{rem.}}$

□

Corollary 3.12. *Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be a ribbon which satisfies the conditions of Hypothesis 1. Let α_π be a permutation of α, where $\pi(1) = 3$. Then $[\alpha_\pi] = [\alpha]$.*

Proof. This corollary follows immediately from the proof of Lemma 3.11 and the fact that rotating a skew shape antipodally preserves its support (Remark 0.3). □

We now apply the previous two lemmas to finally prove Theorem 3.5, which, as we saw in the proof of Theorem 3.6, serves in conjunction with Corollary 1.5 as a necessary and sufficient condition for ribbons with four rows to have full equivalence class.

Theorem 3.5. *Suppose we have a ribbon $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ satisfying the conditions of Hypothesis 1. Then the following hold:*

1. *α has maximal support equivalence class if and only if $\alpha_1 < \lceil n/2 \rceil - 1$.*
2. *For n even, if $\alpha_1 = n/2 - 1$, then $(n/2, n/2) \notin [(\alpha_1, \alpha_2, \alpha_3, \alpha_4)]$, but $(n/2, n/2) \in [(\alpha_2, \alpha_1, \alpha_3, \alpha_4)]$.*
3. *For n odd, if $\alpha_1 = \lceil n/2 \rceil - 1$, then $(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor, 1)$ and $(\lceil n/2 \rceil, \lfloor n/2 \rfloor)$ are in the support of $(\alpha_2, \alpha_1, \alpha_3, \alpha_4)$, but not in the support of $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.*

Proof. We will prove the three statements individually.

Proof of (1): Note that $\alpha_1 < \alpha_2 + \alpha_3 + \alpha_4$ implies that $\alpha_1 \leq \lceil n/2 \rceil - 1$. Thus, (2) and (3) prove the converse of this statement. For the forward direction, let $\alpha_1 < \lceil n/2 \rceil - 1$. We will first use Lemma 3.3 to argue that α and $\alpha_{(1\ 2)}$ have the same 1-2 LR-fillings.

Notice that by Lemma 3.2, there are no long rows occurring below the row of length α_1 in either α or $\alpha_{(1\ 2)}$. Moreover, it is clear that the row of length α_1 is a long row in both α and $\alpha_{(1\ 2)}$. Note that since $\alpha_1 < \lceil n/2 \rceil - 1$, we have

$$\alpha_1 + m - 2 = \alpha_1 + 2 < \lceil n/2 \rceil + 1 \implies \alpha_1 + m - 2 \leq \lceil n/2 \rceil$$

and that

$$\alpha_1 + m - 3 = \alpha_1 + 1 < \lceil n/2 \rceil.$$

Therefore, by Lemma 3.3, both α and $\alpha_{(1\ 2)}$ have 1-2 LR-fillings with p 1's for any p such that $\lceil n/2 \rceil \leq p \leq n - (m-1)$.

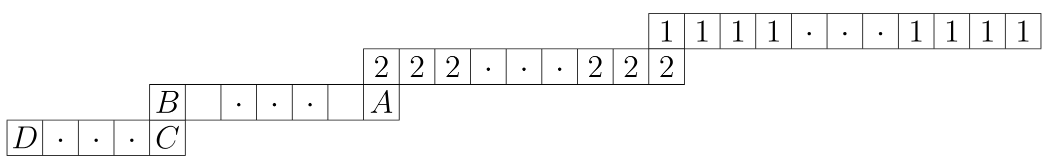
We now show that any filling of α the form $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$, where $\mu_3 > 0$ and $\mu_4 \geq 0$, is also a filling of $\alpha_{(1\ 2)}$. It is clear that $\mu_1 \geq \alpha_1$. Additionally, for 2's and 3's to satisfy the Yamanouchi property and semistandardness, the first box in which a 3 can be placed is the rightmost box of the third row. It follows that the first two rows of α must be filled with exclusively 1's and 2's. Let $\nu = (\nu_1, \nu_2)$ be the filling of the first two rows of α. We can fill the first two rows of $\alpha_{(1\ 2)}$ with ν_1 1's and ν_2 2's as follows:

Fill the first row of $\alpha_{(1\ 2)}$ with α_2 1's. Then, fill the second row of $\alpha_{(1\ 2)}$ (starting from the right) with the second row of α and $\alpha_1 - \alpha_2$ 1's to its left. Since the second row of α is of length α_2, we have that $\nu_2 \leq \alpha_2$, meaning this construction will satisfy the Yamanouchi property. As previously established, $\alpha_1 - \alpha_2 > 0$, meaning the leftmost box in the second row is a 1. Since the rightmost box of the third row of α is at least 2, we can fill the third and fourth rows of $\alpha_{(1\ 2)}$ with exactly the content of the third and fourth rows of α. Therefore, any LR-filling μ of α is also an LR-filling of $\alpha_{(1\ 2)}$.

We now show that any LR-filling of $\alpha_{(1\ 2)}$ of the form $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$, where $\mu_3 > 0$ and $\mu_4 \geq 0$ is also an LR-filling of α. Note that since the first row of $\alpha_{(1\ 2)}$ must consist of α_2 1's, the second row of $\alpha_{(1\ 2)}$ contains at most α_2 2's and at least $\alpha_1 - \alpha_2$ 1's. Thus, using ν as previously defined, $\nu_1 \geq \alpha_1$, so we can begin by filling the first row of α with α_1 1's (taken from the first row of $\alpha_{(1\ 2)}$ and the left side of the second row of $\alpha_{(1\ 2)}$). Place the remaining α_2 1's or 2's from the second row of $\alpha_{(1\ 2)}$ in the second row of α.

If the leftmost box of the second row of α is a 1, then any filling of the third and fourth rows of $\alpha_{(1\ 2)}$ can be directly transfered to α.

Consider the case where all of the second row of α is now 2's. Transfer the filling of the third and fourth rows of $\alpha_{(1\ 2)}$ directly to the third and fourth rows of α, and we will describe how to adjust the filling to obtain a filling which is both semistandard and Yamanouchi. Throughout the remainder of the proof we will reference the following digram of α.



If Box A contains a 3, we are done. So, assume that Box C has a 3, and Box A has a 2. Now, notice that

$$\begin{aligned} n - (\alpha_1 + 1) &= n - \alpha_1 - 1 > n - (\lceil n/2 \rceil - 1) - 1 \\ &= n - \lceil n/2 \rceil = \lfloor n/2 \rfloor > \alpha_1. \end{aligned}$$

Therefore, $\mu \neq (\alpha_1, n-(\alpha_1+1), 1)$, as this would imply that $\mu_2 > \mu_1$, a contradiction. It follows that there is either at least one 1 or at least two 3's remaining in μ after filling the first two rows. We will consider these cases separately.

If there is a 1 remaining in μ, then by semistandardness, there is a 1 in either Box B or Box D (or both). We can then apply the following process:

1. If there is a 1 in Box B, swap the leftmost 3 in the fourth row with the 2 in Box A. This step is valid since we know there to be at least one 3 in the fourth row (namely, in Box C); moreover, since $\alpha_2 \geq 2$, there are at least two 2's in the second row, so moving this 3 preserves the Yamanouchi property.

2. If there is a 1 in Box D but not in Box B, swap the rightmost 1 in row 4 with the 2 in Box B; this clearly does not violate the Yamanouchi property. Then apply step 1.

On the other hand, if μ has at least two remaining 3's, we know that at least the two rightmost boxes of row 4 contain 3's. In this case, swap the leftmost 3 in row 4 with the 2 in Box A. Again, since $\alpha_2 \geq 2$, the tableau remains Yamanouchi.

It now follows from Lemmas 3.9 and 3.11 that α has full support equivalence class if $\alpha_1 < \lceil n/2 \rceil - 1$.

Proof of (2): Let $\alpha_1 = n/2 - 1$. Then $\alpha_1 + m - 2 = \alpha_1 + 2 > n/2$ and $\alpha_1 + m - 3 = \alpha_1 + 1 = n/2$. As argued in the proof of (1), there are no long rows occurring after the row of length α_1 in either α or $\alpha_{(1\,2)}$. It follows from the $j = 1$ case of Lemma 3.3 that $(n/2, n/2) \notin [\alpha]$. Additionally, it follows from the $1 < \alpha_2 < m$ case of Lemma 3.3 that $(n/2, n/2) \in [\alpha_{(1\,2)}]$.

Proof of (3): Let us first show that $(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor, 1) \in [\alpha_{(1\,2)}]$. Since $\alpha_1 = \lceil n/2 \rceil - 1 = \lfloor n/2 \rfloor$, we show that there is an LR-filling of $\alpha_{(1\,2)}$ with α_1 1's, α_1 2's, and one 3. Begin by filling the first row with α_2 1's. Proceed to fill the second row with α_2 2's and $\alpha_1 - \alpha_2$ 1's (with the 1's to the left of the 2's). Notice that conditions (a) and (b) of Hypothesis 1 imply that $\alpha_1 > \alpha_2$, meaning that there is at least one 1 in the second row of α. Thus, we can fill the third row with α_3 2's. The filling thus far is Yamanouchi since $\alpha_1 \geq \alpha_2 + \alpha_3$. Finally, fill the fourth row of α with one 3 and $(\alpha_4 - 1)$ 2's. Thus, we have an LR-filling with $\alpha_1 = \lfloor n/2 \rfloor$ 1's, one 3, and $n - \lfloor n/2 \rfloor - 1 = \lfloor n/2 \rfloor$ 2's, as desired.

We now show that $(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor, 1) \notin [\alpha]$ by showing that an LR-filling with only one 3 requires at least $\lfloor n/2 \rfloor + 1$ 1's. Clearly, we must fill the first row with α_1 1's, and the rightmost box of the second row must be a 2. There are two remaining 2×1 rectangles, which we must fill in one of the following ways, where $x \in \{1, 2\}$:

(a) $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$ (b) $\begin{array}{|c|}\hline x \\ \hline 3 \\ \hline \end{array}$

Moreover, we can only use the filling depicted in (b) once, since our content has only one 3. Therefore, we must use at least $\alpha_1 + 1 = \lfloor n/2 \rfloor + 1$ 1's. It is therefore clear that $(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor, 1) \notin [\alpha]$.

We will now show that $(\lceil n/2 \rceil, \lfloor n/2 \rfloor) \in [\alpha_{(1\ 2)}]$ but that $(\lceil n/2 \rceil, \lfloor n/2 \rfloor) \notin [\alpha]$. As argued in the proof of (1), there are no long rows below the row of length α_1, and the row of length α_1 is long. Since $\alpha_1 + m - 3 = \alpha_1 + 1 = \lceil n/2 \rceil$, we have by Lemma 3.3 that there exists a 1-2 LR-filling of $\alpha_{(1\ 2)}$ with $\lceil n/2 \rceil$ 1's, meaning $(\lceil n/2 \rceil, \lfloor n/2 \rfloor) \in [\alpha_{(1\ 2)}]$. Now, since $\alpha_1 + m - 2 = \alpha_1 + 2 = \lceil n/2 \rceil + 1 > \lceil n/2 \rceil$, we have by Lemma 3.3 that there does not exist a 1-2 LR-filling of α with fewer than $\lceil n/2 \rceil + 1$ 1's. It follows that $(\lceil n/2 \rceil, \lfloor n/2 \rfloor) \notin [\alpha]$. □

We are now ready to prove that Conjecture 2.4 holds in the case of $m = 4$, which amounts to proving Theorem 3.6, which is restated here.

4 Conclusion and Future Directions

4.1 Summary

In this paper, we examined which connected ribbons have full equivalence class. This is easy to determine if the ribbon has a part of length 1 or fewer than 3 rows, so we restricted our focus for most of the paper to ribbons with at least 3 rows, all of length at least 2.

We proved a sufficient condition and a separate necessary condition, and we conjecture that our necessary condition is sufficient in general. We prove that this condition is indeed both necessary and sufficient for $m = 3$ and $m = 4$. If this can be proven to hold in general, this condition would give an easy way to test whether an arbitrary connected ribbon has full equivalence class.

4.2 Open Questions

After our substantial progress towards classifying equality of Schur support among ribbons, several open questions remain. Firstly, to complete the near classification of full Schur support equivalence classes among ribbons, we'd like to answer the following question:

Question 1. Is the condition that was proven in Theorem 2.1 to be necessary for a connected ribbon to have full equivalence class in fact also a sufficient condition (as conjectured in Conjecture 2.4)?

These results could then be extended so as to completely classify Schur support equality among ribbons:

Question 2. For a connected ribbon without full equivalence class, which permutations of the row lengths of the ribbon preserve Schur support?

Beyond equality of Schur supports, we can also ask when we have Schur support containment:

Question 3. What does the poset (ordered by containment) of Schur support look like for connected ribbons?

Finally, we would like to be able to extend these results beyond connected ribbons:

Question 4. Besides connected ribbons, when do two skew shapes have equal Schur support?

5 Appendix

5.1 Miscellaneous Results about Support Equality

The following result pertains to the support of skew shapes that are composed with ribbons, where this composition is as defined in [9].

Proposition 5.1. *For any skew shapes D, D' with $[D] = [D']$ and any ribbon α, $[D \circ \alpha] = [D' \circ \alpha]$.*

Proof. It is known [9, Prop. 7.5] that the map $- \circ s_D$ is an algebra map and that $s_{D\circ\alpha} = s_D \circ s_\alpha$. It follows that if $X = [D] = [D']$, then

$$[D \circ \alpha] = \left[\left(\sum_{\lambda\in X} c_{D,\lambda} s_\lambda\right) \circ s_\alpha\right] = \left[\sum_{\lambda\in X} c_{D,\lambda} s_{\lambda\circ\alpha}\right] = \left[\sum_{\lambda\in X} c_{D',\lambda} s_{\lambda\circ\alpha}\right]$$
$$= \left[\left(\sum_{\lambda\in X} c_{D',\lambda} s_\lambda\right) \circ s_\alpha\right] = [D' \circ \alpha]$$

□

In addition, we have the following result regarding the supports of skew shapes that have been multiplied by a scalar. Here, scalar multiplication of a skew shape D by m indicates that all row lengths of D are increased by a factor of m.

Proposition 5.2. *For any skew shapes D, D', if $[mD] = [mD']$ for some $m > 1$, then also $[D] = [D']$.*

Proof. Say $D = \lambda/\mu$ and $D' = \nu/\gamma$. Then $[mD] = [mD']$ implies that $c^{m\lambda}_{m\mu,\delta} > 0$ if and only if $c^{m\nu}_{m\gamma,\delta} > 0$. In particular, $c^{m\lambda}_{m\mu,m\rho} > 0$ if and only if $c^{m\nu}_{m\gamma,m\rho} > 0$, so by saturation ([4]), we conclude that $c^{\lambda}_{\mu,\rho} > 0$ if and only if $c^{\nu}_{\gamma,\rho} > 0$. Hence, $[D] = [D']$. □

5.2 Equality of Monomial Support for Skew Shapes

As opposed to the question of the equality of Schur supports, the question of equality among monomial supports of skew shapes is quite easily answered. For any skew shape D, recall that $\mathrm{mSupp}(D)$ denotes the monomial support of D, and let $\mathrm{cols}(D)$ be the column partition (i.e. the partition associated to the sequence whose i^{th} entry is the number of boxes in the i^{th} column of D).

Proposition 5.3. *For any partitions $\mu \subset \lambda$ and $\delta \subset \nu$, $\mathrm{mSupp}(\lambda/\mu) = \mathrm{mSupp}(\nu/\delta)$ if and only if $\mathrm{cols}(\lambda/\mu) = \mathrm{cols}(\nu/\delta)$.*

Proof. For any $\mu \subset \lambda$, we can expand

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu,\nu} s_{\nu} = \sum_{\nu} c^{\lambda}_{\mu,\nu} \sum_{\delta} K_{\nu,\delta} m_{\delta} = \sum_{\delta} \left(\sum_{\nu} c^{\lambda}_{\mu,\nu} K_{\nu,\delta} \right) m_{\delta}$$

where $c^{\lambda}_{\mu,\nu}$ is a Littlewood-Richardson coefficient and $K_{\nu,\delta}$ is a Kostka number. In particular, both are nonnegative and $K_{\nu,\delta} > 0$ if and only if $\nu \geq \delta$ in dominance order. Also, $c^{\lambda}_{\mu,\nu} > 0$ if and only if there exists an LR-filling of shape λ/μ and content ν. However, among ν such that there exists an LR-filling of λ/μ of content ν, there is a maximal one, namely $\nu = \mathrm{cols}(\lambda/\mu)^t$, obtained by placing i as the i^{th} entry of each column. Hence, there exists a ν such that the product $K_{\nu,\delta} c^{\lambda}_{\mu,\nu}$ is nonzero if and only if $\delta \leq \mathrm{cols}(\lambda/\mu)^t$ in dominance order. Hence,

$$\mathrm{mSupp}(\lambda/\mu) = \left\{ \delta \colon \delta \leq \mathrm{cols}(\lambda/\mu)^t \right\}.$$

In particular, $\mathrm{mSupp}(\lambda/\mu) = \mathrm{mSupp}(\nu/\delta)$ if and only if $\mathrm{cols}(\lambda/\mu) = \mathrm{cols}(\nu/\delta)$. □

5.3 Sample Code Used to Test Conjecture 2.4

Recall Conjecture 2.4, repeated here for convenience:

Conjecture 2.4. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a ribbon, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$. Then, α has full equivalence class if and only if*

$$N_j < \sum_{i=j+1}^{m} \alpha_i - (m - j - 2) \tag{5}$$

for all $1 \leq j \leq m-2$, where

$$N_j = \max\{k | \sum_{i \leq j:\, \alpha_i < k} (k - \alpha_i) \leq m - j - 2\}. \tag{6}$$

The code below will reference this conjecture as well as Equations 5 and 6 labeled above. This code tests Conjecture 2.4 only for connected ribbons with 6 rows ($m = 6$), but the functions used to test the conjecture for other values of m are very similar. This code was written and run using Sage ([2]). The main function, `conj_test_six`, requires importing `itertools` and requires the following helper function, which takes as input a Python list representing a ribbon α of the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_6)$:

```
def rib_to_support(comp):
    """Compute the Schur support of a ribbon.

    This function takes as input a composition representing
    the row lengths of a ribbon, and first converts the
    composition to the form of a skew shape (i.e. lambda/mu
    where lambda and mu are partitions representing straight
    shapes. The function then utlizes the .support() function
    from Sage's SymmetricFunctions class to output the Schur
    support of the ribbon.

    Input:
        comp (list): a list (composition) indicating the
            row lengths of a ribbon

    Returns the Schur support of the given ribbon.
    """
    l=list()
    m=list()
    n=len(comp)
```

```
    # converts the ribbon from a composition to the form lambda/mu
    l.append(comp[0])
    m.append(0)
    for i in range(1, n):
        l.append(comp[i]+l[i-1]-1)
        m.append(comp[i-1]+m[i-1]-1)
    l.reverse()
    m.reverse()
    s=SymmetricFunctions(QQ).s()
    s=s(Partition(l)/Partition(m))

    # uses a Sage function to compute the support of the ribbon
    return s.support()
```

The code below is our main function, which takes as input an integer n and tests Conjecture 2.4 for all ribbons with six rows and n boxes.

```
def conj_test_six(n):
    """Test Conjecture 2.4.

    This function tests Conjecture 2.4 (i.e. the conjectured
    necessary and sufficient condition for a connected ribbon
    to have full equivalence class). The function iterates
    through partitions of length six of the input integer n,
    and computes the equivalence classes of the partitions as
    interpreted as ribbons (using rib_to_support and itertools).
    For each partition/ribbon, the function computes and tests
    each N_j (defined in Equation 6) to determine whether the
    ribbon should have full equivalence class according to
    Conj. 2.4. If the prediction from Conj. 2.4 is ever wrong,
    the function returns a counterexample. Otherwise, the
    function indicates that the conjecture held for this n.

    Input:
        n (integer): number of boxes in the ribbons

    Returns either a ribbon that is a counterexample to Conj. 2.4
        or indicates that the conjecture holds for this n.
```

```
"""
# initialize an empty set for ribbons Conj. 2.4 would predict to not
# have full equivalence class
not_full = set()

for alpha in Partitions(n):
    # The following line assumes distinct parts for convenience.
    if len(alpha) == 6 and all(alpha[i] != alpha[l] for (i,l) in
        (0,1),(1,2),(2,3),(3,4),(4,5)):

        # compute the equivalence class of alpha
        equiv_class = set()
        s = rib_to_support(list(alpha))
        for perm in list(itertools.permutations(list(alpha))):
            if set(rib_to_support(perm)) == set(s):
                equiv_class.add(perm)

        # computes N_j from Equation 6
        for j in range(4):
            N_j = alpha[j]
            sum = 0
            while sum <= 4-(j+1):
                sum = 0
                for i in range(j+1):
                    if alpha[i] < N_j:
                        sum += (N_j-alpha[i])
                N_j += 1
            N_j -= 2

            # computes the sum from the RHS of Equation 5
            ineq_sum = 0
            for i in range(j+1,6):
                ineq_sum += alpha[i]

            # if the inequality from Equation 5 is not satisfied,
            # but alpha has full equivalence class, returns
            # alpha as a counterexample
            if N_j >= ineq_sum - (4-(j+1)):
```

```
                not_full.add(tuple(alpha))
                if len(equiv_class) == 720:
                    return 'counterexample:', alpha

        # if alpha hasn't been marked as ``not full"
        # and alpha does not have full equivalence class,
        # returns alpha as a counterexample
        if tuple(alpha) not in not_full:
            if len(equiv_class) != 720:
                return 'counterexample:', alpha

    return 'Conjecture 2.4 holds'
```

As an example, `print(conj_test_six(35))` outputs `'Conjecture 2.4 holds'`.

Acknowledgements

This research was performed as a part of the 2017 University of Minnesota, Twin Cities Combinatorics REU, and was supported by NSF RTG grant DMS-1148634 and by NSF grant DMS-1351590. We would like to thank Victor Reiner, Pavlo Pylyavskyy, and Galen Dorpalen-Barry for their advice, mentorship, and support and we are grateful to Sunita Chepuri for her help in finding a reference on R-matrices. Finally, we would like to thank Thomas Hameister for allowing us to include his proofs of Propositions 5.2 and 5.3.

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