Generating Functions for $f$-vectors of Simple Weight Polytopes

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1 Introduction to Polytopes

2 Coxeter Group and Weight Polytopes

3 $f$-polynomials of Simple Weight Polytopes
Definition (f-vector and f-polynomial)

Define the *f*-vector of a *r*-dim Polytope *P* as 
\[ f(P) := (f_0, \ldots, f_r), \]
where \( f_i \) is the number of \( i \)-dimensional faces of *P*.

Define its *f*-polynomial as 
\[ f_P(t) = \sum_{i=0}^{r} f_i t^i. \]

Example:

A cube has 8 vertices, 12 edges and 6 faces.

\[ f(P) = (8, 12, 6, 1) \]

\[ f_P(t) = 8 + 12t + 6t^2 + t^3 \]
**Definition (h-vector and h-polynomial)**

Define the *h-polynomial* of a $r$-dim Polytope $P$ as

$$h_P(t) = f_P(t - 1) = \sum_{i=0}^{r} f_i(t - 1)^i.$$ 

Assume $h_P(t) = \sum_{i=0}^{r} h_i t^i$, then define its *h-vector* as

$$h(P) := (h_0, h_1, \ldots, h_r).$$

**Example:**

A cube has $f_P(t) = 8 + 12t + 6t^2 + t^3$.

Replace $t$ with $t - 1$.

$$h_P(t) = f_P(t - 1) = 1 + 3t + 3t^2 + t^3$$

$$h(P) = (1, 3, 3, 1)$$
**Definition (h-vector and h-polynomial)**

Define the *h-polynomial* of a $r$-dim Polytope $P$ as

$$h_P(t) = f_P(t - 1) = \sum_{i=0}^{r} f_i(t - 1)^i.$$  

Assume $h_P(t) = \sum_{i=0}^{r} h_i t^i$, then define its *h-vector* as

$$h(P) := (h_0, h_1, \ldots, h_r).$$

**Example:**

A cube has $f_P(t) = 8 + 12t + 6t^2 + t^3$. Replace $t$ with $t - 1$.

$$h_P(t) = f_P(t - 1) = 1 + 3t + 3t^2 + t^3$$

$$h(P) = (1, 3, 3, 1)$$

Is this always symmetric?
Definition (Simple Polytope)

A $r$-dimensional polytope is called a *simple polytope* if and only if each vertex has exactly $r$ incident edges.

For example, a cube is a simple polytope.

Theorem (Dehn-Sommerville equation)

*For any simple polytope $P$, its $h$-vector is symmetric.*
Definition (Face Poset)

The face poset of polytope $P$ is the poset \{faces of $P$\} ordered by inclusion of faces.

Example:

*Note: A Face Poset is graded.*
Methods to describe a polytope:

- $f$-polynomial/$h$-polynomial;
- face poset.
Coxeter Group and Weight Polytopes
Definition (Finite Reflection Group)

A *finite reflection group* is a finite subgroup $W \subset \text{GL}_n(\mathbb{R})$ generated by reflections, i.e. elements $w$ such that $w^2 = 1$ and they fix a hyperplane $H$ and negate the line perpendicular to $H$.

**Example:** One example of a finite reflection group is the Dihedral Group $I_n = \{s, t \mid s^2 = t^2 = e, (st)^n = e\}$. 
Definition (Coxeter Group)

A Coxeter Group is a group $W$ of the form

$$W \cong \langle s_1, \ldots, s_n \mid s_i^2 = e, (s_is_j)^{m_{ij}} = e \rangle$$

for some $m_{ij} \in \{2, 3, 4, \ldots \} \cup \{\infty\}$.

If $W$ is finite, then $W$ is called a Finite Coxeter Group.

$S = \{s_1, s_2, \ldots, s_n\}$ is called the Generating Set of $W$. 
Here is a BIG theorem of Coxeter:

**Theorem (Coxeter)**

*Finite Coxeter groups = Finite reflection groups.*
Definition (Coxeter Diagram)

Given a Coxeter presentation \((W, S)\), we can encapsulate it in the \textit{Coxeter Diagram}, denoted \(\Gamma(W)\), a graph with \(V = S\) and if \(m_{ij} = 3\), \(s_i\) and \(s_j\) are connected with no label and if \(m_{ij} > 3\), \(s_i\) and \(s_j\) are connected with label \(m_{ij}\).

Example: The dihedral group \(I_n\) has Coxeter diagram

\[
\begin{array}{c}
\text{\(n\)} \\
\bullet - \bullet
\end{array}
\]
Amazingly, finite Coxeter groups are classified! They come in four infinite families, $A_n$, $B_n$, $D_n$, and $I_n$, as well as a finite collection of exceptional cases. The Coxeter diagrams look as follows:

We will focus our energies on types $A_n$, $B_n$, $D_n$. 

![Coxeter Diagrams](image)
Definition (Weight Polytope)

Given a finite Coxeter group $W$, $\lambda \in \mathbb{R}^n$, we define the Weight Polytope $P_{\lambda}$ to be the convex hull of \{$w \cdot \lambda$ | $w \in W$\}.
Definition (Stabilizer)

Let $J(\lambda) = \{ s \in S \mid s(\lambda) = \lambda \}$ be the stabilizer of $\lambda$.

Theorem (Maxwell)

The $f$-vector and face lattice of a weight polytope $P_\lambda$ is only dependent on $W$, $S$ and $J(\lambda)$. 
Weight Polytopes

Weight Polytope Example 1

**Coxeter Group**

\[ W = A_n = \text{symmetric group } S_{n+1} \]

**Vector \( \lambda \)**

\[ \lambda = (0, \ldots, 0, 1) \]

- \( n \) zeros
Weight Polytope Example 1

Coxeter Group

\[ W = A_n = \text{symmetric group } S_{n+1} \]

Vector \( \lambda \)

\[ \lambda = (0, \ldots, 0, 1) \]

\( n \) zeros

\[ J(\lambda) \]

1 2 3 \( \cdots \) \( n-1 \) \( n \)

Polytope

Name: Simplex

Vertices: Set of vectors with \( n \) zeros and 1 one
Weight Polytope Example 2

Coxeter Group

\[ W = B_n = \text{signed permutation group} \]

Vector \( \lambda \)

\[ \lambda = (1, 1, \ldots, 1) \]

\( n \) ones
Weight Polytope Example 2

Coxeter Group

\[ W = B_n = \text{signed permutation group} \]

\[
\begin{array}{cccccc}
4 & & & & & \\
(-1) & (12) & (23) & (34) & (n - 1, n)
\end{array}
\]

Vector \( \lambda \)

\[ \lambda = (1, 1, \ldots, 1) \]

\[ J(\lambda) \]

\[ 1 \quad 2 \quad 3 \quad 4 \quad \ldots \quad n \]

Polytope

Name: HyperCube

Vertices: Set of vectors with 1 and \(-1\)
Section 3

$f$-polynomials of Simple Weight Polytopes
Theorem (Renner)

A type $A_n$ or $B_n$ weight polytope is simple iff its Coxeter diagram has one of the following structures.

- \( \geq 2 \) points
- \( \leq n \) points
- \( \leq n - 3 \) points
Renner’s Classification of Simple Polytopes

Theorem (Renner)

A type $A_n$ or $B_n$ weight polytope is simple iff its Coxeter diagram has one of the following structures.

- $\geq 2$ points
- $4 \leq n$ points
- $4 \leq n - 3$ points

What are their $f$-polynomials?
Case 1

Theorem (Golubitsky)

Denote $F_{n,k}(t)$ as the $f$-polynomial for the $f$ polytope of

\[ k \text{ points} \]

\[ \begin{array}{ccccccc}
\includegraphics[width=0.5\textwidth]{polytope.png} \\
\text{\textit{n points}}
\end{array} \]

Then,

\[
\sum_{n \geq k \geq 0} F_{n,k}(t) \cdot \frac{x^{n+1}y^k}{(n+1)!} = \frac{e^{xy}}{y-1} \cdot \left( y + \frac{e^{txy} - t - 1}{t + 1 - e^{tx}} \right) - 1.
\]
Case 2

Theorem

Denote $F_{n,a,b}(t)$ as the $f$-polynomial for the $f$ polytope of

\[ a \text{ points} \quad \circ \quad \cdots \quad \circ \quad \cdots \quad \circ \quad \cdots \quad \circ \quad b \text{ points} \]

\[ \begin{array}{c}
\underbrace{\circ \cdots \circ} \quad n \text{ points}
\end{array} \]

Then, 

\[
\sum_{a,b \geq 0} \sum_{n > a+b} F_{n,a,b}(t) \cdot \frac{x^{n+1} y^a z^b}{(n+1)!} = \frac{1}{y^2 - y} \left( x + \frac{(xy - e^{xy} + 1)(xz - e^{xz})}{y} \right)
\]

\[
= \left( tz + (t + 1)e^{xz} - t - e^{(t+1)xz} \right) \left( \frac{ty + (t+1)e^{(tx)} - t - e^{(t+1)tx}}{(t-e(tx)+1)y} - e^{(tx)} \right)
\]

\[
+ \frac{e^{(xy+zx)}}{ty} + \frac{ze^{(tx)} - ye^{(txz)}e^{(xy+zx)}}{t(y-z)y} \right).
\]
Case 3

**Theorem**

Denote $F_{n,k}(t)$ as the $f$-polynomial for the $f$ polytope of $n$ points and $k$ points.

Then,

$$
\sum_{n>k \geq 0} F_{n,k}(t) \cdot \frac{x^n y^k}{n!} =
$$

$$
\frac{1}{y-1} \left( e^{(t+2)xy} + \frac{e^{tx} \cdot (e^{2(t+1)xy} - (t+1) e^{2xy} + t - ty)}{(t + 1 - e^{2tx})y} \right).
$$
Case 4

Theorem

Denote $F_{n,k}(t)$ as the $f$-polynomial for the $f$ polytope of

\[
\begin{array}{c}
\begin{array}{c}
\bullet \quad \bullet \quad \cdots \quad \bullet \\
4 \quad \cdots \\
n \text{points} \quad k \text{ points}
\end{array}
\end{array}
\]

Then,

\[
\sum_{n-2 > k \geq 0} F_{n,k}(t) \frac{x^{n+1} y^k}{(n+1)!} = \frac{1}{y^2 - y} \left( xy + \left( t + 1 \right) e^{(2 xy)} \right) - e^{(2 tx)}.
\]
Ingredients of the Proof

**Definition** (*J*-minimal subset)

For a Coxeter diagram $\Gamma = (W, S)$ and subset $J \subseteq S$, a

*J*-minimal subset is a subset $X \subseteq S$ such that no connected component of $X$ on the Coxeter diagram lies entirely in $J$.

**Example:**

```
1 2 3
```

$J$

All six *J*-minimal subsets

Not *J*-minimal
Ingredients of the Proof

Theorem (Renner, Maxwell)

Consider the action of $W$ on $\{\text{faces of } P_\lambda\}$, then there is a bijection

$$f : \{\text{J(}\lambda\text{-minimal sets)}\} \rightarrow \{\text{orbits of the action}\}.$$  

If $X$ is J(\lambda)-minimal, then all faces in $f(X)$ are called $X$-type face. All $X$-type face has dimension $|X|$, and the number of $X$-type faces is

$$\frac{|W|}{|W_{X^*}|},$$

where $W_{X^*} \subseteq W$ is the subgroup generated by

$$\{s \in S | s \in X \text{ or } s \text{ and } X \text{ are not connected}\}.$$
Example of Renner/Maxwell

| $X$          | Face         | $W_{X^*}$ | $|W|/|W_{X^*}|$ |
|--------------|--------------|-----------|------------------|
| $\emptyset$ | Vertices     | $\{3\}$  | $48/2 = 24$      |
| $4$          | Long Edges   | $\{1, 3\}$ | $48/4 = 12$    |
| $4$          | Triangle Edges | $\{2\}$   | $48/2 = 24$    |
| $4$          | Octagons     | $\{1, 2\}$ | $48/8 = 6$     |
| $4$          | Triangles    | $\{2, 3\}$ | $48/6 = 8$     |
| $4$          | Truncated Cube | $\{1, 2, 3\}$ | $48/48 = 1$   |

$J$-polynomial $f$-vectors and $cd$-index of Weight Polytopes

Gao, McDonald
## Summary: What have we done?

<table>
<thead>
<tr>
<th></th>
<th>$f$-polynomial</th>
<th>Face Poset</th>
</tr>
</thead>
<tbody>
<tr>
<td>General Simple Weight Polytopes</td>
<td>✓</td>
<td>Maxwell (we rewrote ✓)</td>
</tr>
<tr>
<td>Weyl Group Weight Polytopes</td>
<td>✓ (some done by Golubitsky)</td>
<td>Renner</td>
</tr>
<tr>
<td>Simplex</td>
<td>Known</td>
<td>Known</td>
</tr>
</tbody>
</table>
The End!

Thank You!