UMN Twin Cities Problem 2 Draft

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July 25 2018

1 Abstract

The **Abelian Sandpile Model** and its recurrent configurations, known as the **Sandpile group**, are abundant in modern mathematics and have combinatoric, algebraic, and geometric descriptions. Past work has focused on the sandpile group of the n-dimensional hypercube. In this project, we perform a more general analysis on the Cayley graph of the group $\mathbb{F}_2^r$ and any of its generating sets. While the p-Sylow component of the sandpile group has been classified for $p \neq 2$, significantly less is known about the 2-Sylow component. In this paper, we use representation theory and ring theory to prove a sharp upper bound for the largest 2-Sylow subgroup in the sandpile group of an arbitrary Cayley graph. We also partially classify the number of 2-Sylow subgroups in the sandpile group and make further reductions into determining its structure. Using these reductions, we provide a full classification of the sandpile group for the $r = 2$ case and other enlightening results for small r cases.

2 Introduction and Notation

Let $G = (V, E)$ be a connected graph on n vertices with no self-loops and an ordering on the vertices. We define its *Laplacian* $L(G)$ to be the $n \times n$ matrix with entries

$$L(G)_{u,v} = \begin{cases} \deg(u) & u = v \\ -m(u,v) & u \neq v \end{cases}$$

where $m(u, v)$ is the number of edges between u and v. $L(G)$ is an integer matrix, so we can study it as a linear map of $\mathbb{Z}$-modules $\mathbb{Z}^n \to \mathbb{Z}^n$.

Note by the definition of $L(G)$ that the vector $(1, 1, \dots, 1) \in \ker(L(G))$. When the graph G is connected, we have an equality $\ker(L(G)) = \text{span}((1, 1, \dots, 1))$. Therefore, $\text{Im}(L(G)) \cong \mathbb{Z}^{n-1}$, a sublattice. It follows that the cokernel can be written as

$$\mathbb{Z}^n/\text{Im}(L(G)) \cong \mathbb{Z} \oplus K(G)$$

where $K(G)$ is a finite abelian group, known as the *sandpile group* of G. It follows from Kirchoff's Matrix Theorem that $|K(G)|$ is the number of spanning trees of G. This group is our main object of study.

We are interested in computing the sandpile groups of Cayley graphs of the group $\mathbb{F}_2^r$. One motivation for studying this family of graphs is that the hypercube graph Q_n, which has a sandpile group that is not completely determined, is a Cayley graph of $\mathbb{F}_2^r$.

In 2003, H. Bai determined the p-Sylow groups Q_n for $p \neq 2$ [7]. Bai also derive formulae for the number of Sylow-2 cyclic factors and the number of $\mathbb{Z}/2\mathbb{Z}$'s. Meanwhile, Ducey and Jalil [4] computed the Sylow-p groups for the Cayley graphs of any finite group for $p \nmid |G|$ in terms of the eigenvalues of $L(G)$. In 2015, Chandler et. al. [2] determined the cokernel of the *adjacency matrix* of Q_n in terms of n.

However, the 2-Sylow structure of Cayley graphs of $\mathbb{F}_2^r$ is still unknown. During the 2016 Twin Cities REU, Anzis and Prasad made progress in this direction by boundeing the largest 2-Sylow cyclic factor of $K(Q_n)$.

We begin by proving that in a *generic* case, the number of 2-sylow cyclic factors of $d(M)$ is $2^{r-1} - 1$, extending Bai's previous result for the hypercube. We conjecture that this is both a lower bound and this lower bound is only achieved in the generic case. We then use the methods of Anzis-Prasad to extend and improve their upper bound for all Cayley graphs of $\mathbb{F}_2^r$. In the case of Q_n, we go further to explicitly determine the top cyclic factor. We then continue to determine the 2nd through nth cyclic factors, and conjecture a formula for the $(n+1)$st factor. We conclude by completely determining the sandpile group for $r=2$ and for the generic case of $r=3$.

2.1 Background and Previous Results

We first define what a Cayley graph is in our context. Given $G=\mathbb{F}_2^r$ and a set of generators M

$$M=\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$

such that $v_i \in \mathbb{F}_2^r - \{0\}$, we form the *Cayley graph*, $G(\mathbb{F}_2^r, M)$, with vertex set $V=\mathbb{F}_2^r$ and edges $w, w+v_i$ for $w \in V$ and $v_i \in M$. The fact that M is a generating set implies G is connected, and $v_i \neq 0$ ensures there are no self-loops. Since addition is performed in $\mathbb{F}_2^r$, note we also have $w'+v_i=w$. Therefore, we can think of this graph as undirected. If we index the matrix representation of $L(G)$ by the binary tuples $u, v \in \mathbb{F}_2^r$ as opposed to a decimal indexing, then we can say that

$$L(G)_{u,v}=\begin{cases} n & u=v \\ -(\#\text{ of generators, } v_i\text{, such that } u+v_i=v) & u\neq v \end{cases}$$

since G is an n-regular graph.

We are now interested in $K(G)$ for such graphs. As mentioned in the introduction, Kirchoff's Matrix Tree theorem tells us that if $\lambda_1=0<\lambda_2\leq\lambda_3\leq\cdots\leq\lambda_{2^r}$ then $|K(G)|=\frac{1}{2^r}\prod_{i=2}^{2^r}|\lambda_i|$, which is also the number of spanning trees [6]. Note by the structure theorem, $K(G)\cong\bigoplus_p \oplus_{e\geq 1}(\mathbb{Z}/p^e\mathbb{Z})^{m(p^e)}$, where $m(p^e)$ is the order order of $\mathbb{Z}/p^e\mathbb{Z}$ in $K(G)$. Thus, we can try to determine the group prime by prime.

We will now detail some basic properties about the sandpile group of an arbitrary Cayley graph with vertex set $V=\mathbb{F}_2^r$. Much of this is easily derived from other sources such as Ducey-Jalil and Stanley, but we add in statements and proofs for completeness of the story. In particular, we will outline the proofs required to determine the p-primary component for $p\neq 2$.

First off, when regarding these matrices over $\mathbb{R}$ it turns out all there is an eigenbasis for all of these $G(\mathbb{F}_2^r, M)$ at once.

Definition 2.1. For $u\in\mathbb{F}_2^r$, define

$$f_u=\sum_{x\in\mathbb{F}_2^r}(-1)^{u\cdot x}e_x$$

where e_x is the standard basis vector $(0,\ldots,0,1,0,\ldots,0)$ of $\mathbb{R}^{2^r}$ with the only 1 at the xth index.

These vectors have some very special properties. Namely,

Lemma 2.2. *$\{f_u\}$ are an orthogonal 2^r basis for $\mathbb{R}^{2^r}$, and the standard basis $\{e_u\}_{u\in\mathbb{F}_2^r}$ satisfies $e_u=\frac{1}{2^r}\sum_{v\in\mathbb{F}_2^r}(-1)^{u\cdot v}f_v$.*

Proof.

$$f_u=\sum_{v\in\mathbb{F}_2^r}(-1)^{u\cdot v}e_v \implies f_u\cdot f_u=\sum_{v\in\mathbb{F}_2^r}(-1)^{2u\cdot v}e_v\cdot e_v=\sum_{v\in\mathbb{F}_2^r}1=2^r$$

by orthonormality of the $\{e_u\}$. Similarly

$$f_u\cdot f_w=\sum_{v\in\mathbb{F}_2^r}(-1)^{(u+w)\cdot v}$$

Let $z=u+w$, because $u\neq w$, we have that $z\neq 0$ in $\mathbb{F}_2^r$, so that there exists a coordinate index i such that $z_i\neq 0$. From here, we write

$$\begin{aligned} f_u\cdot f_w &= (-1)^{z\cdot e_i}\sum_{v\in\mathbb{F}_2^r}(-1)^{z\cdot(v-e_i)}=-\sum_{v\in\mathbb{F}_2^r}(-1)^{z\cdot(v-e_i)} \\ &= -\sum_{v\in\mathbb{F}_2^r}(-1)^{z\cdot v}=-f_u\cdot f_w \end{aligned}$$

And thus $f_u \cdot f_w = 0$. Given that $|\{f_u\}| = 2^r$ and these vectors are orthogonal, then they must span $\mathbb{R}^{2^r}$. Now consider the orthonormal basis of $\{2^{-r/2} f_v\}$ so that

$$e_u \cdot \frac{f_v}{2^{r/2}} = \frac{f_v}{2^{r/2}} \cdot e_u = \frac{1}{2^{r/2}} \left(\sum_{k \in \mathbb{F}_2^r} (-1)^{v \cdot k} e_k \right) \cdot e_u = (-1)^{v \cdot u} 2^{-r/2}$$

And so $e_u = \frac{1}{2^r} \sum_{v \in \mathbb{F}_2^r} (-1)^{u \cdot v} f_v$, as desired. □

This basis is in fact an eigenbasis, with the eigenvalues only depending on the generating set M.

Lemma 2.3. *For any set of generators of $\mathbb{F}_2^r$ given by $M = (v_1, \ldots, v_n)$, where the $\{v_i\}$ is a collection of column vectors, the Cayley graph $G = G(\mathbb{F}_2^r, M)$ and its graph Laplacian $L(G)$ has every f_u as eigenvector, with eigenvalue $\lambda_{u,M} = n - \sum_{i=1}^n (-1)^{u \cdot v_i}$.*

Proof. By definition, $(L(G))_{i,i} = n$ because every node in the graph has an incident edge for every generator, meaning n edges for each element of $\mathbb{F}_2^r$. Similarly, if we index the states by $u, w \in \mathbb{F}_2^r$, then $(L(G))_{u,w} = -(\sharp \text{ of generators such that } v_i + u = w)$. Note that by nature of generators being involutions, we have that $(L(G))_{u,w} = (L(G))_{w,u}$. With this, we write $L(G) = nI_n - A$ and we see that

$$(Af_u)_w = \sum_{v \in \mathbb{F}_2^r} (\sharp \text{ of generators } v_i \text{ such that } v_i + v = w)(-1)^{u \cdot v} = \sum_{v_i \in M} (-1)^{u \cdot (w - v_i)} = (-1)^{u \cdot w} \sum_{v_i \in M} (-1)^{u \cdot v_i}$$

$$\implies Af_u = \left(\sum_{i=1}^n (-1)^{u \cdot v_i} \right) f_u$$

which gives that f_u is an eigenvector of $L(G)$ with eigenvalue $n - \sum_{i=1}^n (-1)^{u \cdot v_i}$. Note that $\lambda_{\vec{0},M} = 0$ □

We use this information to determine the Sylow-p structure. If we define the ring $R = \mathbb{Z}\left[\frac{1}{2}\right] = \mathbb{Z}_2 = \{a/2^k \ : \ a \in \mathbb{Z},\ k = 0, 1, \ldots, \}$, then the change of basis formula from Lemma 2.2 implies we can diagonalize $L(G)$ over R to a matrix $D = \mathrm{diag}(\lambda_u, u \in \mathbb{F}_2^r)$. Using this fact we prove

Proposition 2.4.

$$Syl_p K(G) = Syl_p \left(\bigoplus_{u \in \mathbb{F}_2^r - \{0\}} \mathbb{Z}/\lambda_{u,M}\mathbb{Z} \right)$$

for $p \neq 2$

Proof. Writing $K(G) = \oplus_{p \text{ prime}} \oplus_{e \geq 1} (\mathbb{Z}/p^e\mathbb{Z})^{m(p^e)}$, we have the exact sequence

$$\mathbb{Z}^{2^r} \xrightarrow{L(G)} \mathbb{Z}^{2^r} \longrightarrow \mathbb{Z} \bigoplus \left(\oplus_{p \text{ prime}} \oplus_{e \geq 1} (\mathbb{Z}/p^e\mathbb{Z})^{m(p^e)} \right) \longrightarrow 0$$

Localization preserves exactness, so localizing at the element 2 yields

$$R^{2^r} \xrightarrow{L(G)} R^{2^r} \longrightarrow R \bigoplus \left(\oplus_{p \text{ prime}} \oplus_{e \geq 1} (\mathbb{Z}/p^e\mathbb{Z})^{m(p^e)} \right)_2 \longrightarrow 0$$

Now we know that

$$\left(\oplus_{p \text{ prime}} \oplus_{e \geq 1} (\mathbb{Z}/p^e\mathbb{Z})^{m(p^e)} \right)_2 = \oplus_{p \text{ prime}} \oplus_{e \geq 1} ((\mathbb{Z}/p^e\mathbb{Z})_2)^{m(p^e)} = \oplus_{p \neq 2 \text{ prime}} \oplus_{e \geq 1} (\mathbb{Z}/p^e\mathbb{Z})^{m(p^e)}$$

where we've noted that $(\mathbb{Z}/2^e\mathbb{Z})_2 = 0$ because $1 \equiv 2 \cdot \frac{1}{2} \equiv 0$ in this module, and also $(\mathbb{Z}/p^e\mathbb{Z})_2 \cong \mathbb{Z}/p^e\mathbb{Z}$ because 2 has an inverse in $\mathbb{Z}/p^e\mathbb{Z}$ for all $p \neq 2$ since $\gcd(p, 2) = 1$ and so localizing does nothing up to isomorphism. But we also know that $L(G)(R^{2^r}) = \oplus_{u \in \mathbb{F}_2^r - \{0\}} \lambda_{u,M}\mathbb{Z}$, and so the cokernel of $L(G)$ must be $R \oplus \left(\oplus_{u \in \mathbb{F}_2^r} \lambda_{u,M} R \right)$, yielding

$$R \bigoplus (\oplus_{p \neq 2 \text{ prime}} \oplus_{e \geq 1} (\mathbb{Z}/p^e\mathbb{Z})^{m(p^e)}) \cong R \bigoplus \left(\oplus_{u \in \mathbb{F}_2^r - \{0\}} \lambda_{u,M} R \right)$$

By classification, the torsion parts must be isomorphic, which yields the desired result. □

Thus, we have a nice description in terms of the eigenvalues for the Sylow-p subgroups for $p \neq 2$. One might hope this classification to also hold for $p = 2$, but data shows it is in general much wilder. In order to deal with the sandpile group when $p = 2$, we first adopt the approach of Reiner et. al. and induce a ring structure on $K(G)$. We once again include the proof for the sake of completeness.

Proposition 2.5.

$$cokerL(G) \cong \mathbb{Z}[x_1, x_2, \ldots, x_r]/(x_1^2 - 1, \ldots, x_r^2 - 1, n - \sum_{i=1}^{n} \prod_{j=1}^{r} x_j^{(v_i)_j})$$

Proof. We first note the isomorphism of abelian groups

$$\mathbb{Z}^{2^r} \cong \mathbb{Z}[x_1, \ldots, x_r]/(x_1^2 - 1, \ldots, x_r^2 - 1)$$

by taking the natural basis of monomial powers.

We know that $L(G) : \mathbb{Z}^{2^r} \to \mathbb{Z}^{2^r}$, and so with this isomorphism of $\mathbb{Z}^{2^r}$, it suffices to show that $L(G)$ acts by multiplying by $n - \sum_{i=1}^{n} \prod_{j=1}^{r} x_j^{(v_i)_j}$. Let $e_v = \prod_{j=1}^{r} x_j^{v_j}$, where $v \in \mathbb{F}_2^r$. We omit the $\overline{x_j}$ notation and assume that any polynomial is considered as an element of the quotient above. Unwrapping the definition of the Laplacian yields

$$L(G)e_v = ne_v - \sum_{w \in \mathbb{F}_2^r} (\sharp\{v_i \mid v_i + v = W\})e_w \qquad = ne_v - \sum_{w \in \mathbb{F}_2^r} (\sharp\{v_i \mid v_i + v = w\})e_{v_i+v} = ne_v - \sum_{v_i \in M} e_{v_i+v}$$

Now we note the following clever fact that

$$e_{v+w} = e_v e_w$$

for all $v, w \in \mathbb{F}_2^r$. This follows from the fact that $x_i^2 - 1$, so that whenever a coordinate i satisfies $(v+w)_i \equiv 0 \mod 2$, there is no x_i term in the monomial $e_v e_w$. Using this, we have that

$$L(G)e_v \equiv \overline{e_v}(n - \sum_{i=1}^{n} e_{v_i}) = e_v \left(\overline{n} - \sum_{i=1}^{n} \prod_{j=1}^{r} x_j^{(v_i)_j} \right)$$

Taking a cokernel then yields our desired result. $\square$

Remark 2.6. Note that by definition of cokernel, the order of an element $\omega = (a_1, \ldots, a_{2^r})$ in the cokernel is equal to the smallest integer C such that there exists a $v \in \mathbb{Z}^{2^r}$ such that $L(G)v = Cw$. This is used to find orders of elements in the polynomial ring (which corresponds to a vector in $\mathbb{Z}^{2^r}$) for determining top cylic factors and their 2-valuations.

Remark 2.7. In the case $G = Q_n$, this polynomial ring is

$$\mathbb{Z}[x_1, \ldots, x_n]/(x_1^2 - 1, x_2^2 - 1, \ldots, x_n^2 - 1, n - (x_1 + x_2 + \cdots + x + n))$$

This group has kernel that is symmetric under the action of S_n, which is an important fact that we will use later.

3 Equivalence of Sandpile groups up to $\mathbf{GL}_r(\mathbb{F}_2)$ action

Before, we move on to, it is very important to prove the following result about these Sandpile groups:

Proposition 3.1. *Let $T \in GL_r(\mathbb{F}_2)$, and M be an $r \times n$ list of generators for $\mathbb{F}_2^r$. Then $L(G(\mathbb{F}_2^r, M)) \sim L(G(\mathbb{F}_2^r, T \circ M))$, so that their sandpile groups are isomorphic.*

Proof. Note that an element $T \in \mathrm{GL}_r(\mathbb{F}_2)$ acts on $\mathbb{F}_2^r - \{0\}$ by permutation of elements. Moreover, it also sends each edge in $G(\mathbb{F}_2^r, M)$ to a unique edge in $G(\mathbb{F}_2^r, T \circ M)$. Moreover, T^{-1} is an inverse map, so we have an isomorphism of graphs.

$$G(\mathbb{F}_2^r, M)) \sim G(\mathbb{F}_2^r, T \circ M)$$

Therefore, up to relabeling the vertices by a permutation denoted by a $2^r \times 2^r$ matrix P_T, they are the same graph. As a result, $P_T L(G(\mathbb{F}_2^r, M))P_T^{-1} = L(G(\mathbb{F}_2^r, T \circ M))$ so these matrices have the same Smith normal form. $\square$

As a result, we only need to think about isomorphism classes of generators.

4 Reduction of the sandpile group in terms of generators

One approach to determining sandpile groups for general families of Cayley graphs is by specifying multiplicities of each nonzero vector in M.

Lemma 4.1. *Let $\{a_1, \ldots, a_{2^r-1}\}$ denote the multiplicites of each non-zero generator of the Cayley graph, G, on $\mathbb{F}_2^r$. Say another matrix N has multiplicities $\{\lambda a_1, \ldots, \lambda a_{2^r-1}\}$ for some $k \in \mathbb{Z}$. If the $L(G(\mathbb{F}_2^r, M))$ has Smith normal form $[0, s_1, \ldots, s_{2^r-1}]$ then $L(G(\mathbb{F}_2^r, N))$ has Smith normal form $[0, \lambda s_1, \ldots, \lambda s_{2^r-1}]$.*

Proof. If

$$L(G) = P \operatorname{diag}(s_1, \ldots, s_{2^r})\, Q$$

for $P, Q \in \mathrm{GL}_{2^r}(\mathbb{Z})$ then

$$L(G') = (\lambda I_{2^r}) L(G) = (\lambda I_{2^r}) P \operatorname{diag}(s_1, \ldots, s_{2^r})\, Q = P \operatorname{diag}(\lambda s_1, \ldots, \lambda s_{2^r})\, Q$$

□

An important corollary is that when

$$K(G) = \bigoplus_{i=1}^{2^r-1} \mathbb{Z}/s_i\mathbb{Z}$$

we know that if $K(G')$ is the sandpile group corresponding to generators with multiplicities $\{\lambda a_1, \ldots, \lambda a_{2^r-1}\}$, then

$$K(G') = \bigoplus_{i=1}^{2^r-1} \mathbb{Z}/(\lambda s_i)\mathbb{Z}$$

With this, we can reduce the problem of finding the sandpile group of $G = \mathbb{F}_2^r$ to the case in which the generator multiplicities satisfy

$$\gcd(a_1, \ldots, a_{2^r-1}) = 1$$

5 Number of even invariant factors depends on parity of generators

In this section we wish to compute the number of even cyclic factors appearing in the sandpile group. Given a sandpile group

$$K(G) \cong \bigoplus_p \bigoplus_{e \geq 1} (\mathbb{Z}/p^e\mathbb{Z})^{m(p^e)}$$

Tensoring with $\mathbb{Z}/2\mathbb{Z}$ yields

$$K(G) \otimes (\mathbb{Z}/2\mathbb{Z}) \cong \bigoplus_{e \geq 1} (\mathbb{Z}/2^e\mathbb{Z})^{m(2^e)} \otimes \mathbb{Z}/2\mathbb{Z} \cong (\mathbb{Z}/2\mathbb{Z})^{\sum_e m(2^e)}$$

where we used the facts that $(\mathbb{Z}/p^e\mathbb{Z}) \otimes (\mathbb{Z}/2\mathbb{Z}) = 0$ for $p \neq 2$ and $(\mathbb{Z}/2^e\mathbb{Z}) \otimes (\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ for all $e \geq 1$. We define

$$d(M) := \sum_{e \geq 1} m(2^e)$$

which is the number of even invariant factors. Now say we have a Cayley graph on $\mathbb{F}_2^r$ with $M = \{v_i\}_{i=1}^n$ a collection of generators with μ_u for each $\lambda_u \in \mathbb{F}_2^r - \{0\}$ with $n := \sum_i \mu_i$. Our first result about $d(M)$ is the following:

Proposition 5.1. *$d(M)$ is only depend on the multiplicities of the generators modulo 2.*

Proof. We use the ring description of the sandpile group. That is, we have that

$$\mathbb{Z} \oplus K(G) \cong \mathbb{Z}[x_1, \ldots, x_r]/\left(x_1^2 - 1, \ldots, x_r^2 - 1, n - \sum_{i=1}^{k} \mu_i \prod_{j=1}^{r} x_j^{(v_i)_j}\right)$$

If we have another Cayley graph G', corresponding to $V = \mathbb{F}_2^r$ and another set of generators with multiplicities $\{\lambda_i\}_{i=1}^{2^r-1}$ and $\{\mu_i\}_{i=1}^{2^r-1}$ such that $\lambda_i \equiv \mu_i \mod 2$ so that $n' = \sum_i \lambda_i \equiv \sum_i \mu_i \mod 2$, then

$$[\mathbb{Z} \oplus K(G)] \otimes (\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus (K(G) \otimes (\mathbb{Z}/2\mathbb{Z})) \cong \mathbb{Z}/2\mathbb{Z}[x_1, \ldots, x_r]/(x_1^2 - 1, \ldots, x_r^2 - 1, n - \sum_{i=1}^{2^r-1} \mu_i \prod_{j=1}^{r} x_j^{(v_i)_j})$$

$$\cong \mathbb{Z}/2\mathbb{Z}[x_1, \ldots, x_r]/(x_1^2 - 1, \ldots, x_r^2 - 1, n' - \sum_{i=1}^{2^r-1} \lambda_i \prod_{j=1}^{r} x_j^{(v_i)_j}) \cong \mathbb{Z}/2\mathbb{Z} \oplus (K(G') \otimes (\mathbb{Z}/2\mathbb{Z}))$$

and thus the $K(G) \otimes \mathbb{Z}/2\mathbb{Z} \cong K(G') \otimes \mathbb{Z}/2\mathbb{Z}$ meaning that $d(M) = d(M')$. $\square$

6 Characterization of the Number of 2-Sylow Components

Our first main theorem derives a formula for the number of even invariant factors for most choices of M. First, we make a definition:

Definition 6.1. Given $M = \{v_1, \ldots, v_n\}$ an $r \times n$ list of generators of $\mathbb{F}_2^r$, we say that M is **generic** if

$$\sum_{i=1}^{n} v_n \neq \vec{0}$$

Remark 6.2. For a fixed r and $1 \leq i \leq r$, the probability that the sum of the ith coordinates is 0 is roughly 1/2. Heuristically, each of the coordinates is about independent (not exactly since not all the coordinates can be 0, but this is just a heuristic). Then the probability that M is not in the generic case is roughly $1/2^r$, which exponentially decays to 0. This is why we use the word 'generic.'

Theorem 6.3. *Suppose that M is in the generic case. Then the number of invariant Sylow-2 cyclic factors of $K(G)$ is $2^{n-1} - 1$.*

Proof. The number of Sylow-2 cyclic factors will be the rank of the vector space $K(G) \otimes \mathbb{Z}/2\mathbb{Z}$, since $\mathbb{Z}/2^e\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$ for $e > 0$ and $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} = 0$ for $2 \nmid m$. Thus, we want the rank of

$$K(G) \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2Z[x_1, \ldots, x_r]/\left(x_1^2 - 1, \ldots, x_r^2 - 1, n - \sum_{i=1}^{n} \prod_{j=1}^{r} x_j^{(v_i)_j}\right)$$

Making the change of variables $u_i := x_i - 1$ yields

$$K(G) \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2Z[u_1, \ldots, u_r]/\left(u_1^2, \ldots, u_r^2, n - \sum_{i=1}^{n} \prod_{j=1}^{r} (u_j - 1)^{(v_i)_j}\right)$$

Call this last relation in the ideal $p(x_1, \ldots, x_r) := n - \sum_{i=1}^{n} \prod_{j=1}^{r} (u_j + 1)^{(v_i)_j}$ Since $\sum_{i=1}^{n} v_i \neq 0$, there exists an index k such that $\sum_{i=1}^{n} (v_i)_k = 1$. As a result, the coefficient of u_k in $p(x_1, \ldots, x_r)$ is 1. Therefore, noting this expression has trivial constant term and that any monomial in the ring can only have degree 1 factors of u_k, we can rewrite $0 \equiv p(u_1, \ldots, u_r) = u_k \cdot f - g$, where f, g are polynomials with no monomials dividing u_k and f has constant term 1. Since f has nonzero constant term and any monomial $u_i^2 = 0$, we in fact have $f^2 = 1$, so f is invertible and $u_k \equiv fg$ in the quotient. Relabel the variables so that $k = r$ (alternatively use GL_r invariance of the sandpile group with the transposition $(k\ r)$). We can now construct a bijection from $T : \mathrm{coker} L(G) \otimes \mathbb{Z}/2 \to \mathbb{Z}/2[u_1, \ldots, u_{r-1}]/(u_1^2, \ldots, u_{r-1}^2) \cong (\mathbb{Z}/2\mathbb{Z})^{2^{r-1}}$, by mapping $u_t \to u_t$ for $t < r$ and $u_r \to g(u_1, \ldots, u_{r-1}) f(u_1, \ldots, u_{r-1})$. Note that as a vector space $\mathrm{rank}(\mathbb{Z}/2 \oplus K(G) \otimes \mathbb{Z}/2) \leq 2^{r-1}$, since all monomials involving u_r can be written in terms of $u_1, \ldots, u_{r-1}$. Thus, the map T is a surjective linear map from a space of dimension $\leq 2^{r-1}$ to a space of dimension 2^{r-1}. It then must be an isomorphism. Therefore, $(\mathbb{Z} \oplus K(G)) \otimes \mathbb{Z}/2\mathbb{Z} \cong (\mathbb{Z}/2\mathbb{Z})^{2^{r-1}}$ so $K(G)$ has $2^{r-1} - 1$ Sylow-2 cyclic factors. $\square$

What about in the **nongeneric cases**? In that case, the final relation no longer has a degree 1 term, so we cannot construct the isomorphism from the proof above. However, we can at least prove a basic structural result as follows:

Proposition 6.4. *Let $(a_{v_1}, \dots, a_{v_{2^r-1}})$ be the multiplicities of the generators associated to M, and assume not all a_{v_i} have the same parity. Let M' be the generating set with $(a_{v_1}+1, \dots, a_{v_{2^r-1}}+1)$. Then $d(M) = d(M')$.*

Proof. Using the techniques from 6.3, we have that $d(M)$ is one less than the dimension of

$$\mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_r]/(x_1^2 - 1, \dots, x_r^2 - 1, g(x_1, x_2, \dots, x_r))$$

where

$$g(x_1, x_2, \dots, x_r) = n - \sum_{i=1}^{n} \prod_{j=1}^{r} x_j^{(v_i)_j} = n - \sum_{i=1}^{2^r-1} a_i e_{w_i}$$

where a_i is the multiplicity of the ith standard generator, $w_i \in \mathbb{F}_2^r$, which is also the binary expansion of i, and $e_{w_i} = \prod_{j=1}^{r} x_j^{(w_i)_j}$ as in 2.5. As before, we make the substitution $u_i = x_i - 1$ to get

$$\mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_r]/(x_1^2 - 1, \dots, x_r^2 - 1, g(x_1, x_2, \dots, x_r)) \cong \mathbb{Z}/2\mathbb{Z}[u_1, \dots, u_r]/(u_1^2, \dots, u_r^2, p(u_1, u_2, \dots, u_r))$$

where

$$p(u_1, u_2, \dots, u_r) = n - \sum_{i=1}^{n} \prod_{j=1}^{r} (u_j - 1)^{(v_i)_j}$$

as in 6.3. Note that $d(M')$ is one less than the dimension of

$$\mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_r]/(x_1^2 - 1, \dots, x_r^2 - 1, g'(x_1, \dots, x_r), \quad \text{s.t.}$$

$$g'(x_1, \dots, x_r) = n - \sum_{i=1}^{2^r-1} (a_i + 1) e_{w_i} = g(x_1, \dots, x_r) - \sum_{i=1}^{2^r-1} e_{w_i}$$

note that under the substitution of u_i's, we get

$$g'(x_1, \dots, x_r) \mapsto p'(u_1, \dots, u_r) = p(u_1, \dots, u_r) - \sum_{i=1}^{2^r-1} \prod_{j=1}^{r} (u_j + 1)^{(w_i)_j}$$

having used $1 \equiv -1 \mod 2$. To each w_i, we define $s(w_i)$ to be the subset of components of w_i which have value 1, so that

$$\prod_{j=1}^{r} (u_j + 1)^{(w_i)_j} = \sum_{K \subseteq s(w_i)} \prod_{k \in K} u_k = \sum_{K \subseteq s(w_i)} u_{\alpha_K}$$

From this we get

$$\sum_{i=1}^{2^r-1} \prod_{j=1}^{r} (u_j + 1)^{(w_i)_j} = \sum_{i=1}^{2^r-1} \sum_{K \subseteq s(w_i)} u_{\alpha_K} = \sum_{K \subseteq \{1, \dots, r\}} \sum_{i \text{ s.t. } K \subseteq s(w_i)} u_{\alpha_K}$$

Now note that given a fixed binary string, γ, with r components, the number of $z \in \mathbb{F}_2^r$ such that $s(\gamma) \subseteq s(z)$ is equal to $2^{r-s(\gamma)}$, and thus

$$\sum_{K \subseteq \{1, \dots, r\}} \sum_{\substack{i \text{ s.t.} \\ K \subseteq s(w_i)}} u_{\alpha_K} = \sum_{K \subseteq \{1, \dots, r\}} 2^{r-s(\gamma)} u_{\alpha_K} \equiv u_{\alpha_{\{1, \dots, r\}}} = \prod_{i=1}^{r} u_i$$

From this, we determine $p' = p + \prod_{i=1}^{r} u_i$, so $d(M')$ is the dimension of

$$\mathbb{Z}/2\mathbb{Z}[u_1, \dots, u_r]/(u_1^2, \dots, u_r^2, p(u_1, u_2, \dots, u_r) + u_1 u_2 \dots u_r))$$

Since M does not have all even or all odd multiplicities, then $p(u_1, \dots, u_r)$ is both nonzero and has coefficient 1 for some monomial $u = u_{b_1} \dots u_{b_k}$ that does contain all of the u_i. Take such a monomial of minimal degree, and we see that

$$(u + [p(u_1, \dots, u_r) - u]) \cdot \frac{u_1 \cdots u_r}{u} \equiv u_1 \cdots u_r + [p(u_1, \dots, u_r) - u] \cdot \frac{u_1 \cdots u_r}{u} \equiv u_1 \cdots u_r$$

This is because every summand in $p(u_1, \dots, u_r) - u$ has a generator, u_i, in common with the monomial, u, and so every summand of $(p - u) \cdot \frac{u_1 \cdots u_r}{u}$ will have u_i^2 for some i, meaning that all summands are equivalent to 0, giving the last equivalence. Therefore $u_1 \dots u_r = 0$ in both rings (as the same conditions and constructions could have been done with M'), and $p(u_1, \dots, u_r) + u_1 \dots u_r \equiv p(u_1, \dots, u_r)$, which implies $d(M) = d(M')$. □

By Proposition 5.1, there are only finitely many possible values for what $d(M)$ could be. We just need to choose a representative from each sequence of evens and odds, up to the $\mathrm{GL}_r(\mathbb{F}_2)$ action. We have compiled all the possible $d(M)$ for $r = 2, 3, 4$ in later sections.

7 Bounding the Largest Cyclic Factor

In their report, Anzis and Prasad proved that for $G = Q_n$, the largest cyclic factor must divide $2^n \text{lcm}(1, \ldots, n)$. As a corollary, they derived that the largest 2-cyclic factor is bounded by $2^{\lfloor \log_2 n \rfloor + n}$. We generalize this result for all Cayley graphs of $\mathbb{F}_2^r$ and improve the bound by a factor of 2. Let $G = G(\mathbb{F}_2^r, M)$ be an arbitrary Cayley graph. Let $\lambda_1 = 0, \lambda_2, \ldots, \lambda_{2^r}$ be the eigenvalues of the Laplacian matrix $L(G)$.

Theorem 7.1. *Let d be the size of the largest cyclic factor in $K(G)$. Then $d \mid 2^{r-2} lcm(\lambda_i : i \geq 2)$.*

We follow the exact same proof outline as in Anzis-Prasad, with minor tweaks to account for the general case. First, using the identification $\mathbb{Z} \oplus K(G) \cong \mathbb{Z}[x_1, \ldots, x_r]/(x_1^2 - 1, \ldots, x_r^2 - 1, n - \sum_{j=1}^n \prod_{i=1}^r x_i^{(v_j)_i})$, we have the following lemmas

Lemma 7.2. *For $0 \neq \overline{p(x_1, \ldots, x_r)} \in \mathbb{Z}[x_1, \ldots, x_r]/(x_1^2 - 1, \ldots, x_r^2 - 1, n - \sum_{j=1}^n \prod_{i=1}^r x_i^{(v_j)_i})$ with finite additive order, let $\mathbf{w}$ be the vector in $\mathbb{Z}^{2^r}$ corresponding to $p(x_1, \ldots, x_r)$ under the isomorphism, $\mathbb{Z}^{2^r} \cong \mathbb{Z}[x_1, \ldots, x_r]/(x_1^2 - 1, \ldots, x_r^2 - 1)$. Let C be the additive order of $\overline{p}$ in the quotient ring, or equivalently, the additive order of $\mathbf{w}$ in coker$(L(G))$. Denote this by $|\mathbf{w}|$. Then*

$$|\mathbf{w}| = \text{ smallest } C \in \mathbb{Z} \quad s.t. \quad \exists v \in \mathbb{Z}^{2^r} \quad s.t. \quad L(G)v = Cw$$

Proof. This follows by the definition of cokernel and considering the cokernel as a $\mathbb{Z}$-module. □

Lemma 7.3. *The largest cyclic factor in $K(G)$ is $\max_{1 \leq k \leq r} ord(x_k - 1)$*

Proof. We follow the same process as Anzis-Prasad. See Lemma 2.3 in their paper. For the sake of completeness, we write out the whole proof. [1] Proposition 5.20 implies that $L(G)$ is the extended McKay Cartan matrix associated to the $\mathbb{F}_2^r$ faithful representation with character $\sum_{i=1}^n \chi_{v_i}$. This representation is faithful by [1] Proposition 5.3 c).

Proposition 5.20 then tells us that, $\mathbb{Z}[x_1, \ldots, x_r]/(x_i^2 - 1, n - \sum \prod x_j^{(v_i)_j})$ is isomorphic to the representation ring of $\mathbb{F}_2^r$ modulo the ideal generated by $n - \sum_i \chi_{v_i}$. By the second part of the proposition, an element has finite additive order in this ring iff it lies in the kernel of the map sending all of the χ_{v_i} to 1, implying that any irreducible character $\chi_v \to 1$. The element corresponding to $x_j - 1$ in the representation ring is $\chi_e - 1$ for some irreducible character χ_e under our isomorphism, and it follows that it has finite additive order. Furthermore, a consequence of this proposition is that any polynomial with finite additive order is a linear combination of $x_I - 1$, where x_I denotes a monomial free of second powers.

Now let $x_I - 1$ be the polynomial such that x_I is the monomial with largest finite order. If $C(x_I - 1) \in (x_1^2 - 1, \ldots, x_r^2 - 1, n - \sum \prod x_i^{(v_j)_i})$ then we wish to show that $C(X_I - 1) \equiv 0$. Indeed suppose $x_j \mid x_I$. Then we have $C(X_I - 1) = C(x_j - 1) \cdot \frac{X_I}{x_j} + C\left(\frac{X_I}{x_j} - 1\right)$, which we can reduce inductively to a sum of $x_I - 1$ with $\deg x_I = 1$. This shows that the largest cyclic factor is determined by the maximal order taken over all $x_i - 1$, the desired result. □

Remark 7.4. This lemma can actually be slightly generalized. Namely, let $w_1, \ldots, w_n$ be any generating set of $\mathbb{F}_2^r$. Then the maximal order element of the set $\{\prod_j x_j^{(w_i)_j} - 1\}$ will have largest possible additive order.

Anzis and Prasad's original argument shows that for the hypercube, we can take any $x_k - 1$ for $1 \leq k \leq n$. The argument relies on showing that $x_i - 1$ and $x_j - 1$ have the same additive order, which follows from symmetry under permutation. However, this is no longer the case. If we took a set of generators M fixed under the action of permutation, then any $x_i - 1$ would have maximal order.

Note now that our sandpile group remains the same under permutation of the variables $\{x_1, \ldots, x_r\}$ (this is a transformation induced by the GL_r action). Therefore, we can assume one of the maximal order elements is $x_1 - 1$.

Proof of Theorem 7.1. We once again follow the same argument as Anzis-Prasad. Namely, that we wish to find the smallest integer C such that there is a solution w to $L(G)\mathbf{v} = C\mathbf{w}$ for $\mathbf{w} = x_1 - 1 \mapsto (-1, 1, 0, \ldots, 0)$.

Define $\mathbf{w}_\lambda = (0, 2^{1-n}, 0, 2^{1-n}, \ldots, 0, 2^{1-n})$. By Lemma 3.2, we change to the eigenbasis and get

$$v = x_1 - 1 \longmapsto \frac{1}{2^r} \left[\sum_{v \in \mathbb{F}_2^r} [(-1)^{v \cdot e_1} - (-1)^{v \cdot 0}] f_v \right] = -\frac{1}{2^{r-1}} \sum_{u_1 = 1} f_u$$

Since f_u is an eigenbasis, we can take the following solution to the equation $L(G)v = Cw$:

$$\mathbf{v} = -\frac{1}{2^{r-1}} \sum_{u_1 = 1} \frac{1}{\lambda_u} f_u$$

Let X_r is the change-of-basis matrix from $\{e_u\}$ to the $\{f_u\}$ where we consider the index, u, as a number written in binary. Upon changing basis, the equation $L(G)\mathbf{v} = \mathbf{w}$ becomes $(X_r^{-1}L(G)X_r)(X_r^{-1}\mathbf{v}) = X_r^{-1}\mathbf{w}$. The right-hand side is equal to $\mathbf{w}_\lambda$ from above, while the LHS is equal to $D(X_r^{-1}\mathbf{v})$ where D is the diagonalization of $L(G)$ by the aforementioned change-of-basis matrix. We thus have $\mathbf{v} = X_r D^{-1}\mathbf{w}_\chi$. Then

$$D^{-1}\mathbf{w}_\chi = \frac{1}{2^{r-1}} \sum_{u_1=1} \frac{f_u}{\lambda_u}$$

Then

$$D^{-1}\mathbf{w}_\chi = \frac{1}{2^{r-1}} \sum_{u_1=1} \frac{1}{\lambda_u} \sum_{v\in\mathbb{F}_2^r} (-1)^{u\cdot v} e_v = \frac{1}{2^{r-1}} \sum_{v\in\mathbb{F}_2^r} \sum_{u_1=1} \left(\frac{(-1)^{u\cdot v}}{\lambda_u}\right) e_v$$

Let $\ell(\lambda)$ be the lcm of the eigenvalues. Multiplying this expression by $2^{r-1}L(\lambda)$ yields

$$2^{r-1}\ell(\lambda)\mathbf{w} = \frac{1}{2} \sum_{v\in\mathbb{F}_2^r} \sum_{u_1=1} \left(\frac{(-1)^{u\cdot v}\ell(\lambda)}{\lambda_u}\right) e_v$$

Let $p(v) = 2^{1-r}\sum_{u_1\neq 0}(\frac{(-1)^{u\cdot v}}{\lambda_u})$, and note that the coefficients $q(v) = \ell(\lambda)2^{r-1}$ are all integers, and that $q(v_1) \equiv q(v_2) \mod 2$, since all of the signs are equivalent modulo 2. If $q(v)$ are all even, then $2^{r-1}\ell(\lambda)w \in \mathbb{Z}^{2^r}$, which yields the result. Otherwise, $\frac{1}{2}q(v) \in \mathbb{Z}+1/2$, so that $2^{r-1}\ell(\lambda)w \in (\mathbb{Z}+1/2)^{2^r}$. But then recall that $L(G)$ has a 1-dimensional kernel spanned by $\mathbf{1} = \sum_i e_i$. But then $2^{r-1}\ell(\lambda)(\lambda)w + \frac{1}{2}\mathbf{1} \in \mathbb{Z}^{2^r}$ and satisfies $L(G)(2^{r-1}\ell(\lambda)w + \frac{1}{2}\mathbf{1}) = 2^{r-1}\ell(\lambda)v$, so the result follows. □

Corollary 7.4.1. *The largest 2-cyclic factor, $\mathbb{Z}/2^e\mathbb{Z}$ has bound*

$$e \leq \lfloor \log_2(n) \rfloor + r - 1$$

Proof. First we note that if d is the order of the largest cyclic factor, then if we write

$$K(G) \cong \mathbb{Z}/\alpha_1\mathbb{Z} \oplus \mathbb{Z}/\alpha_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\alpha_{2^r-1}\mathbb{Z}$$

for $\{\alpha_i\}$ the list of non-zero invariant factors in Smith Normal Form such that $\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_{2^r-1}$, then $d = \alpha_{2^r-1}$ will contain the largest 2-sylow component under the isomorphism

$$\mathbb{Z}/(2^k \cdot b)\mathbb{Z} \cong \mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$$

where $2 \nmid b$. From this, given $d \mid 2^{r-2}\mathrm{lcm}\,(\lambda_i : i \geq 2)$, the largest 2-Sylow component is $v_2(d)$ so that

$$v_2(d) \leq v_2\mathrm{lcm}\,(\lambda_i : i \geq 2) + r - 2$$

Note that $2 \leq \lambda_i \leq 2n$, so

$$\mathrm{lcm}\,(\lambda_i : i \geq 2) \mid \mathrm{lcm}\,(2, 2, \ldots, 2n)$$

which implies that

$$v_2\mathrm{lcm}\,(\lambda_i : i \geq 2) \leq v_2\,[\mathrm{lcm}\,(2, \ldots, 2n)] = 1 + \lfloor \log_2(n) \rfloor$$

giving

$$v_2(d) \leq \lfloor \log_2(n) \rfloor + r - 1$$

□

What is especially nice about this improvement is that it is asymptotically tight: As we will see later, this upperbound is achieved for all hypercubes $Q(2^k), Q(2^k+1)$. This will be an immediate consequence of the main result of the next section, which completely determines the top cyclic factor of the hypercube.

We now start talking about results that will eventually lead to the complete determination of the top cyclic factor of Q_n.

Lemma 7.5. *The order of $x_j - 1$ in $K(G)$ is equal to the minimum integer C, such that for any $v \in \mathbb{F}_2^r$,*

$$\frac{C}{2^{r-2}} \sum_{\substack{u\cdot v=1 \\ u_j=1}} \frac{1}{\lambda_u} \in \mathbb{Z}.$$

Proof. Using the same logic as above but without looking just at a maximal element, C is equal to the minimal integer such that there exist some $k \in \mathbb{R}$,

$$\frac{C}{2^{r-1}} \cdot \left(\sum_{u_j=1} \frac{f_u}{\lambda_u} + k\mathbf{1} \right) = \frac{C}{2^{r-1}} \sum_{v \in \mathbb{F}_2^r} (p(v)+k)e_v \in \mathbb{Z}$$

Denote the coordinate of e_v in $\frac{1}{2^{r-1}} \sum\limits_{u_j=1} \frac{f_u}{\lambda_u}$ as $p(v)$. Finding the minimal C for some value of k is equivalent to finding the smallest C such that $\frac{C}{2^{r-1}}\alpha_i(k) \in \mathbb{Z}$ for all i, which is equivalent to finding the minimal C such that $C \cdot [p(v) - p(\vec{0})] \in \mathbb{Z}$ for all $x \in \{1, \dots, 2^r\}$. This is because if such a C is chosen, then we can choose $k \in \mathbb{R}$ so that

$$C \cdot [p(v) + k \cdot 2^{1-r}] = C \cdot \left([p(v) - p(\vec{0})] + [p(\vec{0}) + k \cdot 2^{1-r}] \right) \in \mathbb{Z}$$

and there always exists a $k \in \mathbb{R}$ so that $C \cdot [p(\vec{0}) + k \cdot 2^{1-r}] \in \mathbb{Z}$. Moreover if $C \cdot [p(w) + k \cdot 2^{1-r}] \in \mathbb{Z}$, then we can take the difference to get

$$C \cdot [p(w) + k \cdot 2^{1-r}] - C \cdot [p(\vec{0}) + k \cdot 2^{1-r}] = C \cdot [p(w) - p(\vec{0})] \in \mathbb{Z}$$

Thus our search for the minimal C is equivalent to finding a C so that $C \cdot [p(v) - p(\vec{0})] \in \mathbb{Z}$ for all v, and is thus independent of k. Unraveling this yields

$$C(p(v) - p(\vec{0})) = \frac{C}{2^{r-1}} \sum_{u_j=1} \frac{(-1)^{u \cdot v} - 1}{\lambda_u} = -\frac{C}{2^{r-2}} \sum_{\substack{u \cdot v = 1 \\ u_j = 1}} \frac{1}{\lambda_u} \in \mathbb{Z}.$$

as desired. □

Remark 7.6. Since we only care about the Sylow-2 factor in these maximal orders, it actually suffices to find the minimal C such that for any $v \in \mathbb{F}_2^r$,

$$\frac{C}{2^{r-2}} \sum_{\substack{u \cdot v = 1 \\ u_j = 1}} \frac{1}{\lambda_u} \in \mathbb{Z}_{(2)}$$

where $\mathbb{Z}_{(2)}$ is the localization of the integers away from the prime ideal (2). This way, we don't actually care about odd denominators.

The sums $\sum_{u \cdot v = 1} \frac{u_r}{\lambda_u}$ are in general hard to handle. In order to deal with this sum more concretely, we prove the following very useful lemma, which lets us break down these sums into much smaller pieces.

Lemma 7.7. *In $\mathbb{C}[a_u : u \in \mathbb{F}_2^r \setminus \{\mathbf{0}\}]$,*

$$\mathit{span}_{\mathbb{Z}} \left\{ \sum_{u \cdot v = 1} a_u \middle| v \in \mathbb{F}_2^r \right\} = \mathit{span}_{\mathbb{Z}} \left\{ 2^{|S|-1} \sum_{u_S = d} a_u \middle| \emptyset \neq S \subseteq [n], d \in \mathbb{F}_2^{|S|} \setminus \{\mathbf{0}\} \right\}.$$

Here, $u_S = d$ means each coordinate u_i for $i \in S$ matches the entries of d. For example, $u_{\{1,4,7\}} = [0,1,0] \iff u_1 = 0, u_4 = 1, u_7 = 0$.

Intuitively, the sum $\sum_{u \cdot v = 1} \frac{u_r}{\lambda_u}$

Proof. We prove by induction on $1 \leq k \leq r$ that

$$\mathit{span}_{\mathbb{Z}} \left\{ \sum_{u \cdot v = 1} a_u \middle| v \in \mathbb{F}_2^r, w(v) \leq k \right\} = \mathit{span}_{\mathbb{Z}} \left\{ 2^{|S|-1} \sum_{u_S = d} a_u \middle| \emptyset \neq S \subseteq [n], |S| \leq k, d \in \mathbb{F}_2^{|S|} \setminus \{\mathbf{0}\} \right\}.$$

Here $w(v)$ is the number of 1's in v. When $k = 1$ it is obvious. If for $k-1$ it holds, we now prove it for k. Denote the two equal spans in $k-1$ case as V. Denote $v_k = \sum_{i=1}^k e_i$. It suffices (why?) to prove that for any $d \in \mathbb{F}_2^k \setminus \{\mathbf{0}\}$,

$$2^{k-1} \sum_{u_{[k]} = d} a_u - \sum_{v_k \cdot u = 1} a_u \in V.$$

It requires three steps. LMAO this part is sooo hard to write up that I will leave it to later. NOT DONE □

There is a clear motivation in lemma 7.5. If we assign $a_u = 1/\lambda_u$ whenever $u_r = 1$ and $a_u = 0$ whenever $u_r = 0$, we can rewrite the definition of C in lemma 7.4 into the following corollary:

Corollary 7.7.1. *The order of $x_j - 1$ in $K(G)$ is equal to the minimum integer C, such that for any $S \subseteq [n], |S| \geq 2, d \in \mathbb{F}_2^{|S|} \setminus \{\mathbf{0}\}$,*

$$\frac{C}{2^{r-|S|}} \sum_{u_S = d} \frac{1}{\lambda_u} \in \mathbb{Z}.$$

This is equivalent to saying, for G, if we draw a r-dimensional hyper cube, and write down $1/\lambda_u$ on each vertex $u \neq \mathbf{0}$, then C will be the largest denominator of the arithmetic mean of vertices of a certain face, among all faces that does not pass through the origin.

8 2 Sylow's of a Cayley Graph

In this section we will classify all sandpile groups for $r = 2$ and n arbitrary. In the case that $r = 2$, we have the following generating matrix:

$$M = \begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 & 1 & \dots & 1 \\ 0 & \dots & 0 & 1 & \dots & 1 & 1 & \dots & 1 \end{pmatrix} = \left(a * \begin{bmatrix} 1 \\ 0 \end{bmatrix}, b * \begin{bmatrix} 0 \\ 1 \end{bmatrix}, c * \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

Set $n = a + b + c$. The eigenvalues are

$$\lambda_{(1,0),M} = (a + b + c) - (-a + b - c) = 2(a + c)$$

$$\lambda_{(0,1),M} = (a + b + c) - [a - b - c] = 2(b + c)$$

$$\lambda_{(1,1),M} = (a + b + c) - [-a - b + c] = 2(a + b)$$

So that by the matrix tree theorem

$$\det \overline{L(G)}^{i,i} = \tau(G) = \frac{\lambda_2 \cdots \lambda_{2^r}}{2^r}$$

$$\frac{\det \overline{L(G)}^{i,i}}{\prod_{u \in \mathbb{F}_2^r - \{0\}} \lambda_{u,M}} = \frac{1}{2^r} \implies |Syl_2(K(G))| = \frac{1}{2^r} Pow_2 \left(\prod_{u \in \mathbb{F}_2^r - \{0\}} \lambda_{u,M} \right)$$

The 2-Sylow structure is then given by a partition of the $\log_2 |Syl_2(K(G))|$ where the length of the partition is bounded but undetermined.

Restricting to when M is generic, the length of the partition is 1, so $Syl_2(K(G)) = \mathbb{Z}/2^e\mathbb{Z}$ for $2^e = |Syl_2(K(G))| = 2Pow_2[(a + b)(a + c)(b + c)]$. The corresponding values of (a, b, c) are

$$(a, b, c) \equiv (\bar{a}, \bar{b}, \bar{c}) \mod 2 \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1), (1, 1, 0)\}$$

noting that when $r = 2$, we can permute all of the generators via $GL_2(\mathbb{F}_2)$-action, we have only two generic cases: $(1, 0, 0)$ and $(1, 1, 0)$, i.e. a odd, c even, and b is either.

$(a, b, c) \equiv (1, 0, 0)$
In this case, the only even factor is $b + c$ so that

$$2^e = 2 \cdot 2^{v_2((a+b)(a+c)(b+c))} = 2^{v_2(b+c)+1}$$

$(a, b, c) \equiv (1, 1, 0)$
In this case, we have that the only even factor is $a + b$, so

$$2^e = 2^{v_2(a+b)+1}$$

For non-generic matroid, $(a, b, c) \equiv (0, 0, 0)$ can be reduced by 4.1, so it suffices to handle the case $(a, b, c) \equiv (1, 1, 1)$ which is not generic. In this case, we can explicitly determine the top cyclic factor using the method in 7.3. By GL-action, we can assume that the maximum order is achieved by $x_2 - 1 \leftrightarrow (-1, 0, 1, 0)$, and so we want to find the smallest C such that

$$C \cdot \frac{1}{2} \left[\frac{\chi_{(0,1)}}{\lambda_{(0,1)}} + \frac{\chi_{(1,1)}}{\lambda_{(1,1)}} \right] \in \mathbb{Z}$$

but we have that

$$\chi_{(0,1)} = \sum_{v \in (\mathbb{Z}/2\mathbb{Z})^2} (-1)^{(0,1)\cdot v} f_v = [f_{(0,0)} - f_{(0,1)} + f_{(1,0)} - f_{(1,1)}]$$

$$\chi_{(1,1)} = \sum_{v \in (\mathbb{Z}/2\mathbb{Z})^2} (-1)^{(1,1)\cdot v} f_v = [f_{(0,0)} - f_{(0,1)} - f_{(1,0)} + f_{(1,1)}]$$

Moreover, from the previous calculations, we have that $\lambda_{(0,1)} = 2(b+c)$ and $\lambda_{(1,1)} = 2(a+b)$, so that

$$C \cdot \frac{1}{2}\left[\frac{\chi_{(0,1)}}{\lambda_{(0,1)}} + \frac{\chi_{(1,1)}}{\lambda_{(1,1)}}\right] = C \cdot \frac{1}{2}\Big[f_{(0,0)}\left(\frac{1}{2(b+c)} + \frac{1}{2(a+b)}\right) - f_{(0,1)}\left(\frac{1}{2(b+c)} + \frac{1}{2(a+b)}\right)$$
$$+ f_{(1,0)}\left(\frac{1}{2(b+c)} - \frac{1}{2(a+b)}\right) + f_{(1,1)}\left(-\frac{1}{2(b+c)} + \frac{1}{2(a+b)}\right)\Big] \in \mathrm{span}_{\mathbb{Z}}\{f_{(0,0)}, f_{(1,0)}, f_{(0,1)}, f_{(1,1)}\}$$

from this, it suffices to take the least common multiple of the four such C_u which guarantee that each coefficient, r_u, of f_u lies in $\mathbb{Z}$, in particular

$$r_{(0,0)} = \frac{a+b+b+c}{2(a+b)(b+c)} = \frac{a+2b+c}{2(a+b)(b+c)} = -r_{(0,1)}$$

$$r_{(1,0)} = -r_{(1,1)} = \frac{a-c}{2(a+b)(b+c)}$$

Instead of taking the least common multiple of this opaque formula, we note that the order of the cyclic factor, C, is invariant under which solution we choose, as $\ker(L(G)) = \mathbb{Z}v_0$, but

$$\forall \lambda \in \mathbb{Z}, \quad L(G)(v + \lambda v_0) = L(G)(v) = w$$

using the notation of Anzis and Prasad [3], where $v_0 = (1, \dots, 1)$ in the f_u basis. In particular, we choose $\lambda = \frac{1}{2(b+c)} + \frac{1}{2(a+c)}$, so that

$$v = \frac{1}{2}\left[\frac{\chi_{(0,1)}}{\lambda_{(0,1)}} + \frac{\chi_{(1,1)}}{\lambda_{(1,1)}}\right]$$

$$\implies v + v_0 = \frac{1}{2}\Big[f_{(0,0)}\left(\frac{1}{(b+c)} + \frac{1}{(a+b)}\right) - f_{(1,0)}(0)$$
$$+ f_{(0,1)}\left(\frac{1}{(b+c)}\right) + f_{(1,1)}\left(\frac{1}{2(a+b)}\right)\Big]$$

for which $C = 2lcm((b+c),(a+c))$ is the minimal such C so that

$$L(G)v = L(G)(v + \lambda v_0) = Cw \in \mathrm{span}_{\mathbb{Z}}\{f_{(0,0)}, f_{(1,0)}, f_{(0,1)}, f_{(1,1)}\}$$

this same technique of adding an element of the kernel to get a more transparent expression for C is used in the general proof of the largest 2-Sylow. WLOG $v_2(b+c) \geq v_2(a+c)$, so that $v_2(C) = v_2(b+c)+1$ and the first factor is $\mathbb{Z}/2^e\mathbb{Z}$ with $e = v_2(b+c)+1$, meaning that the other factor has size $f = v_2((a+c)(a+b))$. Note that given 3 odd numbers, at least 2 of them must be sum to be $2 \mod 4$. In particular, taking the 3 cases of

$$(a,b,c) \in \{(1,1,1),(3,1,1),(3,3,1)\} \mod 4$$

which occur up to permutation equivalence and $gcd(a,b,c) = 1$ reduction, we see that $v_2(b+c) \geq v_2(a+c)$ means that equality implies that $v_2(a+b) = 1$, so that $f = v_2(a+c)+1$ and

$$Syl_2(K(G)) = \mathbb{Z}/2^e\mathbb{Z} \oplus \mathbb{Z}/2^f\mathbb{Z}$$

9 $r = 3$ determination of 2-Sylow structure

We now turn our attention to the case of $r = 3$. Say that our matroid M has multiplicities as follows:

$$M = \left(a * \begin{bmatrix}1\\0\\0\end{bmatrix}, b * \begin{bmatrix}0\\1\\0\end{bmatrix}, c * \begin{bmatrix}0\\0\\1\end{bmatrix}, d * \begin{bmatrix}1\\1\\0\end{bmatrix}, e * \begin{bmatrix}1\\0\\1\end{bmatrix}, f * \begin{bmatrix}0\\1\\1\end{bmatrix}, g * \begin{bmatrix}1\\1\\1\end{bmatrix}\right)$$

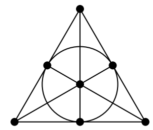

Figure 1: In this diagram, the circle and each straight line represents a line in $\mathbb{F}_2^3$ not passing through the origin. Note that each eigenvalue is 2 times the sum of complements of a line.

Then the 7 nonzero eigenvalues are

$$2(a+b+e+f), 2(b+c+d+f), 2(a+c+d+e), 2(b+d+e+g), 2(c+e+f+g), 2(a+d+f+g), 2(a+b+c+g)$$

One way to think of this is via the Fano plane description of $\mathbb{F}_2^3 - \{0\}$. See figure 1.
Recall from before that the number of Sylow-2 generators, $d(M)$, is only dependent on the parity of the numbers of each generator. We can think of all these cases thus in terms of how many odd multiplicities and how many even multiplicities there are. The cases are

1. all odd: $d(M) = 6$
2. 1 odd, 6 even: $d(M) = 3$
3. 2 odd, 6 even: $d(M) = 3$
4. 3 odd, 4 even, the odd lies on a line: $d(M) = 5$
5. 3 odd, 4 even, the odd multiplicity vectors span the space: $d(M) = 3$

And the mirror images where we switch the number of evens with the number of odds. Note that cases 2, 4, 5 and the switched parity analogues are in the generic case, while 3 and its mirror case are not.

For $r = 3$ and an arbitrary set of generators, we can apply the methodology from Section 7 to get the following result:

Proposition 9.1. *For $r = 3$, let $d_1 \le d_2 \le \cdots \le d_7$ be all the powers of 2 in the nonzero eigenvalues of $L(G(\mathbb{F}_2^3, M)$ for M with reduced multiplicity (gcd of the multiplicities is 1). Let c_{top} be the top Sylow-2 cyclic factor. Then*

$$c_{top} = \begin{cases} 2^{d_7+1} & \text{not all } d_i \text{ equal} \\ 2^{d_7} & d_i = d_j \text{ for all } i, j \in \{1, \dots, 7\} \end{cases}$$

.

Proof. WLOG say that the eigenvalue λ_7 with $v_2(\lambda_7) = d_7$ corresponds to an element $u \in \mathbb{F}_2^3$ with $u_3 = 1$. Then we claim that $x_3 - 1$ has maximal additive order. In particular, we will minimize C over all v such that

$$C \cdot \frac{1}{2} \sum_{\substack{u \cdot v = 1 \\ u_3 = 1}} \frac{1}{\lambda_u} \in \mathbb{Z}_{(2)}$$

First, note $Pow_2(C)$ is bounded from above by 2^{d_7+1}, since we are taking $\frac{1}{2}$ times a sum of reciprocals of eigenvalues. The conditions $u \cdot v = 1, u_3 = 1$ for a fixed $v \ne 0, e_3$ are satisfied by 2 vectors in $\mathbb{F}_2^3$. Assume that λ_u is an eigenvalue with $u_3 = 1$ and $d_u < d_7$. Then there exists a unique vector v such that $u \cdot v = 1$, $u_3 = 1$ is only satisfied by the vector corresponding to λ_7 and u. Our sum then becomes

$$\frac{C}{2} \cdot \left(\frac{1}{\lambda_u} + \frac{1}{\lambda_7}\right) \in \mathbb{Z}_{(2)}$$

Since $v_2(\lambda) > v_2(\lambda_u)$, we must have $C \equiv 0 \mod 2^{d_7+1}$ for this equation to hold. Therefore, we achieve our upper-bound, and have the desired top cyclic factor.

In the case that all the d_i are equal, every choice of $v \ne v'$, yields a sum $\frac{1}{2}\sum_{u \cdot v = 1, u \cdot v' = 1} \frac{1}{\lambda_u} = \frac{1}{2}(\frac{1}{\lambda_1} + \frac{1}{\lambda_2})$ will always have order 2^{d_i}, since $v_2(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}) = v_2(\frac{1}{\lambda_2}) + 1 = -(d_i + 1)$. $\square$

In this generic case, using the method in Section 10 for trying to determine the 2nd largest cyclic factor, we are able to show

Theorem 9.2. *Let $G = G(\mathbb{F}_2^3, M)$ be in the generic case, and let $d_1 \le \cdots \le d_7$ be the powers of 2 in the eigenvalues of $L(G)$. Then*

$$Syl_2(K(G)) = \begin{cases} \mathbb{Z}/2^{d_5-1}\mathbb{Z} \times (\mathbb{Z}/2^{d_7+1}\mathbb{Z})^2 & d_6 = d_7 \\ \mathbb{Z}/2^{d_5}\mathbb{Z} \times \mathbb{Z}/2^{d_6}\mathbb{Z} \times \mathbb{Z}/2^{d_7+1}\mathbb{Z} & d_6 < d_7 \end{cases}$$

First, we prove the following lemma:

Lemma 9.3. *If M is in the generic case, then $d_1 = d_2 = d_3 = d_4 = 1$ and all the other eigenvalues have larger powers of 2.*

Proof. From above we know that the generic case us when there is either 1 odd and 6 even, 2 odd and 5 even, 3 odd and 4 even with the 3 odd being a basis, and their mirrors. In each, in the first case, if we assume a is odd, then there are four eigenvalues that have a as a summand. With all the other multiplicities being even, these eigenvalues are 2 mod 4. For the second case, say a, b are odd. Then $2(b+c+d+f), 2(a+c+d+e), 2(b+d+e+g), 2(a+d+f+g)$ are four eigenvalues containing one of a, b as summand, and must be 2-mod 4. For the third case, since the odds are a basis we can assume a, b, c are odd. Then $2(b+d+e+g), 2(c+e+f+g), 2(a+d+f+g), 2(a+b+c+g)$ are 4 eigenvalues that sum an odd number of odd values, so these eigenvalues are 2 mod 4. These calculations also imply that the other eigenvalues are 0 mod 4, since they are two times an even number. The case of the mirrors follows from adding 1 to each multiplicity, and noting the the eigenvalues remain invariant modulo $2(1+1+1+1) = 8$. □

10 Data for $d(M)$ for $r = 4$

For the $r = 4$ case, we perform some reductions in terms of the number of even multiplicities. Let the number of even multiplicities be denoted by ω, so that $\omega \in \{0, 2, \ldots, 14\}$ as there are $2^4 - 1 = 15$ non-trivial generators in the $r = 4$ case, and the case in which all of the generator multiplicities are even is reduced by section 4. Let the generators be given by

$$\begin{pmatrix} | & \cdots & \cdots & | \\ v_1 & \cdots & \cdots & v_{15} \\ | & \cdots & \cdots & | \end{pmatrix} = \begin{array}{c} \begin{array}{ccccccccccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

10.1 $\omega \le 2$

For $\omega = 0$, we have the complete graph with

$$\{\alpha_i\} = \{1, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16\} \quad \boxed{d(M_0) = 15}$$

In this case, we can assume by GL_4 action that v_1 has even multiplicty so that for our matroid of generators, M, with generator multiplicities satisfying

$$a_1 = 2, \qquad a_n = 1, \quad \forall 2 \le n \le 15$$

$$\implies \{\alpha_i\} = \{1, 1, 1, 1, 1, 1, 1, 9, 36, 288, 288, 288, 288, 288, 288\}, \qquad \boxed{d(M_1) = 7}$$

For $\omega = 2$, again GL action reduces it to the case when v_1, v_2 have odd multiplicity, so $a_1 = a_2 = 2$ and $a_n = 1$ for $3 \le n \le 15$, with

$$\{\alpha_i\} = \{1, 1, 1, 1, 1, 1, 1, 9, 36, 36, 36, 180, 720, 2880, 2880\} \implies \boxed{d(M_2) = 7}$$

10.2 $\omega = 3, 4$

When $\omega > 2$, we have to worry about whether or not the generators which have even multiplicity span a space of dimension 2, 3, 4, or more.

$\omega = 3$

In the $\omega = 3$ case, the vectors either span 2 dimensions or 3, and so it suffices to consider the cases of $a_1 = a_2 = a_4 = 2$ and $a_1 = a_2 = a_3 = 2$ with all other $a_i = 1$. The former case yields

$$a_4 = 2 \implies \{\alpha_i\} = \{1, 1, 1, 1, 1, 1, 1, 1, 2, 90, 360, 360, 360, 3960, 31680\}, \qquad \boxed{d(M_{3,3}) = 7}$$

$$a_3 = 2 \implies \{\alpha_i\} = \{1,1,1,5,20,20,20,20,20,20,20,20,80,320,320\} \qquad \boxed{d(M_{3,2}) = 11}$$

In the $\omega = 4$ case, the vectors with even multiplicity can either span a space of dimension 3 or 4. If they span 4 dimensions, then WLOG, we can assume that they are the standard basis vectors so

$$a_1 = a_2 = a_4 = a_8 = 2 \implies \{\alpha_i\} = \{1,1,1,1,1,1,1,1,2,10,120,3960,3960,3960,15840\} \qquad \boxed{d(M_{4,4}) = 7}$$

when they span 3 dimensions, then by GL equivalence they lie in the space spanned by v_1, v_2, v_4, so assume $a_1 = a_2 = a_4 = 2$. This yields at most 4 cases in which we choose one of a_3, a_5, a_6, or a_7 to be equal to 2. The first three choices are equivalent, as v_3, v_5, v_6 all represent vectors that are sums of two of the standard basis vectors, and hence are GL equivalent, thus we consider

$$a_1 = a_2 = a_4 = a_3 = 2 \implies \{\alpha_i\} = \{1,1,1,1,1,1,1,1,2,110,440,440,440,3960,31680]\} \qquad \boxed{d(M_{4,3,1}) = 7}$$

as well as

$$a_1 = a_2 = a_4 = a_7 = 2 \implies \{\alpha_i\} = \{1,1,1,5,20,20,20,20,20,20,20,20,80,240,960\} \qquad \boxed{d(M_{4,3,2}) = 11}$$

10.3 $\omega = 5$

The 5 generators in question could span either 3 or 4 dimensions.
dim $= 3$
We assume that $a_1 = a_2 = a_4 = 2$, leaving $\binom{4}{2} = 6$ choices to make as to which of the other generator multiplicities from the set $\{a_3, a_5, a_6, a_7\}$ should be 2. Clearly

$$\{a_3 = a_5 = 2\} \cong \{a_3 = a_6 = 2\} \cong \{a_5 = a_6 = 2\}$$

for we can act by the permutations $(3,4)$ and $(2,3,4)$ realized as matrices in $GL_4(\mathbb{F}_2)$ to get equivalence. Similarly

$$\{a_3 = a_7 = 2\} \cong \{a_5 = a_7 = 2\} \cong \{a_6 = a_7 = 2\}$$

by permutations $(2,3)$ and $(3,4)$. Note that the first above cases are equivalent under multiplication by

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so there's actually only one case:

$$\{a_1 = a_2 = a_4 = a_3 = a_5 = 2\} \implies \{\alpha_i\} = \{1,1,1,1,1,1,1,11,44,44,44,220,880,2640,10560\} \qquad \boxed{d(M_{5,3,1}) = 7}$$

dim $= 4$
In this case, we can assume that $a_1 = a_2 = a_4 = a_8 = 2$, so it suffices to consider the 11 cases in which we choose any of the remaining generators to be 2. Again from GL action we see that the choice of $a_i = 2$ is equivalent to $a_j = 2$ if v_i and v_j are both sums of k standard basis vectors for the same number k. Thus (suppressing the notation that $\{a_1 = a_2 = a_4 = a_8 = 2\}$), we have

$$\{a_3 = 2\} \cong \{a_4 = 2\} \cong \{a_6 = 2\} \cong \{a_9 = 2\} \cong \{a_{10} = 2\} \cong \{a_{12} = 2\}$$

and separately

$$\{a_7 = 2\} \cong \{a_{11} = 2\} \cong \{a_{13} = 2\} \cong \{a_{14} = 2\}$$

and the case of $\{a_{15} = 2\}$ is isolated. With this, we have

$$\{a_3 = 2\} \implies \{\alpha_i\} = \{1,1,1,1,1,1,1,1,2,22,132,660,5280,15840,15840\} \qquad \boxed{d(M_{5,4,1}) = 7}$$

$$\{a_7 = 2\} \implies \{\alpha_i\} = \{1,1,1,1,1,1,1,1,2,110,440,440,440,1320,205920\} \qquad \boxed{d(M_{5,4,2}) = 7}$$

$$\{a_{15} = 2\} \implies \{\alpha_i\} = \{1,1,1,1,1,5,20,20,20,20,120,480,480,480,480\} \qquad \boxed{d(M_{5,4,3}) = 9}$$

10.4 $\omega = 6$

dim $= 3$
We assume $a_1 = a_2 = a_4 = 2$, and it remains to choose 3 generators from the set $\{v_3, v_5, v_6, v_7\}$. Note that by a permutation of coordinates $(2,3,4)$ via GL action, we can assume that $a_3 = 2$, leaving only 3 cases.

$$\{a_5 = a_6 = 2\}, \qquad \{a_5 = a_7 = 2\}, \qquad \{a_6 = a_7 = 2\}$$

Note that the latter two cases are equivalent by nature of the permutation $(3,4)$, and the first and last case are equivalent by action by

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

so the only case is

$$\{a_5 = a_6 = 2\} \implies \{\alpha_i\} = \{1,1,1,1,1,1,1,11,44,132,528,528,528,528,2112\}, \qquad \boxed{d(M_{6,3,1}) = 7}$$

dim $= 4$
WLOG, $a_1 = a_2 = a_4 = a_8 = 2$, leaving $\binom{11}{2} = 55$ cases to reduce. From here on we abbreviate the set $\{a_{i_1} = \cdots = a_{i_k} = 2\}$ by the indices $\{i_1, \ldots, i_k\}$and just list groups of indices as opposed to writing the $\cong$ sign. We find all collections of generators whose multiplicities are equivalent under GL action by explicit computation. There are 24 equivalent cases as such

$$\{3,5\},\{3,9\},\{6,10\},\{5,12\},\{3,6\},\{9,10\},\{3,10\},\{6,12\},\{9,12\},\{5,6\},\{10,12\},\{5,9\}$$
$$\{3,7\},\{3,11\},\{6,14\},\{12,13\},\{11,14\},\{6,14\},\{9,11\},\{10,11\},\{12,14\},\{5,13\},\{5,7\},\{9,13\}$$

$$\{3,5\} \implies \{\alpha_i\} = \{1,1,1,1,1,1,1,1,2,66,264,528,528,5280,205920\} \qquad \boxed{d(M_{6,4,1}) = 7}$$

and a separate 3 equivalent cases here

$$\{3,12\},\{6,9\},\{5,10\}$$

$$\{3,12\} \implies \{\alpha_i\} = \{1,1,1,1,1,1,24,24,24,120,120,480,480,480,480\} \qquad \boxed{d(M_{6,4,2}) = 9}$$

and another 22-equivalent cases

$$\{3,14\},\{6,13\},\{11,12\},\{10,13\},\{7,9\},\{7,12\},\{5,11\},\{9,14\},\{3,13\},\{6,11\},\{5,14\}$$
$$\{3,15\},\{5,15\},\{6,15\},\{9,15\},\{10,15\},\{12,15\}$$
$$\{7,15\},\{11,15\},\{13,15\},\{14,15\},\{6,10\}$$

$$\{3,14\} \implies \{\alpha_i\} = \{1,1,1,1,1,1,1,1,2,22,44,220,5280,68640,68640\} \qquad \boxed{d(M_{6,4,3}) = 7}$$

And finally 6 equivalent cases

$$\{7,11\},\{7,13\},\{7,13\},\{7,14\},\{11,13\},\{13,14\}$$

$$\{7,11\} \implies \{\alpha_i\} = \{1,1,1,1,1,1,1,11,44,44,44,44,1320,5280,36960\} \qquad \boxed{d(M_{6,4,5}) = 7}$$

10.5 $\omega = 7$

dim $= 3$
Again, assume $a_1 = a_2 = a_4 = 2$ by GL action, then we must have in fact that $a_i = 2$ for $1 \le i \le 7$, yielding

$$\{\alpha_i\} = \{1,3,24,24,24,24,24,24,24,24,24,24,24,24,192\} \qquad \boxed{d(M_{7,3,1}) = 13}$$

dim $= 4$
WLOG, $a_1 = a_2 = a_4 = a_8 = 2$, so we have $\binom{11}{3} = 165$ cases to reduce. The following 16 cases are equivalent

$$\{6,10,12\},\{5,9,12\},\{3,5,6\},\{3,9,10\},\{10,12,14\},$$
$$\{9,12,13\},\{5,6,7\},\{3,10,11\},\{6,12,14\},\{5,9,13\},\{5,12,13\},$$
$$\{3,6,7\},\{3,9,11\},\{6,10,14\},\{3,5,7\},\{9,10,11\}$$

$$\{6,10,12\} \implies \{\alpha_i\} = \{1,1,1,1,1,1,1,3,6,48,48,528,6864,6864,20592\}, \qquad \boxed{d(M_{7,4,1}) = 7}$$

The following 72 cases are equivalent

$$\{7,10,13\},\{7,9,14\},\{5,11,14\},\{6,11,13\},\{7,11,12\},\{3,13,14\}$$
$$\{6,11,12\},\{5,11,12\},\{3,6,13\},\{3,9,14\},\{3,5,14\},\{6,10,13\},$$
$$\{7,9,12\},\{5,9,14\},\{3,10,13\},\{7,9,10\},\{7,10,12\},\{5,6,11\}$$
$$\{6,10,15\},\{5,9,15\},\{3,5,15\},\{9,10,15\},\{3,6,15\},\{10,12,15\},\{5,12,15\},$$
$$\{9,12,15\},\{5,6,15\},\{3,9,15\},\{6,12,15\},\{3,10,15\},\{7,10,14\},$$
$$\{7,9,13\},\{5,7,11\},\{10,11,13\},\{6,7,11\},\{11,12,14\},\{5,13,14\},\{11,12,13\},$$
$$\{6,7,13\},\{9,11,14\},\{3,11,13\},\{6,13,14\},\{3,11,14\},\{5,7,14\},\{9,13,14\},\{7,12,13\},$$
$$\{7,10,11\},\{6,11,14\},\{10,13,14\},\{3,7,14\},\{7,12,14\},\{5,11,13\},\{7,9,11\},\{3,7,13\}$$
$$\{13,14,15\},\{7,14,15\},\{7,11,15\},\{7,13,15\},\{11,13,15\},\{11,14,15\}$$
$$\{6,13,15\},\{5,14,15\},\{3,14,15\},\{7,9,15\},\{3,13,15\},\{7,10,15\},\{11,12,15\},$$
$$\{5,11,15\},\{9,14,15\},\{10,13,15\},\{6,11,15\},\{7,12,15\}$$

$$\{7,10,13\} \implies \{\alpha_i\} = \{1,1,1,1,1,1,1,1,2,22,264,528,528,13728,480480\} \qquad \boxed{d(M_{7,4,2}) = 7}$$

A separate 63 cases occur:

$$\{6,9,15\},\{5,10,15\},\{3,12,15\}\{10,14,15\},$$
$$\{9,13,15\},\{5,7,15\},\{10,11,15\},\{6,7,15\},\{12,14,15\},$$
$$\{5,13,15\},\{12,13,15\},\{9,11,15\},\{3,11,15\},\{6,14,15\},\{3,7,15\}$$
$$\{3,5,11\},\{3,6,11\},\{9,10,13\},\{5,12,14\},\{9,10,14\},\{3,9,13\},\{6,7,10\},\{3,10,14\},\{9,12,14\},$$
$$\{6,12,13\},\{5,7,12\},\{5,7,9\},\{6,7,12\},\{10,12,13\},\{6,10,11\},$$
$$\{3,6,14\},\{5,6,14\},\{3,7,9\},\{5,9,11\},\{10,11,12\},\{3,5,13\},\{3,7,10\},\{5,6,13\},\{9,11,12\}$$
$$\{6,7,9\},\{5,7,10\},\{3,11,12\},\{6,9,13\},\{5,10,11\},\{3,12,14\},$$
$$\{6,9,11\},\{5,10,13\},\{5,10,14\},\{3,12,13\},\{6,9,14\},\{3,7,12\}$$
$$\{5,10,12\},\{6,9,12\},\{5,6,10\},\{3,5,10\},\{5,6,9\},\{3,6,12\},$$
$$\{5,9,10\},\{3,5,12\},\{3,6,9\},\{3,10,12\},\{6,9,10\},\{3,9,12\}$$

$$\{6,9,15\} \implies \{\alpha_i\} = \{1,1,1,1,1,1,1,1,2,2,24,6864,6864,68640,68640\} \qquad \boxed{d(M_{7,4,3}) = 7}$$

And a separate 4 cases

$$\{11,13,14\},\{7,13,14\},\{7,11,14\},\{7,11,13\}$$

$$\{11,13,14\} \implies \{\alpha_i\} = \{1,1,1,1,1,1,1,3,66,528,528,528,528,528,2640\} \qquad \boxed{d(M_{7,4,3}) = 7}$$

And a separate 10 cases

$$\{3,7,11\},\{9,11,13\},\{12,13,14\},\{10,11,14\},\{6,7,14\},$$
$$\{5,7,13\},\{9,10,12\},\{5,6,12\},\{3,6,10\},\{3,5,9\}$$

$$\{3,7,11\} \implies \{\alpha_i\} = \{1,1,1,1,6,24,24,24,24,24,24,48,240,480,3360\} \qquad \boxed{d(M_{7,4,5}) = 11}$$

This accounts for all 165 cases.

10.6 $\omega \geq 8$

The same process can be repeated to collect further data. There is a symmetry that should be noted: it suffices to use to above equivalences and check the above cases *when* 1 *and* 2 *are switched in the multiplicities*, so all of the case work and equivalence groupings has been done, and we present the few cases of interest. Via proposition 7.4, we immediately learn the value of $d(M)$ for $\omega \geq 8$, and in the case that $\omega = 15$, we apply lemma 5.1 to get that

$d(M) = 15$. As an example, here is $\omega = 8$ with the odd vectors spanning a space of dimension 3. The lemma dictates that $d(M) = 3$ and indeed

$$\{a_i = 1 \; \forall 1 \leq i \leq 7\} \implies \{\alpha_i\} = \{1, 3, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 384\} \quad \boxed{d(M_{8,3,1}) = 13}$$

The proposition is used to complete the rest of the data, indicating the following conjectures

Conjecture 10.1. *For a matroid, M, yielding a connected Cayley graph on $\mathbb{F}_2^r$, $d(M) \geq 2^{r-1} - 1$ with equality occurring iff $\sum_{i=1}^n v_i \neq \vec{0}$.*

Conjecture 10.2. *$d(M)$ is odd unless all of the eigenvalues have the same power of 2, in which case $d(M) = 2^n - 2$.*

11 The Complete Description of the Top Cyclic Factor of Q_n

In this section, we will use the techniques developed in the last section to prove the following theorems:

Theorem 11.1. *For $n \geq 2$, let c_n be the size of the largest cyclic factor in $K(Q_n)$. Then,*

$$v_2(c_n) = \max\{\max_{x<n}\{v_2(x) + x\}, v_2(n) + n - 1\}.$$

Theorem 11.2. *For $n \geq 3$, the 2^{nd} to the $(n-1)^{th}$ largest cyclic factor in $K(Q_n)$ all have the same size d_n. Moreover,*

$$v_2(d_n) = \max_{x<n}\{v_2(x) + x\}.$$

Conjecture 11.3. *For $n \geq 3$, let e_n be the size of the n^{th} largest cyclic factor in $K(Q_n)$. Then,*

$$v_2(e_n) = \max\{\max_{x<n-1}\{v_2(x) + x\}, v_2(n-1) + n - 3\}.$$

Similarly, for $n \geq 4$, let f_n be the size of the $(n+1)^{th}$ largest cyclic factor in $K(Q_n)$. Then,

$$v_2(f_n) = \max_{x<n-1}\{v_2(x) + x\}.$$

Lemma 11.4. *c_n is the minimum integer such that for any $2 \leq a \leq n$ and $1 \leq b \leq a$,*

$$\frac{c_n}{2^{n-a}} \sum_{i=0}^{n-a} \frac{\binom{n-a}{i}}{2(b+i)} \in \mathbb{Z}.$$

Proof. This follows from Corollary 7.7.1 and the fact that when $G = Q_n$, $\lambda_u = 2w(u)$ (recall $w(u)$ is the number of 1s in vector $u \in \mathbb{F}_2^r$). In particular, there are $\binom{n-a}{i}$ ways to choose i 1's in a vector given at first a size a subvector. $\square$

Theorem 11.5. *(Kummer's Theorem) For any non-negative integers $a \geq b$,*

$$v_2\left(\binom{a}{b}\right) = \text{number of carries when adding } a - b \text{ to } b \text{ in binary.}$$

For example,

$$\begin{array}{rccccccccccl} & & & & ^1 & ^1 & ^1 & ^1 & ^1 & ^1 & & \\ & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & & 317 \\ + & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & & 183 \\ \hline & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & & 500 \end{array}$$

Therefore, $v_2(\binom{500}{317}) = 6$, since there are 6 carries.

Lemma 11.6. *For any $p \geq 1$, $q \geq 0$, assume u is the unique element in the interval $[p, p+q]$ that maximizes $v_2(u)$, then*

$$v_2\left(\sum_{i=0}^{q} \frac{\binom{q}{i}}{p+i}\right) = v_2\left(\frac{\binom{q}{u-p}}{u}\right).$$

Proof. First we claim that for any $c \in [p, p+q]$,

$$v_2\left(\frac{\binom{q}{c-p}}{c}\right) \leq v_2\left(\frac{\binom{q}{u-p}}{u}\right).$$

This is because the binary form of $c-p$ and $u-p$ are the same in the last $v_2(c)$ bits. Denote $k := v_2(u)$. Then $q < 2^{k+1}$. Therefore, in the remaining $k - v_2(c)$ bits, the maximal number of carries possible is $u-p$, with a carry on every single bit and no carries from $c-p$. By Kummer's Theorem, this worst scenario exactly results in equality. For example, when $p = 134$, $q = 101$, we have $u = 192$ and $k = 6$. Now we analyze the case when $c = 168$, $v_2(c) = 3$. In the two vertical additions, last $v_2(c) = 3$ bits (indicated by the red box) are identical, and therefore the first 3 carries (indicated by the blue box) are identical. The number of carries differ in at most 3, which is the same as $k - v_2(c)$.

$$\begin{array}{cccccccc|c}
 & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & {\scriptstyle 1} & & & \\
 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & u-p=58 \\
+ & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 43 \\
\hline
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & q=101
\end{array}
\qquad
\begin{array}{cccccccc|c}
 & & & & & {\scriptstyle 1} & & & \\
 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & c-p=34 \\
+ & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 67 \\
\hline
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & q=101
\end{array}$$

Now we proceed to show that equality will be achieved an odd numbers of times. According to the above analysis, equality occurs when $c-p$ does not carry in the highest $k - v_2(c)$ bits. However, we can negate the top bit of $c-p$. We reconsider the example above where $c = 168$. In such case, we can switch the top bit (indicated by the orange box) of $c = 168$ to get $c' = c + 2^k = 232$, such that $v_2(\binom{101}{34}) = v_2(\binom{101}{98})$.

$$\begin{array}{cccccccc|c}
 & & & & & {\scriptstyle 1} & & & \\
 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & c-p=34 \\
+ & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 67 \\
\hline
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & q=101
\end{array}
\qquad
\begin{array}{cccccccc|c}
 & & & & & {\scriptstyle 1} & & & \\
 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & c'-p=98 \\
+ & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 \\
\hline
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & q=101
\end{array}$$

Therefore, there is a pairing of $c + 2^k - p$ and $c - p$ when $c \neq u$. This ends the proof. $\square$

Now we have all the tools we need to calculate c_n. Assume $u = 2^k$ is the largest power of 2 smaller or equal to n. We have

$$\begin{aligned}
v_2(c_n) &= \max_{\substack{2\leq a\leq n\\ 1\leq b\leq a}} \left\{ -v_2\left(\sum_{i=0}^{n-a} \frac{\binom{n-a}{i}}{b+i}\right) - a + n + 1 \right\} && \text{from Lemma 11.4} \\
&= \max\left\{ v_2(c_{n-1}), \max_{2\leq a\leq n} \left\{ -v_2\left(\sum_{i=0}^{n-a} \frac{\binom{n-a}{i}}{a+i}\right) - a + n + 1 \right\}, -v_2\left(\sum_{i=0}^{n-2} \frac{\binom{n-2}{i}}{1+i}\right) + n - 1 \right\} && \text{by induction} \\
&= \max\left\{ v_2(c_{n-1}), \max_{2\leq a\leq u} \left\{ -v_2\left(\sum_{i=0}^{n-a} \frac{\binom{n-a}{i}}{a+i}\right) - a + n + 1 \right\} \right\} && (*) \\
&= \max\left\{ v_2(c_{n-1}), \max_{2\leq a\leq u} \left\{ -v_2\left(\frac{\binom{n-a}{u-a}}{u}\right) - a + n + 1 \right\} \right\} && \text{from Lemma 11.6} \\
&= \max\left\{ v_2(c_{n-1}), n + k + 1 - \min_{2\leq a\leq u} \left\{ a + v_2\left(\binom{n-a}{u-a}\right) \right\} \right\}
\end{aligned}$$

The justification of the $(*)$ comes from the following 2 facts:

- $v_2\left(\sum_{i=0}^{n-a} \frac{\binom{n-a}{i}}{a+i}\right) \geq v_2\left(\sum_{i=0}^{n-u} \frac{\binom{n-u}{i}}{u+i}\right)$.
  This inequality is true since the right side of the equation equals $-k$ according to Lemma 11.6, and each term in the sum on the left side has $v_2 \geq -k$. This helps rule out all cases in $(*)$ where $a > u$.
- $v_2\left(\sum_{i=0}^{n-2} \frac{\binom{n-2}{i}}{1+i}\right) \geq v_2\left(\sum_{i=0}^{n-2} \frac{\binom{n-2}{i}}{2+i}\right)$. When $n = u$, the right side of the inequality equals $-k$ and is definitely no larger than the left side. When $n > u$, according to Lemma 11.6, the left side is $v_2(\binom{n-2}{u-1}) - k$, and the right

side is $v_2(\binom{n-2}{u-2}) - k$. Since $\binom{n-2}{u-1} = \frac{n-u}{u-1}\binom{n-2}{u-2}$, we have the inequality is true, and thus it helps ruling out the last term in $(*)$.

Claim 11.7.

$$\min_{2\le a\le u}\left\{a + v_2\left(\binom{n-a}{u-a}\right)\right\} = \min_{2\le a\le u}\{a+k-v_2(n-a+1), 2+k-v_2(n)\}.$$

Assume the binary expansion of n is $n = 2^{p_1}+2^{p_2}+\cdots+2^{p_d}$ for $0 \le p_1 < p_2 < \cdots < p_d$. Denote $n_i = 2^{p_1}+2^{p_2}+\cdots+2^{p_i}$ for $i = 1, 2, \ldots, d-1$ and $n_d = u = 2^{p_d}$. We only need to prove the following two subclaims:

1. For any $i = 1, 2, \ldots, d-1$, we have

$$\min_{n_i<a\le n_{i+1}}\left\{a + v_2\left(\binom{n-a}{u-a}\right)\right\} = n_i+1+k-p_{i+1} = \min_{n_i<a\le n_{i+1}}\{a+k-v_2(n-a+1)\}.$$

   The second equation comes from the fact that $v_2(n-a+1) \le p_{i+1}$ for $a \in (n_i, n_{i+1}]$ and minimum is reached when $a = n_i+1$. The first equation comes from the fact that when subtracting $u-a$ from $n-a$, since $n-a$ and $n-u$ are the same in all but the last $p_{i+1}+1$ bits, and $u-a$ have all 1 except in the last $p_{i+1}+1$ bits, there is always $k-p_{i+1}$ carries in the first $k-p_{i+1}$ bits. By Kummer's Theorem, $v_2\left(\binom{n-a}{u-a}\right) \ge k-p_{i+1}$, and equality is achieved when $a = n_i+1$.

2. When n is even, we have

$$\min_{2\le a\le 2^{p_1}}\left\{a + v_2\left(\binom{n-a}{u-a}\right)\right\} = 2+k-p_1 = \min_{2\le a\le 2^{p_1}}\{a+k-v_2(n-a+1), 2+k-v_2(n)\}.$$

   The second equation comes from the fact that $v_2(n-a+1) \le p_1$ for $a \in [2, 2^{p_1}]$ and the minimum is reached when $a = 2$. For the first equation with the same argument as in case 1, we have $v_2\left(\binom{n-a}{u-a}\right) \ge k-p_1$, and equality is achieved when $a = 2$.

By combining the two cases, we have the claim, and using the formula from before:

$$v_2(c_n) = \max_{2\le a\le n}\{v_2(x)+x, v_2(n)+n-1\}$$

Corollary 11.7.1. *For $n = 2^k-1$, the top $n-1$ Sylow-2 cyclic factors have exponent 2^k-1*
For $n = 2^k$, the top Sylow-2 cyclic is 2^k+k-1. The 2nd through $n-1$st Sylow-2 cyclic factors are 2^k-1. For $n = 2^k+1$, the top $n-1$ Sylow-2 cyclic factors are 2^k+k. These last two statements imply the bound of $r+\lfloor\log_2 n\rfloor-1$ is asymptotically sharp over all Cayley graphs.

We would now like to work towards the proof of 11.2. We already have a strategy for finding the top cyclic factor of $K(G)$, but now we would like to find a strategy for finding the 2nd largest cyclic factor of $K(G)$. Let G be any Cayley graph of $\mathbb{F}_2^r$. Note that we are try to find the maximal additive order in $\mathbb{Z}[x_1, \ldots, x_r]/(x_1^2-1, \ldots, x_r^2-1, n-\sum x_{v_i})/\langle x_1-1\rangle$. The elements of finite order will still be elements that are in the kernel of the map $x_i \to 1$, and we will be able to write any element as a sum of the elements of the form x_i-1 for $2 \le i \le r$. Thus, the second largest cyclic factor will correspond again to the maximal order element of the for x_i-1 in the new quotient group. In general, knowing which i to choose is very hard, but for the hypercube symmetry implies that we can just choose x_2-1.
To find the order of the 2nd largest cyclic factor, we solve the equation $L(G)v = C(x_1-1)+D(x_2-1)$ be $C, D \in \mathbb{Z}$ and D as small as possible. So we have this strange new parameter C. However, the symmetry of the hypercube can yield the following nice result

Lemma 11.8. *For $2 \le k \le n-1$, the kth largest cyclic factor will correspond to the largest D such that there exists an integer vector v with $L(G)v = D(x_k-x_1)$. In the notation above, this means we can choose $C = -D$.*

Proof. First we will deal with the case $k = 2$. The second largest cyclic is the smallest positive integer C such that $k(x_1-1)+C(x_2-1) \in \mathrm{Im}(L(G))$. Note, however, by symmetry that if $k(x_1-1)+C(x_2-1) \in \mathrm{Im}(L(G))$ then $k(x_1-1)+C(x_3-1) \in \mathrm{Im}(L(G))$. Therefore, $C(x_2-x_1) \in \mathrm{Im}(L(G))$. Conversely, if $C(x_2-x_1) = -C(x_1-1)+C(x_2-1) \in \mathrm{Im}(L(G))$ then we can take $k = C$ and so the second largest factor must be the order of x_2-x_1.

For general k, We wish to solve for the minimal C_k such that there exists costants $r_1, \ldots, r_{k-1}$ such that $r_1(x_1 - 1) + \cdots + r_{k-1}(x_{k-1} - 1) + C_k(x_k - 1) \in \mathrm{Im}(L(G))$. Since $k \leq n-1$, x_n is not amongst $x_1, \ldots, x_k$, and so by symmetry we have that $r_1(x_n - 1) + r_2(x_2 - 1) + r_3(x_3 - 1) + \cdots + r_{k-1}(x_{k-1} - 1) + C_k(x_k - 1) \in \mathrm{Im}(L(G))$ and $r_1(x_n - 1) + r_2(x_2 - 1) + r_3(x_3 - 1) + \cdots + r_{k-1}(x_{k-1} - 1) + C_k(x_1 - 1) \in \mathrm{Im}(L(G))$. subtracting yields $C_k(x_k - x_1) \in \mathrm{Im}(L(G))$, which implies that C_k must just be the order of $x_k - x_1$, as desired.

□

Note that the order of $x_k - x_1$ is just the order of $x_k x_1 - 1$, since $x_1^2 = 1$. By symmetry, all these elements have the same additive order, so this lemma implies that the 2nd through $(n-1)$st largest cyclic factors are all the same. It would thus suffice to compute the 2nd largest cyclic factor.

Proof of Theorem 11.2. Using 7.5, we want to find the minimal C such that

$$\frac{C}{2^{n-2}} \sum_{\substack{u \cdot v = 1 \\ u \cdot (e_1 + e_k = 1}} \frac{1}{\lambda_u}$$

. We will be once again using Lemma 7.7, which tells us we need to find minimal C such that $\frac{C}{2^n - |S|} \sum_{\substack{u_S = d \\ u \cdot (e_1 + e_k)}} \frac{1}{\lambda_u} \in \mathbb{Z}_{(2)}$. We now want to find an analogue of 11.4. We can rewrite this relation as

$$\frac{C}{2^n - |S|} \sum_{\substack{u_S = d \\ u \cdot (e_1 + e_k) = 1}} \frac{1}{\lambda_u} = \frac{C}{2^n - |S|} \sum_{\substack{u_S = d \\ u \cdot e_1 = 1 \\ u \cdot e_k = 0}} \frac{1}{\lambda_u} + \frac{C}{2^n - |S|} \sum_{\substack{u_S = d \\ u \cdot e_1 = 0 \\ u \cdot e_k = 1}} \frac{1}{\lambda_u}$$

Then note that when choosing a specific fixed subvector, the conditions $u \cdot e_1 = 0, u \cdot e_k = 1$ and $u \cdot e_1 = 1, u \cdot e_k = 0$ cannot both happen at the same time, so one of the sums will be empty. For the other sum if we let our fixed subvector have size a with b 1's, then the number of vectors corresponding to eigenvalue $2(b+i)$ is the number of ways to choose i 1's from $n-a$ slots. This calculation yields the same sum as in 11.4:

$$\frac{C}{2^{n-a}} \sum_{i=0}^{n-a-1} \frac{\binom{n-a}{i}}{2(b+i)}$$

However, in this case we must restrict to the case where either $a > 2$ and $b \geq 1$, or $a = 2$ and $b = 1$. A fixed subvector with $a = b = 2$ is impossible because we need to either specify $u \cdot e_1 = 1, u \cdot e_k = 0$ or $u \cdot e_1 = 0, u \cdot e_k = 1$, both of which only have a single 1. Then following the calculation for Theorem 7.7.1, we include all cases except $a = b = 2$, which yields the number $v_2(n) + n - 1$. Therefore, our factor is just equal to the max over the cases when $a > 2$ and $b = 1$, which is

$$\max_{x<n} \{v_2(x) + x\}$$

as desired. □

12 Lower bounds on the top cyclic factor

Let c_{top} be the largest cyclic factor of the sandpile group associated to some Cayley graph $G(\mathbb{F}_2^r, M)$. In the case that $G = Q_n$, our above result implies that $c_{top} \leq n$, with equality holding when $n = 2^k - 1$. One may ask if we can find a more general bound for arbitrary Cayley graphs. We first have the easy lemma from Rushanan:

Lemma 12.1 (Rushanan [8], Theorem 1)**.** *Let M be an $n \times n$ nonsingular integer matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and its Smith Normal Form $S = diag(s_1, \ldots, s_n)$. Then $\lambda_i \mid s_n$ for all i.*

Proof. There exists invertible integer matrices P, Q such $M = PSQ$, and since M is nonsingular we can write

$$M^{-1} = Q^{-1} S^{-1} P^{-1}$$

in $\mathrm{GL}_n(\mathbb{Q})$. By definition of S we then have that $s_n M^{-1}$ is an integer matrix, with rational eigenvalues s_n / λ_i. Since the matrix is integral, these eigenvalues must be integral, so $\lambda_i \mid s_n$. □

So by passing to the reduced Laplacian, $\overline{L(G)}^{i,i}$, we have $2^{\max_i v_2(\lambda_i)} \mid c_{top}$, with equality achieved for $G = Q_{2^k}$, so this is asymptotically sharp.

13 Inductive Growth of the Sandpile groups under Contraction

Instead of writing $K(Q_n) \cong \bigoplus_p \oplus_{e\geq 1}(\mathbb{Z}/p^e\mathbb{Z})^{m(p^e)}$, we return to the ring-theoretic decomposition so that

$$\mathbb{Z} \oplus K(Q_n) \cong \mathbb{Z}[x_1,\dots,x_n]/(x_1^2-1,\dots,x_n^2-1, n-\sum_{i=1}^n x_i)$$

In particular, we have the surjection

$$\mathbb{Z}[x_1,\dots,x_n]/(x_1^2-1,\dots,x_n^2-1,n-\sum_{i=1}^n x_i) \overset{"x_n=1"}{\twoheadrightarrow} \mathbb{Z}[x_1,\dots,x_{n-1}]/(x_1^2-1,\dots,x_{n-1}^2-1,(n-1)-\sum_{i=1}^{n-1} x_i)$$

$$\implies \mathbb{Z}\oplus K(Q_n) \overset{"x_n=1"}{\twoheadrightarrow} \mathbb{Z}\oplus K(Q_{n-1})$$

and so comparing the torsion components, we see that $K(Q_n)$ maps surjectively onto $K(Q_{n-1})$. In particular, let $S_n = \mathbb{Z}[x_1,\dots,x_n]/(x_1^2-1,\dots,x_n^2-1,n-\sum_{i=1}^n x_i)$, then returning to the decomposition with respect to p-primary components, we see that each component, $\mathbb{Z}/p^e\mathbb{Z}$, is generated by the image of some polynomial $\overline{f}\in S_n$, so that the cyclic factor that $\overline{f}$ is contained in must be of the form $\mathbb{Z}/p^{e+k}\mathbb{Z}$ for $k\geq 0$. In a sense, this tells us that the sand pile group is "growing". The same argument can be made in the general Cayley graph situation, for if we have two matrices of generators

$$M = \begin{pmatrix} | & \dots & | \\ v_1 & \dots & v_n \\ | & \dots & | \end{pmatrix}$$

where each $v_i \in \mathbb{F}_2^r$, then we consider the corresponding matrix achieved by projecting each generator to its first $r-1$ components.

$$M' = \pi_{r-1}(M) = \begin{pmatrix} | & \dots & | \\ \pi_{r-1}(v_1) & \dots & \pi_{r-1}(v_n) \\ | & \dots & | \end{pmatrix} = \begin{pmatrix} | & \dots & | \\ v_1' & \dots & v_n' \\ | & \dots & | \end{pmatrix}$$

which yields the following surjection

$$\mathbb{Z}[x_1,\dots,x_r]/\left(x_1^2-1,\dots,x_n^2-1,n-\sum_{i=1}^n\prod_{j=1}^r x_i^{(v_i)_j)}\right) \overset{"x_r=1"}{\twoheadrightarrow} \mathbb{Z}[x_1,\dots,x_{r-1}]/\left(x_1^2-1,\dots,x_{r-1}^2-1,n-\sum_{i=1}^n\prod_{j=1}^{r-1} x_i^{(v_i)_j}\right)$$

in which case we again compare the torsion components to see that cyclic factors of the sandpile group to which the map surjects, $\mathbb{Z}/p^e\mathbb{Z}$, can be viewed as subgroups of a larger cyclic factor, $\mathbb{Z}/p^{e+k}\mathbb{Z}$ in the domain sandpile group. The process of evaluating at $x_r = 1$, which is akin to removing the final row of the matroid, is called matroid *contraction*.

14 A Note on Groebner Bases

Part of this research effort was to explore the ring theoretic description of the sandpile group by looking at groebner bases of the ideal

$$I = (x_1^2-1,\dots,x_r^2-1,n-\sum_{i=1}^n\prod_{j=1}^r x_j^{(v_i)_j})$$

Grobner bases provide a potential approach to decomposing the sandpile group in the case of a Cayley graph on $\mathbb{F}_2^r$ with an aribtrary collection of n generators.

As an example, we have

$$r=2,\ n=4, \quad M = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \implies \mathbb{Z}\oplus K(G) \cong \mathbb{Z}[x_1,x_2]/(x_1^2-1,x_2^2-1,4-(2x_1+x_2+x_1x_2))$$

$$\overset{\mathbb{Z}\text{-mod}}{\cong} \mathbb{Z}[x_1,x_2]/(\underline{x_1^2}-1,\underline{x_1x_2}+2x_1+x_2-4,\underline{3x_1}+6x_2-9,\underline{12x_2}-12) \overset{\mathbb{Z}\text{-mod}}{\cong} \mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}/12\mathbb{Z}$$

This example is suggestive, so we explain the theory of groebner bases to justify the correspondence betwee "leading terms" of the groebner basis generators and the sandpile groups.

14.1 Groebner Bases over a field

For k a field, let $S = k[x_1, \ldots, x_n]$ and define

Definition 14.1. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ a multi-index, we define the monomial

$$x_\alpha := \prod_{i=1}^{n} x_i^{\alpha_i}$$

Definition 14.2. The graded reverse lexographic order, "degrevlex", is an ordering on the collection of monomials in S such that $I < J$ if:
a) $|I| < |J|$ for $|I| = \sum_i \alpha_i$, **OR**
b) $|I| = |J|$ and the first non-zero entry from the end of $J - I$ is positive.

Example:

$$x_1^2 > x_1x_2 > x_2^2 > x_1x_3 > x_2x_3 > x_3^2 > x_1 \leftrightarrow (2,0,0) > (1,1,0) > (0,2,0) > (1,0,1) > (0,1,1) > (0,0,2) > (1,0,0)$$

degrevlex is the standard monomial ordering taken for groebner bases due to ease of calculation.

Definition 14.3. For $f \in S$ with $f = \sum_\alpha a_\alpha x_\alpha = a_{\alpha_0} x_{\alpha_0} + g$ with α_0 corresponding to the largest multiindex under degrevlex, we define the leading term of f

$$LT(f) := a_{\alpha_0} x_{\alpha_0}$$

Definition 14.4. Given $I \subseteq S$, we define the leading term ideal

$$LT(I) = \langle \{LT(f) \mid f \in I\} \rangle$$

Definition 14.5. For $I \subseteq S$, a Grobner basis of I is a set $\{g_1, \ldots, g_m\} \subseteq S$ such that
a) $I = \langle g_1, \ldots, g_m \rangle$
b) $LT(I) = \langle LT(g_1), \ldots, LT(g_m) \rangle$

Theorem 14.6. *Given $\emptyset \neq I \subseteq S$ a Grobner basis always exists.*

Proposition 14.7. *Given $I \subseteq S$, $S/I \overset{k-mod}{\cong} S/LT(I)$ holds, where the isomorphism is as free k-modules, i.e. as vector spaces*

Note: this is significantly weaker than an isomorphism as quotient rings, which does not hold in general. See this work by Francis and Dukkipati [5] for a proof of these results.

14.2 Groebner bases over $\mathbb{Z}$

Groebner bases over rings that are not fields are much more difficult to calculate. In order for the theorems pertaining to polynomial rings over a field to hold for polynomial rings over $\mathbb{Z}$, we must have that the polynomials generating the ideal in question are monic. For if not, an analogous isomorphism to 14.7 over $\mathbb{Z}$ with finitely many finite cyclic factors is difficult, as seen in the following example

$$\mathbb{Z}[x]/(2x-1) \ncong \mathbb{Z}[x]/(2x)$$

as the former ring is isomorphic as a $\mathbb{Z}$-module to $\mathbb{Z}\left[\frac{1}{2}\right] \subseteq \mathbb{Q}$, which has no torsion, while the latter is isomorphic to $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$. With the above example, constructing an isomorphism to $\mathbb{Z}[x]/J \cong \mathbb{Z} \oplus K(G)$ for $K(G)$ a finite group and some ideal $J \subseteq \mathbb{Z}[x]$ will be impossible, and the leading coefficient of 2 lies at the heart of the problem. Moreover, due to generator multiplicity in the context of matroids of generators, we will rarely have all monic polynomials generating the ideal of interest in our ring theoretic approach to the sandpile group. Nonetheless, Francis and Dukkipati present the following statement in section 3.2:

Theorem 14.8. *For A a PID, consider an ideal $s \subseteq A[x_1, \ldots, x_n]$. Let $G = \{g_i\}_{i=1}^t$ be a groebner basis for a. Recall that*

$$J_{x_\alpha} := \{i : \ lm(g_i) \mid x_\alpha, \ g_i \in G\}, \qquad I_{J_{x_\alpha}} := \langle \{lc(g_i) : \ i \in J_{x_\alpha}\} \rangle$$

as in Adams and Loustaunau, chapter 4, p. 226 [9]. Refer to $I_{J_{x_\alpha}}$ as the leading coefficient ideal, and assume that A has effective coefficient representatives. Let $\{C_{J_{x_\alpha}}\}$ represent a set of coset representatives of the equivalence *classes in* $A/I_{J_{x_\alpha}}$. *Let* $f \in A[x_1, \ldots, x_n]$ *so that* $f \equiv \sum_{i=1}^{m} a_i x_{\alpha_i} \mod \langle G \rangle$ *with* $a_i \in A$. *If* $A[x_1, \ldots, x_n]/\langle G \rangle$ *is a finitely generated* A*-module of size* m*, then corresponding to coset representatives,* $C_{J_{x_{\alpha,1}}}, \ldots, C_{J_{x_{\alpha,m}}}$*, there exists an isomorphism*

$$\begin{aligned} \phi : & A[x_1, \ldots, x_n]/\langle G \rangle \to A/I_{J_{x_{\alpha,1}}} \times \cdots \times A/I_{J_{x_{\alpha,m}}} \quad s.t. \\ & \sum_{i=1}^{m} a_i x_{\alpha_i} + \langle G \rangle \mapsto (c_1 + I_{J_{x_{\alpha,1}}}, \ldots, c_m I_{J_{x_{\alpha,m}}}) \end{aligned}$$

For a proof of this, see p. 4 and 5 of [5], and note the modified definition of groebner basis in the paper.

A nice example in which this works is the following:

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned} \implies I_1 &= (x_1^2 - 1, x_2^2 - 1, x_3^2 - 1, x_4^2 - 1, -x_1 * x_2 * x_3 - x_1 - x_2 - x_3 - x_4 + 5) \\ \langle G_1 \rangle &= ([x_1^2 - 1, x_1x_2 + x_1x_3 + x_2x_3 + x_3x_4 - 5x_3 + 1, \\ & x_2^2 - 1, x_3^2 - 1, x_1x_4 - x_3x_4 - 5x_1 + 5x_3, x_2x_4 - x_3x_4 - 5x_2 + 5x_3, \\ & 2x_3x_4 + 2990x_3 + 21340x_4 - 24332, x_4^2 - 1, 24x_1 + 312x_3 + 354x_4 - 690, \\ & 24x_2 - 24x_3, 48x_3 - 3318x_4 + 3270, 60x_4 - 60) \end{aligned}$$

but sage (which was used to calculate the above), does not always produce the correct groebner bases under the definition used in Francis and Dukkipati's paper, as seen from this example

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned} I_2 &= x_1^2 - 1, x_2^2 - 1, x_3^2 - 1, x_4^2 - 1, -x_2x3x_4 - x_1 - x_2 - x_3 - x_4 + 5 \\ \langle G_2 \rangle &= (x_1^2 - 1, x_1x_2 - x_1x_4 - 5x_2 + 5x_4, x_2^2 - 1, x_1x_3 - x_1x_4 - 5x_3 + 5x_4, \\ & x_2x_3 + x_1x_4 + x_2x_4 + x_3x_4 - 5x_4 + 1, x_3^2 - 1, 2x_1x_4 + 16x_1 + 14x_4 - 32, \\ & x_4^2 - 1, 6x_1 + 624x_4 - 630, 24x_2 - 3384x_4 + 3360, 24x_3 - 3384x_4 + 3360, 480x_4 - 480) \end{aligned}$$

here, the two matroids produce the same sandpile group due to the matroid of generators being equivalent under $GL_4(\mathbb{F}_2)$ action, and indeed the underlying sandpile group is

$$K(G) = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/6\mathbb{Z}) \oplus (\mathbb{Z}/24\mathbb{Z})^4 \oplus (\mathbb{Z}/480\mathbb{Z})$$

However, if we were to apply the theorem 13.8 under the assumption that sage produced a groebner basis in accordance with Francis and Dukkipati's definition, then we'd have that by looking at all relevant leading terms of generating polynomial

$$\mathbb{Z}[x_1, \ldots, x_n]/I_1 \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/24\mathbb{Z})^4 \oplus (\mathbb{Z}/48\mathbb{Z}) \oplus (\mathbb{Z}/60\mathbb{Z}) \ncong \mathbb{Z} \oplus K(G)$$

as the former $\mathbb{Z}$-module has no element of order 480.

15 Remaining Conjectures

We list remaining conjectures we've gathered based on data

Conjecture 15.1. *When the greatest common divisor of all generator multiplicities is* 1*, the sandpile group depends only on the collection of eigenvalues, not their labellings.*

Conjecture 15.2. *Two Cayley graphs have the same sandpile group if and only if their generator multiplicities are the same up to GL-equivalence.*

Conjecture 15.3. *The 2-Sylow component of the sandpile group for Q_{2^n-1} and Q_{2^n} differs as follows: $Syl_2(K(Q_{2^n}))$ equals a top cyclic factor as determined in section 11 and then the remaining factors come from taking $Syl_2(K(Q_{2^n-1}))$ and doubling the multiplicity of each factor. That is, we have*

$$Syl_2(K(Q_{2^k}) \cong Syl_2(K(Q_{2^k-1}))^2 \times \mathbb{Z}/2^{2^k+k-1}\mathbb{Z}$$

16 Acknowledgements

This research was carried out as part of the 2018 REU program at the School of Mathematics at University of Minnesota, Twin Cities. The authors are grateful for the support of NSF RTG grant DMS-1745638 and making the program possible. The authors would like to thank Professor Victor Reiner for providing both guidance and independence in their research efforts. The authors would also like to thank Eric Stucky for his edits to this paper. Finally, the authors would like to especially thank Amal Mattoo for his contributions to our research.

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