Introduction

The sandpile group of a graph is a finite abelian group with combinatorial, algebraic, and geometric interpretations. We are interested in the sandpile group of hypercube graphs and their generalizations. Cayley graphs of the group $\mathbb{F}_2^d$ with arbitrary generating sets. While the Sylow-$p$ component of these sandpile groups has been classified for $p \neq 2$ [1], the Sylow-2 subgroup remains a mystery. In this project, we use linear algebra and ring theory to achieve the following results:

- A sharp upper bound for the largest Sylow-2 cyclic factor in the sandpile group of an arbitrary Cayley graph.
- An exact formula for the largest few Sylow-2 cyclic factors of the sandpile group of a hypercube graph.
- A full classification of the sandpile group for $r = 2$ and other enlightening results for small $r$.

Definitions

The Laplacian of an undirected graph $G$, denoted $L(G)$, has entries

$$L(G)_{ij} = \begin{cases} \deg(v_i) & i = j \\ \#edges from v_i to v_j & i \neq j \end{cases}$$

Fixing $r = 2$, and a set of generators $M = \{e_1, \ldots, e_n\}$ for $G$, we define the Cayley graph $G(\mathbb{F}_2^d, M)$ with vertices $V = \mathbb{F}_2^d$ and $u, v \in V(G)$ share an edge if $u - v = v_i$ for a generator $v_i \in M$. Multiple edges are allowed. When $r = 2$ and $M = \{v_1, \ldots, v_n\}$, $G(\mathbb{F}_2^d, M) = \mathbb{Q}_n$ is the hypercube graph.

By definition, the Laplacian $L(G)_{ij}$ is the number of edges between vertices $i$ and $j$ in the graph $G$. For $r = 2$, the Laplacian is always an integral matrix, so we can view it as an endomorphism of $\mathbb{Z}$-modules $\mathbb{Z}^d \to \mathbb{Z}^d$. When $G$ is connected, the kernel is span($1$), so the Laplacian is invertible. The determinant of the Laplacian is a polynomial in the adjacency matrix of $G$, and it provides important information about the graph.

Previous Results

Previously, it has been proven:

- For $p \neq 2$ we have $Syl_p(K(G)) \cong \mathbb{Z}/(p^{\deg(v_i)})\mathbb{Z}$.
- Let $c_i(G)$ denote the exponent of the $i$th largest cyclic factor in $Syl_i(K(G))$. Then $c_i(G) \leq n + \lfloor \log_2 n \rfloor$.

Main Results I: Largest Sylow-2 Factor

First, we generalize the Anzai-Prasad bound [2].

**Theorem (General Sharp Upper Bound):**

Given an arbitrary Cayley graph $G(\mathbb{F}_2^d, M)$, we have $c_i(G) \leq \lfloor \log_2(n) \rfloor + r - 1$, which is sharp when $G = K_{2^n}$.

This is proven by using the technique in Anzai-Prasad [2] to view $K(G)$ as a ring $\mathbb{Z} \oplus K(G) \cong \mathbb{Z}[x_1, \ldots, x_n]/(x_i^2 - 1, n - \prod_{i=1}^n x_i^{r_i})$.

Then $c_i(G)$ is the additive order of some $x_i$ in $K(G)$. By translating $x_i - 1$ back as vector in $\mathbb{Z} \oplus K(G)$, we have the following lemma:

**Lemma:** $c_i(G)$ is the smallest $C$ such that for any $S \subseteq \{i\}$, $|S| > 2$, $C \in \mathbb{F}_2^d \setminus \{0\}$

$$C = \sum_{v_i \in S} 1 \in \mathbb{Z}.$$

Note that when $G = K_{2^n}$, the values of $\lambda_i$ are $n - 2k$ with multiplicity $1$. By fully exploiting the lemma above, we determine the exact value of $c_i(G)$ using elementary number theory:

**Theorem (Exact Max Factor for Hypercubes):** For $G = K_{2^n}$, we have the exact formula $c_i(G) = \max \{\text{gcd}(v_2(x) + x, v_2(n) + n - 1)\}$.

We further generalized this result and proved that $c_i(G)$ is the additive order of some $x_i - 1$ for $2 \leq k \leq n$, and we use this to answer the following questions:

- Is the sandpile group of a hypercube group isomorphic to any other group?
- Does it have any interesting properties?

Main Results II: Sandpile group for small $r$

We can use the interpretation of $c_i(G)$ as a minimal $C$ to completely determine the sandpile group in small cases. For example, when $r = 2$, we have a complete classification:

**Theorem (Classification for $r = 2$):** For $M = \{1, \phi\}$ with $\text{gcd}(a, b, c) = 1$ then $Syl_2(K(G)) = \left\{ \begin{array}{ll} \mathbb{Z}_{2^{a+b+c+1}} & \text{odd } a, \text{ odd } b, \text{ even } c \\ \mathbb{Z}_{2^{a+b+c+1}} & \text{a odd, b even, c even} \\ \mathbb{Z}_{2^{a+b+c}} & \text{a odd, b odd, c odd} \end{array} \right.$

The Sylow-2 group for $r = 2$ has at most 2 factors, so it suffices to compute $c_i(G)$, which is what the proof amounts to.

For $r = 3$, $Syl_3(K(G))$ can have many more cyclic factors, but when $M$ is generic, i.e. when $\Sigma_{i=1}^n v_i \neq 0$, it turns out there are only 3 cyclic factors, which is the smallest number of factors. In this case, we have:

**Theorem (r = 3 ‘generic case’):** Suppose that $\sum_{i=1}^n v_i \neq 0$, and let $d_1 \leq \ldots \leq d_r$ be the powers of $2$ in the eigenvalues $\lambda_i$. Then $Syl_r(K(G)) = \left\{ \begin{array}{ll} \mathbb{Z}_{2^{d_r}} & d_r \leq d_i \\ \mathbb{Z}_{2^{d_r}} \times \mathbb{Z}_{2^{d_{r-1}}} & d_r > d_i \end{array} \right.$

The proof of this theorem involves an explicit computation of the largest and second largest cyclic factors for each of these sandpile groups, applying the similar techniques used for the hypercube. These computations uniquely determine the 3rd factor since we already know the order of $Syl_2(K(G))$.

Conclusion and Remaining Questions

- The Sylow 2 component of the Sandpile group appears to be extremely complex, based on our results about the top factors for the hypercubes.
- We conjecture that the eigenvalues of the Laplacian (given that the graph is reduced) uniquely determine the group.
- A mysterious conjecture is that $Syl_2(K(G_{2^n})) \cong Syl_2(K(G_{2^{n-1}}))^2 \times \mathbb{Z}_{2^{n}}$.

This could fit into an interpretation via graph coverings.

- Potential future approaches to this problem include Grobner bases, matroid deletion and contraction, and graph coverings.

References


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