FACTORIZATIONS OF COXETER ELEMENTS IN COMPLEX REFLECTION GROUPS

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ABSTRACT. In [CS12], Chapuy and Stump studied factorizations of Coxeter elements into products of reflections in well-generated, irreducible complex reflection groups, giving a simple closed form expression for their exponential generating function depending on certain natural parameters. In this work, their methods are used to consider a more general multivariate generating function $\mathrm{FAC}_W(u_1, \ldots, u_\ell)$ in which the number of reflections from each hyperplane orbit is recorded. This generating function takes a similar simple form which can be stated uniformly for well-generated irreducible complex reflection groups in terms of certain data associated to orbits of reflecting hyperplanes, and specializes to that of Chapuy and Stump. A consequence of this more refined generating function – that the Coxeter number of any well-generated complex reflection group W is determined by certain data associated to any W-orbit of reflecting hyperplanes– is stated, and a case-free argument is given for real reflection groups.

1. Introduction

A factorization of an element c in a complex reflection group W is a tuple of reflections $(r_1, \ldots, r_m)$ such that $c = r_1 \cdots r_m$. Let f_m denote the number of factorizations of length m of a fixed element c in the complex reflection group W. In [CS12], Chapuy and Stump used representation-theoretic machinery to compute the numbers f_m for c a Coxeter element in a well-generated irreducible complex reflection group, W, and to give a closed form for the exponential generating function

$$\mathrm{FAC}_W(t) := \sum_{n \geq 0} f_m \frac{t^m}{m!}.$$

In particular, by considering cases they concluded that for any well-generated complex reflection group W,

$$\mathrm{FAC}_W(t) = \left(e^{Nx/n} - e^{N^*x/n}\right)^n$$

for n the rank of W, N the number of reflections in W, and N^* the number of hyperplanes fixed by a reflection in W. In the crystallographic real case, Jean Michel also produced a case-free derivation of their identity using properties of Deligne-Lusztig representations [Mic16].

For a given well-generated, irreducible complex refleciton group W, let $\mathcal{R}$ be the set of reflections in W and let $\mathcal{R}^*$ be the set of hyperplanes fixed by some reflection in $\mathcal{R}$. In this note, we study the refinements

$$f_{m_1,\ldots,m_\ell} := \#\left\{(r_1, \ldots, r_m) \in \mathcal{R}_1^{m_1} \times \cdots \times \mathcal{R}_\ell^{m_\ell} : c = r_1 \cdots r_m\right\}$$

for $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_\ell$ a partition of $\mathcal{R}$ corresponding to the decomposition of $\mathcal{R}^*$ into W-orbits. More precisely, we consider $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_\ell$ obtained as follows.

Definition 1.1. Let W act on the set of hyperplanes $\mathcal{R}^*$ by right multiplication, and let $\mathcal{O}_1, \ldots, \mathcal{O}_\ell$ be the W-orbits of $\mathcal{R}^*$ under this action. Then, let

$$\mathcal{R}_i = \bigcup \{\mathrm{Stab}_W(H) \mid H \in \mathcal{O}_i\}$$

be the set of reflections which fix some hyperplane in $\mathcal{O}_i$.

We consider the more general multivariate generating function

$$\mathrm{FAC}_W(u_1, \ldots, u_\ell) = \sum_{m_1,\ldots,m_\ell \geq 0} f_{m_1,\ldots,m_\ell} \prod_{i=1}^{\ell} \frac{u_i^{m_i}}{m_i!}.$$

Our main result is the following closed form expression for $\mathrm{FAC}_W(u_1, \ldots, u_\ell)$.

Theorem 3.7. *Let W be a well-generated, irreducible complex reflection group. For invariants n_i, N_i, N_i^* associated to the classes $\mathcal{R}_i$ in W (see Section 3 for precise definitions),* $\mathrm{FAC}_W(u_1, \ldots, u_\ell)$ *has the closed form*

$$\mathrm{FAC}_W(u_1, \ldots, u_\ell) = \frac{1}{|W|} \prod_{i=1}^{\ell} \left(e^{\frac{N_i u_i}{n_i}} - e^{-\frac{N_i^* u_i}{n_i}} \right)^{n_i}$$

In section 2, we give some background on complex reflection groups and make explicit the representation-theoretic techniques used in the proof of the main result. Section 3 provides the statement of the main theorem, including definitions of the invariants n_i, N_i, and N_i^*. A consequence of this decomposition–that the Coxeter number of any well-generated irreducible complex reflection group W is determined by certain data associated to any W-orbit of reflections–is motivated and case-free arguments are supplied. Finally, the proof of Theorem 3.7 is given in cases, using the Shepard-Todd classification of well-generated, complex reflection groups.

2. Background

Here, we recall some basic notions relevant to the exposition. See [BMR97] or [LT09] for further details on complex reflection groups.

2.1. Complex Reflection Groups.

Definition 2.1. Let V be a finite-dimensional complex vector space of dimension n. A *complex reflection* is a linear transformation $T \in \mathrm{GL}(V)$ with finite order whose fixed space is a hyperplane in V. A *complex reflection group*, W, is a finite subgroup of $\mathrm{GL}(V)$ which is generated by complex reflections.

Definition 2.2. A complex reflection group $W \subseteq \mathrm{GL}(V)$ of rank n is *well-generated* if and only if it is generated by exactly n reflections. W is *irreducible* if there is no nontrivial W-invariant subspace $V_0 \subseteq V$.

Example 2.3. Let $W = G(r, p, n)$ for $p \mid r$ be the group of $n \times n$ monomial matrices with nonzero entries rth roots of unity whose product is a pth root of unity. For example, $B = G(2, 1, n)$ is the group of signed permutation matrices. The group $W = G(r, p, n)$ is well-generated and irreducible if and only if $p = 1$ or $r = p$.

Recall that $\mathcal{R}$ denotes the sets of reflections in W, and $\mathcal{R}^*$ denotes the set of hyperplanes in V fixed by some reflection in W. Note that $\mathcal{R}$ is closed under conjugation in W and so decomposes into a disjoint union of conjugacy classes. Let $\mathcal{C}_{\mathcal{R}}(W)$ be the set of conjugacy classes of reflections in W. Put $N = |\mathcal{R}|$ and $N^* = |\mathcal{R}^*|$.

Definition 2.4. Let V be an n-dimensional complex vector space, and let $W \subseteq \mathrm{GL}(V)$ be a well-generated, irreducible complex reflection group. The *Coxeter number* of W is the constant

$$h = \frac{N + N^*}{n}$$

Definition 2.5. A *ζ-regular element* is an element $c \in W$ with eigenvalue ζ such that, for some eigenvector v corresponding to c, v is not contained in any hyperplane in $\mathcal{R}^*$. A *Coxeter element* is a ζ_h-regular element of W where ζ_h is a primitive hth root of unity.

Remark 2.6. There are several inequivalent definitions of Coxeter elements in the literature. The one given here has the advantage of being more general than that offered for example in [Bes06].

Proposition 2.7 ([CS12], prop 2.1)**.** *Let W be a complex reflection group. An element $c \in W$ is a Coxeter element if and only if it has a primitive hth root of unity as an eigenvalue.*

Definition 2.8. A *Coxeter group* is a group W together with elements $r_1, \dots, r_n \in W$ and $m_{ij} \in \mathbb{N}$ with $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$ such that W admits the presentation

$$W = \langle r_1, \dots, r_n \mid (r_i r_j)^{m_{ij}} = 1 \rangle$$

A *Shephard group* is a reflection group W with generators $r_1, \dots, r_n \in W$ and associated integers $p_1, \dots, p_n, q_1, \dots, q_{n-1}$ with relations

$$r_i^{p_i} = 1 \text{ for } i = 1, \dots, n$$

$$r_i r_j = r_j r_i \text{ for } |i - j| > 1$$

$$r_i r_{i+1} r_i r_{i+1} \cdots = r_{i+1} r_i r_{i+1} r_i \cdots, \text{ where products on both sides have } q_i \text{ terms}$$

The Coxeter-Shephard groups consist of the union of the Coxeter groups and Shephard groups. The Coxeter-Shephard groups form a subclass of the well-generated complex reflection groups.

For every Coxeter-Shephard group, W, there is an associated diagram $\mathcal{D}(W) = (\mathcal{D}, \alpha, \beta)$, where $\mathcal{D}$ is a tree, α is a labeling of the vertices, and β is a labeling of the edges of $\mathcal{D}$. Then, $\mathcal{D}(W)$ indicates a presentation of W as follows. To each vertex v of the diagram $\mathcal{D}(W)$, there exists a simple reflection s_v of order $\alpha(v)$, and for each pair of vertices (v, w), s_v and s_w commute when (v, w) is not an edge of $\mathcal{D}(W)$ and satisfy the interlacing relation

$$s_v s_w s_v s_w \cdots = s_w s_v s_w s_v \cdots$$

with exactly m terms on each side of the above product when (v, w) is an edge of $\mathcal{D}(W)$ with label m. Then, W is isomorphic to the group given by the generators

$$\{s_v \colon v \text{ is a vertex of } \mathcal{D}(W)\}$$

with the aforementioned relations.

Definition 2.9. An *admissible subdiagram* of $\mathcal{D}(W)$ is a full subdiagram of $\mathcal{D}(W)$ which is also a Coxeter-Shephard diagram.

To any admissible subdiagram, $\mathcal{D}' \subset \mathcal{D}(W)$, there is an associated complex reflection group, $W(\mathcal{D}') < W$, whose generating reflections are taken to be the generators of W associated to the vertices of $\mathcal{D}'$. As a complex reflection group, [BMR97] shows that $W(\mathcal{D}')$ has the associated diagram $\mathcal{D}$.

Irreducible well-generated complex reflection groups can be classified completely; namely, the groups $G(r,1,n)$, $G(p,p,n)$, cyclic groups, and a finite list of exceptional groups. Moreover, we can check the following by cases.

Proposition 2.10. *Consider the action of W on $\mathcal{R}^*$ by right multiplication. $\mathcal{R}^*$ decomposes nontrivially into W-orbits if and only if W is Coxeter-Shephard and its the corresponding diagram $\mathcal{D}(W)$ has an edge with an even label.*

2.2. **A Character Theoretic Counting Technique.** Let W be a complex reflection group. Let $C_1, \dots, C_m$ be unions of conjugacy classes of reflections of W and let $\mathcal{C} = (C_1, \dots, C_m)$ where for each i, $C_i \subset \mathcal{R}$ is closed under conjugation in W. Fix a Coxeter element c. We follow the approach of [CS12] to compute the numbers

$$g(\mathcal{C}) = \#\left\{(r_1, \dots, r_m) \colon c = r_1 \cdots r_m \text{ and } r_i \in C_i \text{ for all } i\right\}$$

We first compute a character-theoretic formula to compute $g(\mathcal{C})$. Consider the representation $(\mathbb{C}[G], \rho)$ where $\mathbb{C}[G]$ is the group algebra with G acting on the left. Set $R_i = \sum_{\tau \in C_i} v_\tau \in \mathbb{C}[G]$ in the group algebra. Then,

$$\begin{aligned} g(\mathcal{C}) &= \text{coefficient of } v_c \text{ in } R_1 R_2 \cdots R_m \\ &= \text{coefficient of } \mathbf{1} \text{ in } R_1 R_2 \cdots R_m c^{-1} \\ &= \frac{1}{|G|} \chi_\rho (R_1 R_2 \cdots R_m c^{-1}) \end{aligned}$$

The R_i are in the center of $\mathbb{C}[G]$ so by Schur's lemma R_i acts by a scalar on any irreducible representation. Hence, for any m-tuple $\mathcal{C}$ with m_i copies of the class C_i for each i,

$$g(\mathcal{C}) = \frac{1}{|G|} \sum_{V \in \mathrm{Irr}(G)} \dim(V) \chi_V(c^{-1}) \tilde{\chi}_V(R_1)^{m_1} \cdots \tilde{\chi}_V(R_k)^{m_k} \tag{1}$$

where $\tilde{\chi}_V := \frac{\chi_V}{\deg(\chi_V)}$. In particular, this expression implies the following result.

Corollary 2.11. *Let $\mathfrak{S}_m$ act on m-tuples $\mathcal{C}$ by permutation of the entries. Then for all $\sigma \in \mathfrak{S}_m$, we have $g(\mathcal{C}) = g(\sigma \cdot \mathcal{C})$.*

Remark 2.12. Corollary 2.11 can be seen more directly by the transitivity of the Hurwitz action of the braid group on the set of factorizations $c = r_1 \cdots r_m$. See [Bes03] for more details.

3. Main Results

We recall the definition of hyperplane-induced partitions here. (also see Definition 1.1)

Definition 3.1. Let $W \subseteq \mathrm{GL}(V)$ be a complex reflection group. For $r \in \mathcal{R}$ a reflection in W, let $H_r = \ker(r - \mathbf{1})$ denote the hyperplane in V fixed pointwise by the action of r. Furthermore, for C a conjugacy class of reflections in W, let

$$\mathcal{H}_C = \{H_r \subset V : r \in C\}$$

be the set of hyperplanes in V fixed by some reflection in C.

For any complex reflection group W, define the hyperplane-induced relation on conjugacy classes by $C_i \sim C_j$ if and only if $\mathcal{H}_{C_i} = \mathcal{H}_{C_j}$. Let $\mathcal{R}_1, \ldots, \mathcal{R}_\ell$ consist of the unions of conjugacy classes of reflections in the same equivalence classes under the relation $\sim$. That is, $\mathcal{R}_1, \ldots, \mathcal{R}_\ell$ partitions the set of reflections of W and for each $\mathcal{R}_i$, there is some conjugacy class of reflections C_i such that

$$\mathcal{R}_i = \bigcup \{C \in \mathcal{C}_{\mathcal{R}}(W) : C \sim C_i\}.$$

Call the partition $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_\ell$ the *hyperplane-induced partition of reflections* in W.

Remark 3.2. An equivalent formulation of hyperplane-induced partitions can be stated as follows. Let W act on the set of hyperplanes $\mathcal{R}^*$ by right multiplication, and let $\mathcal{O}_1, \ldots, \mathcal{O}_\ell$ be the W-orbits of $\mathcal{R}^*$ under this action. Then, let

$$\mathcal{R}_i = \bigcup \{\mathrm{Stab}_W(H) \mid H \in \mathcal{O}_i\}$$

be the set of reflections which fix some hyperplane in $\mathcal{O}_i$.

Remark 3.3. By the Shepard-Todd classification of well-generated complex reflection groups, one can check there are at most two classes in any hyperplane-induced partition of reflections in a well-generated irreducible complex reflection group W.

Let W be a well-generated complex reflection group with hyperplane-induced partition of reflections $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_\ell$. Set $f_{m_1,\ldots,m_\ell}$ to be the value $g(\mathcal{C})$ for any permutation of the multiset $\{\mathcal{R}_1^{m_1}, \ldots, \mathcal{R}_\ell^{m_\ell}\}$. By corollary 2.11, we can assume with loss of generality

$$\mathcal{C} = (\underbrace{\mathcal{R}_1, \ldots, \mathcal{R}_1}_{m_1 \text{ times}}, \ldots, \underbrace{\mathcal{R}_\ell, \ldots, \mathcal{R}_\ell}_{m_\ell \text{ times}}).$$

Then, for $R_i = \sum_{\tau \in \mathcal{R}_i} v_\tau \in \mathbb{C}[G]$, equation (1) gives the identity

$$f_{m_1,\ldots,m_\ell} = \frac{1}{|G|} \sum_{V \in \mathrm{Irr}(G)} \dim(V) \chi_V(c^{-1}) \tilde{\chi}_V(R_1)^{m_1} \cdots \tilde{\chi}_V(R_\ell)^{m_\ell} \tag{2}$$

For W as above, define the generating function

$$\mathrm{FAC}_W(u_1, \ldots, u_\ell) = \sum_{m_i \geq 0} f_{m_1,\ldots,m_\ell} \prod_{i=1}^{\ell} \frac{u_i^{m_i}}{m_i!}$$

Definition 3.4. Let $W \subseteq \mathrm{GL}(V)$ be an irreducible, well-generated complex reflection group of rank n with $\mathcal{R}_i$ a class in the hyperplane-induced partition of reflections. Then, for any Coxeter element c with factorization into reflections $c = t_1 \cdots t_n$, let n_i denote the number of reflections t_i in the factorization whose fixed hyperplane lies in the W-orbit of hyperplanes associated to the class $\mathcal{R}_i$. That is,

$$n_i = \#\{i \mid t_i \in \mathcal{R}_i\}.$$

Bessis showed in [Bes03] that the Hurwitz action of the braid group of type A_{n-1} on factorizations of c of size n is transitive and preserves the multiset of conjugacy classes $\{C_i \mid t_i \in C_i\}$. Therefore, the numbers n_i defined above are independent of the choice of factorization of c.

Remark 3.5. If W is Coxeter or Shephard, then one can associate a reflection t_v to each vertex v of the diagram $\mathcal{D} = \mathcal{D}(W)$ such that the product $c = \prod_{v\in V(\mathcal{D})} t_v$ is a Coxeter element. Moreover, the hyperplane-induced partition of reflections further partitions these generating reflections into the connected components of the diagram $\overline{\mathcal{D}}$ obtained from $\mathcal{D}$ by removing all even edges. Hence, writing $\overline{\mathcal{D}} = \mathcal{D}_1 \sqcup \cdots \sqcup \mathcal{D}_\ell$, where $\mathcal{D}_i$ are the connected components of $\overline{\mathcal{D}}$, gives the alternate interpretation

$$n_i = \text{rank } W(\mathcal{D}_i)$$

where $\mathcal{D}_i$ is viewed as an admissible subdiagram of $\mathcal{D}$. In particular, by Proposition 2.10, this holds whenever the hyperplane-induced partition of reflections is nontrivial.

Definition 3.6. Let W and $\mathcal{R}_i$ be as above, and let $\mathcal{H}_i$ be the set of hyperplanes fixed by reflections in $\mathcal{R}_i$. Then, put

$$N_i := \#R_i \qquad \text{and} \qquad N_i^* := \#\mathcal{H}_i$$

We now come the the main result.

Theorem 3.7. *Let $W \subseteq \mathrm{GL}(V)$ be an irreducible, well-generated complex reflection group with hyperplane-induced partition of reflections $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_\ell$. Then,*

$$\mathrm{FAC}_W(u_1,\dots,u_\ell) = \frac{1}{|W|}\prod_{i=1}^{\ell}\left(e^{\frac{N_i u_i}{n_i}} - e^{-\frac{N_i^* u_i}{n_i}}\right)^{n_i}$$

This formula specializes to the one derived in [CS12] as follows. Let W be a well-generated complex reflection group with nontrivial hyperplane-induced partition of reflections $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_\ell$. If $\hat{f}_n$ denotes the number of factorizations of a Coxeter element c into a product of n reflections, then

$$\hat{f}_n = \sum_{m_1+\cdots+m_\ell=n}\binom{m_1+\cdots+m_\ell}{m_1,\dots,m_\ell} f_{m_1,\dots,m_\ell}$$

Hence, the exponential generating function $\sum_{n\geq 0}\hat{f}_n\frac{x^n}{n!}$ can be expressed

$$\begin{aligned}\sum_{n\geq 0}\hat{f}_n\frac{x^n}{n!} &= \sum_{n\geq 0}\left(\sum_{m_1+\cdots+m_\ell=n}\binom{m_1+\cdots+m_\ell}{m_1,\dots,m_\ell} f_{m_1,\dots,m_\ell}\right)\frac{x^{m_1}\cdots x^{m_\ell}}{(m_1+\cdots+m_\ell)!} \\ &= \sum_{m_1,\dots,m_\ell\geq 0} f_{m_1,\dots,m_\ell}\prod_{i=1}^{\ell}\frac{u_i^{m_i}}{m_i!}\Bigg|_{u_1=\cdots=u_\ell=x}\end{aligned}$$

Therefore, equating the result of Theorem 3.7 with the formulation in [CS12] gives the identity

$$\left(e^{x|\mathcal{R}|/n} - e^{-x|\mathcal{R}^*|/n}\right)^n = \prod_{i=1}^{\ell}\left(e^{N_i x/n_i} - e^{-N_i^* x/n_i}\right)^{n_i}.$$

In particular, this implies the equalities $n_1+\cdots+n_\ell = n$ and $h = \frac{|\mathcal{R}|+|\mathcal{R}^*|}{n} = \frac{N_i+N_i^*}{n_i}$ for all i.

Remark 3.8. Let W have nontrivial hyperplane-induced partition of reflections. By 2.10, W is Coxeter-Shephard and so has associated diagram $\mathcal{D} = \mathcal{D}(W)$. Let $\mathcal{D}_i, \ldots, \mathcal{D}_\ell$ be the connected components corresponding to $\mathcal{R}_1, \ldots, \mathcal{R}_\ell$ as in Remark 3.5. Then, $n_i = \text{rank } W(\mathcal{D}_i)$ is equal to the number of vertices in $\mathcal{D}_i$. The identity $n_1 + n_2 = n$ reflects the fact that the corresponding subdiagrams $\mathcal{D}_i$ partition the vertices of $\mathcal{D}(W)$.

Corollary 3.9. *Let W be a well-generated, irreducible complex reflection group with hyperplane-induced partition $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_\ell$, and set $h_i = \frac{N_i+N_i^*}{n_i}$ for N_i, N_i^*, n_i as above. Let h denote the Coxeter number of W. Then for any index $1 \le i \le \ell$, $h_i = h$.*

Remark 3.10. If W is a real reflection group, then Corollary 3.9 follows from the following proposition of Bourbaki.

Proposition 3.11 ([Bou02], Ch VI, Section 11, Prop 33)**.** *Let R be an irreducible, reduced root system with basis $\{\alpha_1, \ldots, \alpha_l\}$ and let $s_i = s_{\alpha_i} = 1 - \langle \alpha_i, \alpha_i^\vee \rangle$ be the reflection corresponding to α_i. Let Γ be the cyclic subgroup of order h generated by the Coxeter element $c = s_1 s_2 \ldots s_l$. Then, the action of Γ on the set of roots R is free, and there exist representatives $\theta_1, \ldots, \theta_n$ of the Γ-orbits of R such that each θ_i is in the W-orbit of the simple root α_i.*

Case-Free Proof of Corollary 3.9 for Real Reflection Groups. For W a real irreducible reflection group, its associated root system is irreducible and reduced. Fix the notation in proposition 3.2. Let $\mathcal{O}_1, \ldots, \mathcal{O}_\ell$ be the W-orbits of $\mathcal{R}^*$ (corresponding to some hyperplane-induced partition of reflections $\mathcal{R} = \mathcal{R}_i \cup \cdots \cup \mathcal{R}_\ell$). Then, n_i counts the number of α_j whose corresponding hyperplane is in the W-orbit $\mathcal{O}_i$. Furthermore, since the representative θ_i is in the W-orbits of the root α_i, it follows that n_i also counts the number of θ_j having corresponding hyperplane in the W-orbit $\mathcal{O}_i$.

Since the reflection groups are real, $N_i = N_i^*$, and the number of roots in R with corresponding hyperplane in $\mathcal{O}_i$ is equal to $2N_i$. Because Γ acts freely, the size of each Γ-orbit is exactly h. Therefore, the action of Γ on the set of hyperplanes $\mathcal{O}_i$ has $2N_i/h$ orbits. But the previous paragraph implies this number is also n_i. Therefore $h = 2N_i/n_i = (N_i + N_i^*)/n_i = h_i$. □

Jean-Michel kindly provided a case-free proof of Corollary 3.9 for well-generated complex reflection groups using results of Bessis and Broué-Malle-Rouquier.

4. Proof of Theorem 3.7

We use the Shephard-Todd classification of well-generated, irreducible complex reflection groups to reduce the proof of Theorem 3.7 to cases. Furthermore, only the cases in which a hyperplane-induced partition of reflections is nontrivial are considered. In particular, this leaves the Type B reflection groups $G(r, 1, n)$, dihedral groups $I_2(n) = G(2, 2, n)$, and exceptional groups.

4.1. **Type B Reflection Groups.** Consider the group $W = G(r, 1, n)$. Recall that W consists of all monomial $n \times n$ matrices whose entries are rth roots of unity. Observe that the reflections in $G(r, 1, n)$ consist of the following conjugacy classes:

(1) The conjugacy class of transpositions and ζ_r-transpositions (those having a diagonal two by two minor equal to $\begin{pmatrix} 0 & \zeta_r^k \\ \zeta_k^{-r} & 0 \end{pmatrix}$ for some r), which has size $r\binom{n}{2}$.
(2) The diagonal matrices with all but the one diagonal entry equal to 1 and one being equal to ζ_r^k. Each class of this type consists of n elements, and there is one class for each $0 < k \leq r-1$.

We group together conjugacy classes which share a set of reflecting hyperplanes. Let $\mathcal{R}_0$ be the conjugacy class of transpositions and let $\mathcal{C}_\ell$ be the conjugacy class with a single ζ_r^ℓ on the diagonal for $1 \leq \ell \leq r-1$. Then, set $\mathcal{R}_1 = \cup_{\ell=1}^{r-1}\mathcal{C}_\ell$. Set $f_{\ell,m}$ to be the value of $g(\mathcal{C})$ for any $(\ell+m)$-tuple $\mathcal{C}$ with entries in $\{\mathcal{R}_0, \mathcal{R}_1\}$ with exactly ℓ copies of $\mathcal{R}_0$ and m copies of the $\mathcal{R}_1$. The formula 1 implies $f_{\ell,m}$ is well-defined and that, for $R_i = \sum_{\tau\in\mathcal{R}_i} v_\tau \in \mathbb{C}[G]$,

$$f_{\ell,m} = \frac{1}{|G|}\sum_{V\in\mathrm{Irr}(G)} \dim(V)\chi_V(c^{-1})\tilde{\chi}_V(R_0)^\ell\tilde{\chi}_V(R_1)^m$$

Now, to finish the computation, we make use of some representation-theoretic machinery described in [CS12]. For $w = \sigma \wr (i_1,\dots,i_n)$ the element of $G(r,1,n)$ with permutation matrix shape σ and nonzero entry $\zeta_r^{i_k}$ in column k, define $|w| = \sigma$ and $\|w\| = i_1 + \cdots i_n \mod r$. Also, $\mathfrak{h}_k^n$ will denote the hook $(n-k, 1^k)$ and ${}_q\mathfrak{h}_k^n$ denotes the n-tuple $(0,\dots,0,\mathfrak{h}_k^n,0,\dots,0)$ where the hook appears in the qth coordinate.

Proposition 4.1 ([CS12], 5.3)**.** *Every irreducible character of $W = G(r,1,n)$ is of the following form. Let $\lambda = (\lambda^{(0)},\dots,\lambda^{(r-1)})$ be an r-tuple of of partitions of total size n, $k_\ell = |\lambda^{(\ell)}|$, and*

$$B = G(r,1,k_0)\times G(r,1,k_1)\times\cdots\times G(r,1,k_{r-1})$$

be the subgroup of block-diagonal matrices of appropriate dimensions inside of W. Then, for each λ as above, there corresponds exactly one irreducible character χ_λ given by

$$\chi_\lambda(w) = \frac{1}{|B|}\sum_{\substack{s\in W\\ s^{-1}ws\in B}}\prod_{\ell=0}^{r-1}\chi_{\lambda^{(\ell)}}(|w_\ell|)\zeta_r^{\ell\|w_\ell\|}$$

where $s^{-1}ws = (w_0,\dots,w_{r-1})$ in the above sum.

Lemma 4.2 ([CS12], 5.4)**.** *The character χ_λ as above vanishes on c^{-1} unless $\lambda =_q \mathfrak{h}_k^n$ for some $0 \leq q < r$ and $0 \leq k < n$. In this case, $\deg(\chi_{{}_q\mathfrak{h}_k^n}) = \binom{n-1}{k}$ and $\chi_{{}_q\mathfrak{h}_k^n}(c^{-1}) = (-1)^k\zeta_r^{-q}$.*

Lemma 4.3 ([CS12])**.** *The evaluation of the character on the elements $C_\ell = \sum_{\tau\in\mathcal{C}_\ell} v_\tau$ is as follows.*

$$\chi_{{}_q\mathfrak{h}_k^n}(R_\ell) = \begin{cases} n\zeta_r^{q\ell} & \text{if } 1 \leq \ell < r \\ \frac{nr(n-2k-1)}{2} & \text{if } \ell = 0 \end{cases}$$

Now, we can express $f_{\ell,m}$ as follows.

$$\begin{aligned}
f_{\ell,m} &= \frac{1}{|G|}\sum_{V\in\mathrm{Irr}(G)} \dim(V)\chi_V(c^{-1})\tilde{\chi}_V(R_0)^{\ell}\tilde{\chi}_V(R_1)^m \\
&= \frac{1}{|G|}\sum_{q=0}^{r-1}\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\zeta_r^{-q}\left(\frac{nr(n-2k-1)}{2}\right)^{\ell}(-n+rn\delta_{q\equiv 0\ (r)})^m \\
&= \frac{1}{|G|}\left\{\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\left(\frac{nr(n-2k-1)}{2}\right)^{\ell}\right\}\left\{\sum_{q=0}^{r-1}\zeta_r^{-q}(-n)^m+\big((r-1)n\big)^m\right\} \\
&= \frac{1}{|G|}\Big(\big((r-1)n\big)^m-(-n)^m\Big)\left\{\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\left(\frac{nr(n-2k-1)}{2}\right)^{\ell}\right\}
\end{aligned}$$

Therefore, the exponential generating function $\mathrm{FAC}_G(t) = \sum_{\ell,m\geq 0} f_{\ell,m}\frac{t^{\ell}u^m}{\ell!m!}$ can be manipulated to take the following form

$$\begin{aligned}
\mathrm{FAC}_G(u,t) &= \sum_{\ell,m\geq 0}\frac{1}{|G|}\Big(\big((r-1)n\big)^m-(-n)^m\Big)\left\{\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\left(\frac{nr(n-2k-1)}{2}\right)^{\ell}\right\}\frac{t^{\ell}u^m}{\ell!m!} \\
&= \frac{1}{|G|}\sum_{m\geq 0}\Big(\big((r-1)n\big)^m-(-n)^m\Big)\left\{\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\exp\left(\frac{nrt(n-2k-1)}{2}\right)\right\}\frac{u^m}{m!} \\
&= \frac{1}{|G|}\Big(e^{(r-1)nu}-e^{-nu}\Big)e^{\frac{n(n-1)rt}{2}}\left\{\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k e^{-nrkt}\right\} \\
&= \frac{1}{|G|}\Big(e^{(r-1)nu}-e^{-nu}\Big)e^{\frac{n(n-1)rt}{2}}\left(1-e^{-nrt}\right)^{n-1} \\
&= \frac{1}{|G|}\Big(e^{(r-1)nu}-e^{-nu}\Big)\Big(e^{nrt/2}-e^{-nrt/2}\Big)^{n-1}
\end{aligned}$$

4.2. **Factorizations in the Dihedral Group $I_2(n)$ for Even n.** Now consider the case of the dihedral group $I_2(n)$ on n vertices for n even. $I_2(n)$ has the presentation

$$I_2(n) = \langle a, x \mid a^n = x^2 = 1,\ ax = xa^{-1}\rangle,$$

By inspection, we have that $I_2(n)$ has n reflecting hyperplanes and n reflections, so $h = \frac{n+n}{n} = 2$. Hence, the Coxeter elements of $I_2(n)$ have eigenvalue $\zeta_2 = -1$. Therefore, Coxeter elements in $I_2(n)$ are given by long cycles. Let $\mathcal{R}_0/\mathcal{R}_1$ be the conjugacy classes of even/odd reflections, respectively, and put $R_i = \sum_{g\in\mathcal{R}_i} v_g \in \mathbb{C}[I_2(n)]$. Then, we can compute the number of factorizations of c by

$$f_{\ell,m} = \frac{1}{|I_2(n)|}\sum_{V\in\mathrm{Irr}(G)}\dim(V)\chi_V(c^{-1})\tilde{\chi}_V(R_0)^{\ell-m}\tilde{\chi}_V(R_1)^m$$

We have the following character evaluations:

Character	Value on $c^{-1}=c$	Value on R_0	Value on R_1
$\mathbf{1}$	1	$n/2$	$n/2$
det, $\substack{a\mapsto 1\\ x\mapsto -1}$	1	$-n/2$	$-n/2$
$\substack{a\mapsto -1\\ x\mapsto 1}$	-1	$-n/2$	$n/2$
$\substack{a\mapsto -1\\ x\mapsto -1}$	-1	$n/2$	$-n/2$
$\chi_{2,r}$, $\substack{a\mapsto \mathrm{rot}(2\pi r/n)\\ x\mapsto \text{refl. mat}}$	$2(-1)^r$	0	0

Hence, for $\ell \geq m$, we compute

$$
\begin{aligned}
f_{\ell,m} &= \frac{1}{2n}\left(\left(\frac{n}{2}\right)^{\ell+m} + \left(-\frac{n}{2}\right)^{\ell}\left(-\frac{n}{2}\right)^{m} - \left(-\frac{n}{2}\right)^{\ell}\left(\frac{n}{2}\right)^{m} - \left(\frac{n}{2}\right)^{\ell}\left(-\frac{n}{2}\right)^{m}\right) \\
&= \frac{n^{\ell+m}}{2^{\ell+m}|I_2(n)|}\left(1 + (-1)^{\ell+m} + (-1)^{1+\ell} + (-1)^{1+m}\right)
\end{aligned}
$$

So therefore, the generating function is

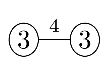
$$
\begin{aligned}
\mathrm{FAC}_G(u,t) &= \sum_{\ell \geq 0} \frac{n^{\ell+m}}{2^{\ell+m}|I_2(n)|}\left(1 + (-1)^{\ell+m} + (-1)^{1+\ell} + (-1)^{1+m}\right)\frac{t^\ell u^m}{\ell! m!} \\
&= \frac{1}{|I_2(n)|}\left(e^{nt/2} - e^{-nt/2}\right)\sum_{m \geq 0}\left(\left(\frac{n}{2}\right)^m - \left(-\frac{n}{2}\right)^m\right)\frac{u^m}{m!} \\
&= \frac{1}{|I_2(n)|}\left(e^{nu/2} - e^{-nu/2}\right)\left(e^{nt/2} - e^{-nt/2}\right)
\end{aligned}
$$

4.3. **Factorizations of Coxeter Elements in Exceptional Groups.** We record here our computations for exceptional groups whose set of reflections decomposes nontrivially into unions of conjugacy classes with no common reflecting hyperplanes. For our computations, an interface from Sage to GAP3 was used to enumerate Coxeter elements and compute character tables for each Shephard-Todd exceptional group. In particular, we compute these results for the rank 2 exceptional groups, G_5, G_6, G_9, G_{10}, G_{14}, G_{17}, G_{18}, and G_{21}, and the higher rank exceptional groups G_{26} and G_{28}. Below is listed the tables of computations recording the values $\deg(\chi)$, $\chi(c^{-1})$, and $\chi(R_i)$, where $R_i = \sum_{\tau \in \mathcal{R}_i} v_\tau$ for conjugacy classes of reflections $\mathcal{R}_i$, as well as the quantities $\mathrm{FAC}_G(u,t)$.

For all other well-generated complex reflection groups, Sage was used to enumerate the set of hyperplanes associated to each conjugacy class of reflections and check the hyperplane-induced partition of reflections has only one class. Hence, [CS12] gives the result in these cases.

The rank two exceptional group
$W = G_5$

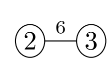
$\deg(\chi)$	$\chi(c^{-1})$	$\chi(R_0)$	$\chi(R_1)$	$\chi(R_2)$	$\chi(R_3)$
1	1	4	4	4	4
1	ζ_3	$4\zeta_3^2$	$4\zeta_3$	4	4
1	ζ_3^2	$4\zeta_3$	$4\zeta_3^2$	4	4
1	ζ_3	4	4	$4\zeta_3$	$4\zeta_3^2$
1	ζ_3^2	4	4	$4\zeta_3^2$	$4\zeta_3$
1	ζ_3^2	$4\zeta_3^2$	$4\zeta_3$	$4\zeta_3$	$4\zeta_3^2$
1	ζ_3	$4\zeta_3$	$4\zeta_3^2$	$4\zeta_3^2$	$4\zeta_3$
1	1	$4\zeta_3$	$4\zeta_3^2$	$4\zeta_3$	$4\zeta_3^2$
1	1	$4\zeta_3^2$	$4\zeta_3$	$4\zeta_3^2$	$4\zeta_3$
2	0	-4	-4	-4	-4
2	0	$-4\zeta_3$	$-4\zeta_3^2$	$-4\zeta_3$	$-4\zeta_3^2$
2	0	$-4\zeta_3^2$	$-4\zeta_3$	$-4\zeta_3^2$	$-4\zeta_3$
2	0	$-4\zeta_3$	$-4\zeta_3^2$	$-4\zeta_3^2$	$-4\zeta_3$
2	0	$-4\zeta_3^2$	$-4\zeta_3$	$-4\zeta_3$	$-4\zeta_3^2$
2	0	-4	-4	$-4\zeta_3$	$-4\zeta_3^2$
2	0	-4	-4	$-4\zeta_3^2$	$-4\zeta_3$
2	0	$-4\zeta_3^2$	$-4\zeta_3$	-4	-4
2	0	$-4\zeta_3$	$4\zeta_3^2$	-4	-4
3	-1	0	0	0	0
3	$-\zeta_3$	0	0	0	0
3	$-\zeta_3^2$	0	0	0	0

Diagram: 

$$\mathrm{FAC}_{G_5}(u,t) = \frac{1}{|G_5|}\left(e^{8u} - e^{-4u}\right)\left(e^{8t} - e^{-4t}\right)$$

The rank two exceptional group
$W = G_6$

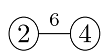
$\deg(\chi)$	$\chi(c^{-1})$	$\chi(R_0)$	$\chi(R_1)$	$\chi(R_2)$
1	1	6	3	3
1	-1	-6	3	3
1	$-\zeta_3$	-6	$3\zeta_3$	$3\zeta_3^2$
1	$-\zeta_3^2$	-6	$3\zeta_3^2$	$3\zeta_3$
1	ζ_3	6	$3\zeta_3$	$3\zeta_3^2$
1	ζ_3^2	6	$3\zeta_3^2$	$3\zeta_3$
2	$-i$	0	-3	-3
2	i	0	-3	-3
2	$\zeta_{12}^3 - \zeta_{12}$	0	$3\zeta_{12}^2$	$-3\zeta_{12}^2 + 3$
2	$-\zeta_{12}^3 + \zeta_{12}$	0	$3\zeta_{12}^2$	$-3\zeta_{12}^2 + 3$
2	ζ_{12}	0	$-3\zeta_{12}^2 + 3$	$3\zeta_{12}$
2	$-\zeta_{12}$	0	$-3\zeta_{12}^2 + 3$	$3\zeta_{12}$
3	0	-6	0	0
3	0	6	0	0

Diagram: 

$$\mathrm{FAC}_{G_6}(u,t) = \frac{1}{|G_6|}\left(e^{8u} - e^{-4u}\right)\left(e^{6t} - e^{-6t}\right)$$

The rank two exceptional group $W = G_9$

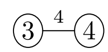
$\deg(\chi)$	$\chi(c^{-1})$	$\chi(R_0)$	$\chi(R_1)$	$\chi(R_2)$	$\chi(R_3)$
1	1	12	6	6	6
1	1	-12	-6	-6	6
1	-1	-12	6	6	6
1	-1	12	-6	-6	6
1	$-i$	-12	$-6i$	$6i$	-6
1	i	-12	$6i$	$-6i$	-6
1	i	12	$-6i$	$6i$	-6
1	$-i$	12	$6i$	$-6i$	-6
2	-1	0	0	0	12
2	1	0	0	0	12
2	i	0	0	0	-12
2	$-i$	0	0	0	-12
2	ζ_8	0	$-6i-6$	$6i-6$	0
2	$-\zeta_8$	0	$-6i-6$	$6i-6$	0
2	ζ_8^3	0	$6i-6$	$-6i-6$	0
2	$-\zeta_8^3$	0	$6i-6$	$-6i-6$	0
2	ζ_8	0	$6i+6$	$-6i+6$	0
2	$-\zeta_8$	0	$6i+6$	$-6i+6$	0
2	ζ_8^3	0	$-6i-6$	$6i-6$	0
2	$-\zeta_8^3$	0	$-6i-6$	$6i-6$	0
3	0	12	-6	-6	-6
3	0	-12	6	6	-6
3	0	-12	-6	-6	-6
3	0	12	6	6	-6
3	0	12	$-6i$	$6i$	6
3	0	12	$6i$	$-6i$	6
3	0	-12	$6i$	$-6i$	6
3	0	-12	$-6i$	$6i$	6
4	$-\zeta_8$	0	0	0	0
4	ζ_8	0	0	0	0
4	$-\zeta_8^3$	0	0	0	0
4	ζ_8^3	0	0	0	0

Diagram: 

$$\mathrm{FAC}_{G_9}(u,t) = \frac{1}{|G_9|}\Big(e^{18u} - e^{-6u}\Big)\Big(e^{12t} - e^{-12t}\Big)$$

The rank two exceptional group $W = G_{10}$

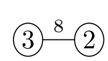
$\deg(\chi)$	$\chi(c^{-1})$	$\chi(R_0)$	$\chi(R_1)$	$\chi(R_2)$	$\chi(R_3)$	$\chi(R_4)$
1	1	8	8	6	6	6
1	-1	8	8	-6	-6	6
1	i	8	8	$-6i$	$6i$	-6
1	$-i$	8	8	$6i$	$-6i$	-6
1	$-\zeta_3$	$8\zeta_3^2$	$8\zeta_3$	-6	-6	6
1	$-\zeta_3^2$	$8\zeta_3$	$8\zeta_3^2$	-6	-6	6
1	ζ_3	$8\zeta_3^2$	$8\zeta_3$	6	6	6
1	ζ_3^2	$8\zeta_3$	$8\zeta_3^2$	6	6	6
1	$-\zeta_{12}$	$-8\zeta_{12}^2$	$8\zeta_{12}^2-8$	$-6\zeta_{12}^3$	$6\zeta_{12}^3$	-6
1	ζ_{12}	$-8\zeta_{12}^2$	$8\zeta_{12}^2-8$	$6\zeta_{12}^3$	$-6\zeta_{12}^3$	-6
1	$-\zeta_{12}^3+\zeta_{12}$	$8\zeta_{12}^2-8$	$-8\zeta_{12}^2$	$-6\zeta_{12}^3$	$6\zeta_{12}^3$	-6
1	$\zeta_{12}^3-\zeta_{12}$	$8\zeta_{12}^2-8$	$-8\zeta_{12}^2$	$6\zeta_{12}^3$	$-6\zeta_{12}^3$	-6
2	0	-8	-8	0	0	12
2	0	-8	-8	0	0	-12
2	0	$-8\zeta_3$	$-8\zeta_3^2$	0	0	12
2	0	$-8\zeta_3^2$	$-8\zeta_3$	0	0	12
2	0	$-8\zeta_3$	$-8\zeta_3^2$	0	0	-12
2	0	$-8\zeta_3^2$	$-8\zeta_3$	0	0	-12
2	0	-8	-8	$-6i-6$	$6i-6$	0
2	0	-8	-8	$6i-6$	$-6i-6$	0
2	0	-8	-8	$6i+6$	$-6i+6$	0
2	0	-8	-8	$-6i+6$	$6i+6$	0
2	0	$-8\zeta_{12}^2+8$	$8\zeta_{12}^2$	$-6\zeta_{12}^3-6$	$6\zeta_{12}^3-6$	0
2	0	$-8\zeta_{12}^2+8$	$8\zeta_{12}^2$	$6\zeta_{12}^3-6$	$-6\zeta_{12}^3-6$	0
2	0	$8\zeta_{12}^2$	$-8\zeta_{12}^2+8$	$-6\zeta_{12}^3-6$	$6\zeta_{12}^3-6$	0
2	0	$8\zeta_{12}^2$	$-8\zeta_{12}^2+8$	$6\zeta_{12}^3-6$	$-6\zeta_{12}^3-6$	0
2	0	$-8\zeta_{12}^2+8$	$8\zeta_{12}^2$	$6\zeta_{12}^3+6$	$-6\zeta_{12}^3+6$	0
2	0	$-8\zeta_{12}^2+8$	$8\zeta_{12}^2$	$-6\zeta_{12}^3+6$	$6\zeta_{12}^3+6$	0
2	0	$8\zeta_{12}^2$	$-8\zeta_{12}^2+8$	$6\zeta_{12}^3+6$	$-6\zeta_{12}^3+6$	0
2	0	$8\zeta_{12}^2$	$-8\zeta_{12}^2+8$	$-6\zeta_{12}^3+6$	$6\zeta_{12}^3+6$	0
3	1	0	0	-6	-6	-6
3	-1	0	0	6	6	-6
3	$-i$	0	0	$-6i$	$6i$	6
3	i	0	0	$6i$	$-6i$	6
3	ζ_3^2	0	0	-6	-6	-6
3	ζ_3	0	0	-6	-6	-6
3	$-\zeta_3^2$	0	0	6	6	-6
3	$-\zeta_3$	0	0	6	6	-6
3	$\zeta_{12}^3-\zeta_{12}$	0	0	$-6\zeta_{12}^3$	$6\zeta_{12}^3$	6
3	$-\zeta_{12}^3+\zeta_{12}$	0	0	$6\zeta_{12}^3$	$-6\zeta_{12}^3$	6
3	ζ_{12}	0	0	$-6\zeta_{12}^3$	$6\zeta_{12}^3$	6
3	$-\zeta_{12}$	0	0	$6\zeta_{12}^3$	$-6\zeta_{12}^3$	6
4	0	8	8	0	0	0
4	0	8	8	0	0	0
4	0	$8\zeta_{12}^2-8$	$-8\zeta_{12}^2$	0	0	0
4	0	$8\zeta_{12}^2-8$	$-8\zeta_{12}^2$	0	0	0
4	0	$-8\zeta_{12}^2$	$8\zeta_{12}^2-8$	0	0	0
4	0	$-8\zeta_{12}^2$	$8\zeta_{12}^2-8$	0	0	0

Diagram: 

$$\mathrm{FAC}_{G_{10}}(u,t) = \frac{1}{|G_{10}|}\Big(e^{18u} - e^{-6u}\Big)\Big(e^{16t} - e^{-8t}\Big)$$

The rank two exceptional group $W = G_{14}$

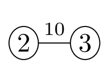
$\deg(\chi)$	$\chi(c^{-1})$	$\chi(R_0)$	$\chi(R_1)$	$\chi(R_2)$
1	1	12	8	8
1	-1	-12	8	8
1	$-\zeta_3$	-12	$8\zeta_3^2$	$8\zeta_3$
1	$-\zeta_3^2$	-12	$8\zeta_3$	$8\zeta_3^2$
1	ζ_3	12	$8\zeta_3^2$	$8\zeta_3$
1	ζ_3^2	12	$8\zeta_3$	$8\zeta_3^2$
2	0	0	-8	-8
2	0	0	$-8\zeta_3^2$	$-8\zeta_3$
2	0	0	$-8\zeta_3$	$-8\zeta_3^2$
2	$\zeta_8^3+\zeta_8$	0	-8	-8
2	$-\zeta_8^3-\zeta_8$	0	-8	-8
2	$\zeta_{24}^7-\zeta_{24}^5-\zeta_{24}^3$	0	$8\zeta_{24}^4$	$-8\zeta_{24}^4+8$
2	$-\zeta_{24}^7+\zeta_{24}^5+\zeta_{24}^3$	0	$8\zeta_{24}^4$	$-8\zeta_{24}^4+8$
2	$-\zeta_{24}^7+\zeta_{24}$	0	$-8\zeta_{24}^4+8$	$8\zeta_{24}^4$
2	$\zeta_{24}^7-\zeta_{24}$	0	$-8\zeta_{24}^4+8$	$8\zeta_{24}^4$
3	-1	12	0	0
3	1	-12	0	0
3	$-\zeta_3$	12	0	0
3	$-\zeta_3^2$	12	0	0
3	ζ_3	-12	0	0
3	ζ_3^2	-12	0	0
4	0	0	8	8
4	0	0	$8\zeta_3^2$	$8\zeta_3$
4	0	0	$8\zeta_3$	$8\zeta_3^2$

Diagram: 

$$\mathrm{FAC}_{G_{14}}(u,t) = \frac{1}{|G_{14}|}\Big(e^{16u} - e^{-8u}\Big)\Big(e^{12t} - e^{-12t}\Big)$$

The rank two exceptional group $W = G_{21}$

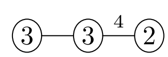
$\deg(\chi)$	$\chi(c^{-1})$	$\chi(R_0)$	$\chi(R_1)$	$\chi(R_2)$
1	1	30	20	20
1	-1	-30	20	20
1	$-\zeta_3^2$	-30	$20\zeta_3$	$20\zeta_3^2$
1	$-\zeta_3$	-30	$20\zeta_3^2$	$20\zeta_3$
1	ζ_3^2	30	$20\zeta_3$	$20\zeta_3^2$
1	ζ_3	30	$20\zeta_3^2$	$20\zeta_3$
2	$\zeta_{20}^7 + \zeta_{20}^3$	0	-20	-20
2	$-\zeta_{20}^7 + \zeta_{20}^5 - \zeta_{20}^3$	0	-20	-20
2	$-\zeta_{20}^7 - \zeta_{20}^3$	0	-20	-20
2	$\zeta_{20}^7 - \zeta_{20}^5 + \zeta_{20}^3$	0	-20	-20
2	$-\zeta_{60}^{15} - \zeta_{60}^{13} + \zeta_{60}^7 + \zeta_{60}^5$	0	$-20\zeta_{60}^{10} + 20$	$20\zeta_{60}^{10}$
2	$\zeta_{60}^{13} - \zeta_{60}^7$	0	$-20\zeta_{60}^{10} + 20$	$20\zeta_{60}^{10}$
2	$\zeta_{60}^{15} + \zeta_{60}^{13} - \zeta_{60}^{11} - \zeta_{60}^9 - \zeta_{60}^7 - \zeta_{60}^5 + \zeta_{60}$	0	$20\zeta_{60}^{10}$	$-20\zeta_{60}^{10} + 20$
2	$-\zeta_{60}^{15} - \zeta_{60}^{13} + \zeta_{60}^{11} + \zeta_{60}^9 + \zeta_{60}^7 - \zeta_{60}$	0	$20\zeta_{60}^{10}$	$-20\zeta_{60}^{10} + 20$
2	$\zeta_{60}^{15} + \zeta_{60}^{13} - \zeta_{60}^7 - \zeta_{60}^5$	0	$-20\zeta_{60}^{10} + 20$	$20\zeta_{60}^{10}$
2	$-\zeta_{60}^{13} + \zeta_{60}^7$	0	$-20\zeta_{60}^{10} + 20$	$20\zeta_{60}^{10}$
2	$-\zeta_{60}^{15} - \zeta_{60}^{13} + \zeta_{60}^{11} + \zeta_{60}^9 + \zeta_{60}^7 + \zeta_{60}^5 - \zeta_{60}$	0	$20\zeta_{60}^{10}$	$-20\zeta_{60}^{10} + 20$
2	$\zeta_{60}^{15} + \zeta_{60}^{13} - \zeta_{60}^{11} - \zeta_{60}^9 - \zeta_{60}^7 + \zeta_{60}$	0	$20\zeta_{60}^{10}$	$-20\zeta_{60}^{10} + 20$
3	$\zeta_5^3 + \zeta_5^2 + 1$	-30	0	0
3	$-\zeta_5^3 - \zeta_5^2$	-30	0	0
3	$-\zeta_{15}^5 + \zeta_{15}^4 + \zeta_{15} - 1$	-30	0	0
3	$-\zeta_{15}^4 - \zeta_{15}$	-30	0	0
3	$-\zeta_{15}^7 + \zeta_{15}^5 - \zeta_{15}^4 + \zeta_{15}^3 - \zeta_{15}^2 - \zeta_{15} + 1$	-30	0	0
3	$\zeta_{15}^7 + \zeta_{15}^4 - \zeta_{15}^3 + \zeta_{15}^2 + \zeta_{15} - 1$	-30	0	0
3	$-\zeta_5^3 - \zeta_5^2 - 1$	30	0	0
3	$\zeta_5^3 + \zeta_5^2$	30	0	0
3	$\zeta_{15}^5 - \zeta_{15}^4 - \zeta_{15} + 1$	30	0	0
3	$\zeta_{15}^4 + \zeta_{15}$	30	0	0
3	$\zeta_{15}^7 - \zeta_{15}^5 + \zeta_{15}^4 - \zeta_{15}^3 + \zeta_{15}^2 + \zeta_{15} - 1$	30	0	0
3	$-\zeta_{15}^7 - \zeta_{15}^4 + \zeta_{15}^3 - \zeta_{15}^2 - \zeta_{15} + 1$	30	0	0
4	-1	0	20	20
4	1	0	20	20
4	$-\zeta_3^2$	0	$20\zeta_3$	$20\zeta_3^2$
4	$-\zeta_3$	0	$20\zeta_3^2$	$20\zeta_3$
4	ζ_3^2	0	$20\zeta_3$	$20\zeta_3^2$
4	ζ_3	0	$20\zeta_3^2$	$20\zeta_3$
4	i	0	20	20
4	$-i$	0	20	20
4	$-\zeta_{12}^3 + \zeta_{12}$	0	$20\zeta_{12}^2 - 20$	$-20\zeta_{12}^2$
4	$-\zeta_{12}$	0	$-20\zeta_{12}^2$	$20\zeta_{12}^2 - 20$
4	$\zeta_{12}^3 - \zeta_{12}$	0	$20\zeta_{12}^2 - 20$	$-20\zeta_{12}^2$
4	ζ_{12}	0	$-20\zeta_{12}^2$	$20\zeta_{12}^2 - 20$
5	0	30	-20	-20
5	0	-30	-20	-20
5	0	30	$-20\zeta_3$	$20\zeta_3 + 20$
5	0	30	$-20\zeta_3^2$	$-20\zeta_3$
5	0	-30	$-20\zeta_3$	$20\zeta_3 + 20$
5	0	-30	$-20\zeta_3^2$	$-20\zeta_3$
6	$-i$	0	0	0
6	i	0	0	0
6	$\zeta_{12}^3 - \zeta_{12}$	0	0	0
6	ζ_{12}	0	0	0
6	$-\zeta_{12}^3 + \zeta_{12}$	0	0	0
6	$-\zeta_{12}$	0	0	0

Diagram: 

$$\mathrm{FAC}_{G_{21}}(u,t) = \frac{1}{|G_{21}|}\left(e^{40u} - e^{-20u}\right)\left(e^{30t} - e^{-30t}\right)$$

The rank three exceptional group $W = G_{26}$

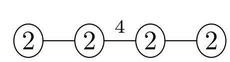
$\deg(\chi)$	$\chi(c^{-1})$	$\chi(R_0)$	$\chi(R_1)$	$\chi(R_2)$
1	1	9	12	12
1	-1	-9	12	12
1	$-\zeta_3^2$	-9	$12\zeta_3^2$	$12\zeta_3$
1	$-\zeta_3$	-9	$12\zeta_3$	$12\zeta_3^2$
1	ζ_3^2	9	$12\zeta_3^2$	$12\zeta_3$
1	ζ_3	9	$12\zeta_3$	$-12\zeta_3 - 12$
2	-1	-18	-12	-12
2	1	18	-12	-12
2	$-\zeta_3$	-18	$-12\zeta_3$	$-12\zeta_3^2$
2	$-\zeta_3^2$	-18	$-12\zeta_3^2$	$-12\zeta_3$
2	ζ_3	18	$-12\zeta_3$	$-12\zeta_3^2$
2	ζ_3^2	18	$-12\zeta_3^2$	$-12\zeta_3$
3	0	27	0	0
3	0	-27	0	0
3	0	-9	$12\zeta_3 + 24$	$-12\zeta_3 + 12$
3	0	-9	$-12\zeta_3 + 12$	$12\zeta_3 + 24$
3	0	-9	$-24\zeta_3 - 12$	$24\zeta_3 + 12$
3	0	-9	$24\zeta_3 + 12$	$-24\zeta_3 - 12$
3	0	-9	$12\zeta_3 - 12$	$-12\zeta_3 - 24$
3	0	-9	$-12\zeta_3 - 24$	$12\zeta_3 - 12$
3	0	9	$12\zeta_3 + 24$	$-12\zeta_3 + 12$
3	0	9	$-12\zeta_3 + 12$	$12\zeta_3 + 24$
3	0	9	$-24\zeta_3 - 12$	$24\zeta_3 + 12$
3	0	9	$24\zeta_3 + 12$	$-24\zeta_3 - 12$
3	0	9	$12\zeta_3 - 12$	$-12\zeta_3 - 24$
3	0	9	$-12\zeta_3 - 24$	$12\zeta_3 - 12$
6	0	18	$-12\zeta_3 + 12$	$12\zeta_3 + 24$
6	0	18	$12\zeta_3 + 24$	$-12\zeta_3 + 12$
6	0	18	$-12\zeta_3 - 24$	$12\zeta_3 - 12$
6	0	18	$12\zeta_3 - 12$	$-12\zeta_3 - 24$
6	0	18	$24\zeta_3 + 12$	$-24\zeta_3 - 12$
6	0	18	$-24\zeta_3 - 12$	$24\zeta_3 + 12$
6	0	-18	$-12\zeta_3 + 12$	$12\zeta_3 + 24$
6	0	-18	$12\zeta_3 + 24$	$-12\zeta_3 + 12$
6	0	-18	$-12\zeta_3 - 24$	$12\zeta_3 - 12$
6	0	-18	$12\zeta_3 - 12$	$-12\zeta_3 - 24$
6	0	-18	$24\zeta_3 + 12$	$-24\zeta_3 - 12$
6	0	-18	$-24\zeta_3 - 12$	$24\zeta_3 + 12$
8	-1	0	24	24
8	1	0	24	24
8	$-\zeta_3^2$	0	$24\zeta_3^2$	$24\zeta_3$
8	$-\zeta_3$	0	$24\zeta_3$	$24\zeta_3^2$
8	ζ_3^2	0	$24\zeta_3^2$	$24\zeta_3$
8	ζ_3	0	$24\zeta_3$	$24\zeta_3^2$
9	0	-27	0	0
9	0	-27	0	0
9	0	27	0	0
9	0	27	0	0

Diagram: 

$$\mathrm{FAC}_{G_{26}}(u,t) = \frac{1}{|G_{26}|}\left(e^{12u} - e^{-6u}\right)^2\left(e^{9t} - e^{-9t}\right)$$

The rank four exceptional group $W = G_{28}$

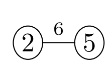
$\deg(\chi)$	$\chi(c^{-1})$	$\chi(R_0)$	$\chi(R_1)$
1	1	12	12
1	1	-12	-12
1	1	-12	12
1	1	12	-12
2	-1	-24	0
2	-1	24	0
2	-1	0	-24
2	-1	0	24
4	0	-24	24
4	0	-24	-24
4	0	24	24
4	0	24	-24
4	1	0	0
6	-1	0	0
6	-1	0	0
8	0	0	48
8	0	0	-48
8	0	-48	0
8	0	48	0
9	0	-36	-36
9	0	-36	36
9	0	36	-36
9	0	36	36
12	1	0	0
16	0	0	0

Diagram: 

$$\mathrm{FAC}_{G_{28}}(u,t) = \frac{1}{|G_{28}|}\left(e^{6u} - e^{-6u}\right)^2\left(e^{6t} - e^{-6t}\right)^2$$

The rank two exceptional group $W = G_{17}$

$\deg(\chi)$	$\chi(c^{-1})$	$\chi(R_0)$	$\chi(R_1)$	$\chi(R_2)$	$\chi(R_3)$	$\chi(R_4)$
1	1	30	12	12	12	12
1	-1	-30	12	12	12	12
1	$\zeta_5^3+\zeta_5^2+\zeta_5+1$	-30	$12\zeta_5$	$-12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$12\zeta_5^2$	$12\zeta_5^3$
1	$-\zeta_5^3$	-30	$12\zeta_5^2$	$12\zeta_5^3$	$-12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$12\zeta_5$
1	$-\zeta_5^2$	-30	$12\zeta_5^3$	$12\zeta_5^2$	$12\zeta_5$	$-12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$
1	$-\zeta_5$	-30	$-12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$12\zeta_5$	$12\zeta_5^3$	$12\zeta_5^2$
1	$-\zeta_5^3-\zeta_5^2-\zeta_5-1$	30	$12\zeta_5$	$-12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$12\zeta_5^2$	$12\zeta_5^3$
1	ζ_5^3	30	$12\zeta_5^2$	$12\zeta_5^3$	$-12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$12\zeta_5$
1	ζ_5^2	30	$12\zeta_5^3$	$12\zeta_5^2$	$12\zeta_5$	$-12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$
1	ζ_5	30	$-12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$12\zeta_5$	$12\zeta_5^3$	$12\zeta_5^2$
2	ζ_{20}^5	0	$12\zeta_{20}^6-12\zeta_{20}^4-12$	$12\zeta_{20}^6-12\zeta_{20}^4-12$	$-12\zeta_{20}^6+12\zeta_{20}^4$	$-12\zeta_{20}^6+12\zeta_{20}^4$
2	ζ_{20}^5	0	$-12\zeta_{20}^6+12\zeta_{20}^4$	$-12\zeta_{20}^6+12\zeta_{20}^4$	$12\zeta_{20}^6-12\zeta_{20}^4-12$	$12\zeta_{20}^6-12\zeta_{20}^4-12$
2	$-\zeta_{20}^5$	0	$12\zeta_{20}^6-12\zeta_{20}^4-12$	$12\zeta_{20}^6-12\zeta_{20}^4-12$	$-12\zeta_{20}^6+12\zeta_{20}^4$	$-12\zeta_{20}^6+12\zeta_{20}^4$
2	$-\zeta_{20}^5$	0	$-12\zeta_{20}^6+12\zeta_{20}^4$	$-12\zeta_{20}^6+12\zeta_{20}^4$	$12\zeta_{20}^6-12\zeta_{20}^4-12$	$12\zeta_{20}^6-12\zeta_{20}^4-12$
2	ζ_{20}	0	$12\zeta_{20}^6-12\zeta_{20}^4+12\zeta_{20}^2$	$-12\zeta_{20}^2+12$	$-12\zeta_{20}^6+12$	$12\zeta_{20}^4+12$
2	$-\zeta_{20}^7$	0	$-12\zeta_{20}^6+12$	$12\zeta_{20}^4+12$	$-12\zeta_{20}^2+12$	$12\zeta_{20}^6-12\zeta_{20}^4+12\zeta_{20}^2$
2	$-\zeta_{20}^3$	0	$12\zeta_{20}^4+12$	$-12\zeta_{20}^6+12$	$12\zeta_{20}^6-12\zeta_{20}^4+12\zeta_{20}^2$	$-12\zeta_{20}^2+12$
2	$\zeta_{20}^7-\zeta_{20}^5+\zeta_{20}^3-\zeta_{20}$	0	$-12\zeta_{20}^2+12$	$12\zeta_{20}^6-12\zeta_{20}^4+12\zeta_{20}^2$	$12\zeta_{20}^4+12$	$-12\zeta_{20}^6+12$
2	$-\zeta_{20}$	0	$12\zeta_{20}^6-12\zeta_{20}^4+12\zeta_{20}^2$	$-12\zeta_{20}^2+12$	$-12\zeta_{20}^6+12$	$12\zeta_{20}^4+12$
2	ζ_{20}^7	0	$-12\zeta_{20}^6+12$	$12\zeta_{20}^4+12$	$-12\zeta_{20}^2+12$	$12\zeta_{20}^6-12\zeta_{20}^4+12\zeta_{20}^2$
2	ζ_{20}^3	0	$12\zeta_{20}^4+12$	$-12\zeta_{20}^6+12$	$12\zeta_{20}^6-12\zeta_{20}^4+12\zeta_{20}^2$	$-12\zeta_{20}^2+12$
2	$-\zeta_{20}^7+\zeta_{20}^5-\zeta_{20}^3+\zeta_{20}$	0	$-12\zeta_{20}^2+12$	$12\zeta_{20}^6-12\zeta_{20}^4+12\zeta_{20}^2$	$12\zeta_{20}^4+12$	$-12\zeta_{20}^6+12$
2	ζ_{20}	0	$-12\zeta_{20}^6-12\zeta_{20}^2$	$12\zeta_{20}^6+12\zeta_{20}^2-12$	$12\zeta_{20}^4-12\zeta_{20}^2$	$-12\zeta_{20}^4+12\zeta_{20}^2-12$
2	$-\zeta_{20}^7$	0	$12\zeta_{20}^4-12\zeta_{20}^2$	$-12\zeta_{20}^4+12\zeta_{20}^2-12$	$12\zeta_{20}^6+12\zeta_{20}^2-12$	$-12\zeta_{20}^6-12\zeta_{20}^2$
2	$-\zeta_{20}^3$	0	$-12\zeta_{20}^4+12\zeta_{20}^2-12$	$12\zeta_{20}^4-12\zeta_{20}^2$	$-12\zeta_{20}^6-12\zeta_{20}^2$	$12\zeta_{20}^6+12\zeta_{20}^2-12$
2	$\zeta_{20}^7-\zeta_{20}^5+\zeta_{20}^3-\zeta_{20}$	0	$12\zeta_{20}^6+12\zeta_{20}^2-12$	$-12\zeta_{20}^6-12\zeta_{20}^2$	$-12\zeta_{20}^4+12\zeta_{20}^2-12$	$12\zeta_{20}^4-12\zeta_{20}^2$
2	$-\zeta_{20}$	0	$-12\zeta_{20}^6-12\zeta_{20}^2$	$12\zeta_{20}^6+12\zeta_{20}^2-12$	$12\zeta_{20}^4-12\zeta_{20}^2$	$-12\zeta_{20}^4+12\zeta_{20}^2-12$
2	ζ_{20}^7	0	$12\zeta_{20}^4-12\zeta_{20}^2$	$-12\zeta_{20}^4+12\zeta_{20}^2-12$	$12\zeta_{20}^6+12\zeta_{20}^2-12$	$-12\zeta_{20}^6-12\zeta_{20}^2$
2	ζ_{20}^3	0	$-12\zeta_{20}^4+12\zeta_{20}^2-12$	$12\zeta_{20}^4-12\zeta_{20}^2$	$-12\zeta_{20}^6-12\zeta_{20}^2$	$12\zeta_{20}^6+12\zeta_{20}^2-12$
2	$-\zeta_{20}^7+\zeta_{20}^5-\zeta_{20}^3+\zeta_{20}$	0	$12\zeta_{20}^6+12\zeta_{20}^2-12$	$-12\zeta_{20}^6-12\zeta_{20}^2$	$-12\zeta_{20}^4+12\zeta_{20}^2-12$	$12\zeta_{20}^4-12\zeta_{20}^2$
3	0	-30	$12(\zeta_5^3+\zeta_5^2+1)$	$12(\zeta_5^3+\zeta_5^2+1)$	$-12(\zeta_5^3+\zeta_5^2)$	$-12(\zeta_5^3+\zeta_5^2)$
3	0	-30	$-12(\zeta_5^3+\zeta_5^2)$	$-12(\zeta_5^3+\zeta_5^2)$	$12(\zeta_5^3+\zeta_5^2+1)$	$12(\zeta_5^3+\zeta_5^2+1)$
3	0	-30	$12(\zeta_5^2+\zeta_5+1)$	$-12(\zeta_5^2+\zeta_5)$	$-12(\zeta_5^3+\zeta_5)$	$12(\zeta_5^3+\zeta_5+1)$
3	0	-30	$-12(\zeta_5^3+\zeta_5)$	$12(\zeta_5^3+\zeta_5+1)$	$-12(\zeta_5^2+\zeta_5)$	$12(\zeta_5^2+\zeta_5+1)$
3	0	-30	$12(\zeta_5^3+\zeta_5+1)$	$-12(\zeta_5^3+\zeta_5)$	$12(\zeta_5^2+\zeta_5+1)$	$-12(\zeta_5^2+\zeta_5)$
3	0	-30	$-12(\zeta_5^2+\zeta_5)$	$12(\zeta_5^2+\zeta_5+1)$	$12(\zeta_5^3+\zeta_5+1)$	$-12(\zeta_5^3+\zeta_5)$
3	0	-30	$-12(\zeta_5^2+1)$	$-12(\zeta_5^3+1)$	$12(\zeta_5^3+\zeta_5^2+\zeta_5)$	$-12\zeta_5-12$
3	0	-30	$12(\zeta_5^3+\zeta_5^2+\zeta_5)$	$-12(\zeta_5+1)$	$-12(\zeta_5^3+1)$	$-12(\zeta_5^2+1)$
3	0	-30	$-12(\zeta_5+1)$	$12(\zeta_5^3+\zeta_5^2+\zeta_5)$	$-12(\zeta_5^2+1)$	$-12(\zeta_5^3+1)$
3	0	-30	$-12(\zeta_5^3+1)$	$-12(\zeta_5^2+1)$	$-12(\zeta_5+1)$	$12(\zeta_5^3+\zeta_5^2+\zeta_5)$
3	0	30	$12(\zeta_5^3+\zeta_5^2+1)$	$12(\zeta_5^3+\zeta_5^2+1)$	$-12(\zeta_5^3+\zeta_5^2)$	$-12(\zeta_5^3+\zeta_5^2)$
3	0	30	$-12(\zeta_5^3+\zeta_5^2)$	$-12(\zeta_5^3+\zeta_5^2)$	$12(\zeta_5^3+\zeta_5^2+1)$	$12(\zeta_5^3+\zeta_5^2+1)$
3	0	30	$12(\zeta_5^2+\zeta_5+1)$	$-12(\zeta_5^2+\zeta_5)$	$-12(\zeta_5^3+\zeta_5)$	$12(\zeta_5^3+\zeta_5+1)$
3	0	30	$-12(\zeta_5^3+\zeta_5)$	$12(\zeta_5^3+\zeta_5+1)$	$-12(\zeta_5^2+\zeta_5)$	$12(\zeta_5^2+\zeta_5+1)$
3	0	30	$12(\zeta_5^3+\zeta_5+1)$	$-12(\zeta_5^3+\zeta_5)$	$12(\zeta_5^2+\zeta_5+1)$	$-12(\zeta_5^2+\zeta_5)$
3	0	30	$-12(\zeta_5^2+\zeta_5)$	$12(\zeta_5^2+\zeta_5+1)$	$12(\zeta_5^3+\zeta_5+1)$	$-12(\zeta_5^3+\zeta_5)$
3	0	30	$-12(\zeta_5^2+1)$	$-12(\zeta_5^3+1)$	$12(\zeta_5^3+\zeta_5^2+\zeta_5)$	$-12\zeta_5-12$
3	0	30	$12(\zeta_5^3+\zeta_5^2+\zeta_5)$	$-12(\zeta_5+1)$	$-12(\zeta_5^3+1)$	$-12(\zeta_5^2+1)$
3	0	30	$-12(\zeta_5+1)$	$12(\zeta_5^3+\zeta_5^2+\zeta_5)$	$-12(\zeta_5^2+1)$	$-12(\zeta_5^3+1)$
3	0	30	$-12(\zeta_5^3+1)$	$-12(\zeta_5^2+1)$	$-12(\zeta_5+1)$	$12(\zeta_5^3+\zeta_5^2+\zeta_5)$
4	1	0	-12	-12	-12	-12
4	-1	0	-12	-12	-12	-12
4	$-\zeta_5^3-\zeta_5^2-\zeta_5-1$	0	$-12\zeta_5$	$12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$-12\zeta_5^2$	$-12\zeta_5^3$
4	ζ_5^3	0	$-12\zeta_5^2$	$-12\zeta_5^3$	$12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$-12\zeta_5$
4	ζ_5^2	0	$-12\zeta_5^3$	$-12\zeta_5^2$	$-12\zeta_5$	$12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$
4	ζ_5	0	$12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$-12\zeta_5$	$-12\zeta_5^3$	$-12\zeta_5^2$
4	$\zeta_5^3+\zeta_5^2+\zeta_5+1$	0	$-12\zeta_5$	$12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$-12\zeta_5^2$	$-12\zeta_5^3$
4	$-\zeta_5^3$	0	$-12\zeta_5^2$	$-12\zeta_5^3$	$12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$-12\zeta_5$
4	$-\zeta_5^2$	0	$-12\zeta_5^3$	$-12\zeta_5^2$	$-12\zeta_5$	$12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$
4	$-\zeta_5$	0	$12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$-12\zeta_5$	$-12\zeta_5^3$	$-12\zeta_5^2$
4	$-zeta4$	0	-12	-12	-12	-12
4	$zeta4$	0	-12	-12	-12	-12
4	$-\zeta_{20}$	0	$-12\zeta_{20}^4$	$12\zeta_{20}^6$	$-12\zeta_{20}^6+12\zeta_{20}^4-12\zeta_{20}^2+12$	$12\zeta_{20}^2$
4	ζ_{20}^7	0	$-12\zeta_{20}^6+12\zeta_{20}^4-12\zeta_{20}^2+12$	$12\zeta_{20}^2$	$12\zeta_{20}^6$	$-12\zeta_{20}^4$
4	ζ_{20}^3	0	$12\zeta_{20}^2$	$-12\zeta_{20}^6+12\zeta_{20}^4-12\zeta_{20}^2+12$	$-12\zeta_{20}^4$	$12\zeta_{20}^6$
4	$-\zeta_{20}^7+\zeta_{20}^5-\zeta_{20}^3+\zeta_{20}$	0	$12\zeta_{20}^6$	$-12\zeta_{20}^4$	$12\zeta_{20}^2$	$-12\zeta_{20}^6+12\zeta_{20}^4-12\zeta_{20}^2+12$
4	ζ_{20}	0	$-12\zeta_{20}^4$	$12\zeta_{20}^6$	$-12\zeta_{20}^6+12\zeta_{20}^4-12\zeta_{20}^2+12$	$12\zeta_{20}^2$
4	$-\zeta_{20}^7$	0	$-12\zeta_{20}^6+12\zeta_{20}^4-12\zeta_{20}^2+12$	$12\zeta_{20}^2$	$12\zeta_{20}^6$	$-12\zeta_{20}^4$
4	$-\zeta_{20}^3$	0	$12\zeta_{20}^2$	$-12\zeta_{20}^6+12\zeta_{20}^4-12\zeta_{20}^2+12$	$-12\zeta_{20}^4$	$12\zeta_{20}^6$
4	$\zeta_{20}^7-\zeta_{20}^5+\zeta_{20}^3-\zeta_{20}$	0	$12\zeta_{20}^6$	$-12\zeta_{20}^4$	$12\zeta_{20}^2$	$-12\zeta_{20}^6+12\zeta_{20}^4-12\zeta_{20}^2+12$
5	-1	30	0	0	0	0
5	1	-30	0	0	0	0
5	$\zeta_5^3+\zeta_5^2+\zeta_5+1$	30	0	0	0	0
5	$-\zeta_5^3$	30	0	0	0	0
5	$-\zeta_5^2$	30	0	0	0	0
5	$-\zeta_5$	30	0	0	0	0
5	$-\zeta_5^3-\zeta_5^2-\zeta_5-1$	-30	0	0	0	0
5	ζ_5^3	-30	0	0	0	0
5	ζ_5^2	-30	0	0	0	0
5	ζ_5	-30	0	0	0	0
6	0	0	12	12	12	12
6	0	0	12	12	12	12
6	0	0	$12\zeta_{20}^4$	$-12\zeta_{20}^6$	$12\zeta_{20}^6-12\zeta_{20}^4+12\zeta_{20}^2-12$	$-12\zeta_{20}^2$
6	0	0	$12\zeta_{20}^6-12\zeta_{20}^4+12\zeta_{20}^2-12$	$-12\zeta_{20}^2$	$-12\zeta_{20}^6$	$12\zeta_{20}^4$
6	0	0	$-12\zeta_{20}^2$	$12\zeta_{20}^6-12\zeta_{20}^4+12\zeta_{20}^2-12$	$12\zeta_{20}^4$	$-12\zeta_{20}^6$
6	0	0	$-12\zeta_{20}^6$	$12\zeta_{20}^4$	$-12\zeta_{20}^2$	$12\zeta_{20}^6-12\zeta_{20}^4+12\zeta_{20}^2-12$
6	0	0	$12\zeta_{20}^4$	$-12\zeta_{20}^6$	$12\zeta_{20}^6-12\zeta_{20}^4+12\zeta_{20}^2-12$	$-12\zeta_{20}^2$
6	0	0	$12\zeta_{20}^6-12\zeta_{20}^4+12\zeta_{20}^2-12$	$-12\zeta_{20}^2$	$-12\zeta_{20}^6$	$12\zeta_{20}^4$
6	0	0	$-12\zeta_{20}^2$	$12\zeta_{20}^6-12\zeta_{20}^4+12\zeta_{20}^2-12$	$12\zeta_{20}^4$	$-12\zeta_{20}^6$
6	0	0	$-12\zeta_{20}^6$	$12\zeta_{20}^4$	$-12\zeta_{20}^2$	$12\zeta_{20}^6-12\zeta_{20}^4+12\zeta_{20}^2-12$

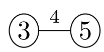
Diagram: 

$$\mathrm{FAC}_{G_{17}}(u,t) = \frac{1}{|G_{17}|}\left(e^{48u} - e^{-12u}\right)\left(e^{30t} - e^{-30t}\right)$$

The rank two exceptional group $W = G_{18}$

$\deg(\chi)$	$\chi(c^{-1})$	$\chi(R_0)$	$\chi(R_1)$	$\chi(R_2)$	$\chi(R_3)$	$\chi(R_4)$	$\chi(R_5)$
1	1	20	20	12	12	12	12
1	ζ_3	$20\zeta_3^2$	$20\zeta_3$	12	12	12	12
1	$-\zeta_3-1$	$20\zeta_3$	$20\zeta_3^2$	12	12	12	12
1	$-\zeta_5^3-\zeta_5^2-\zeta_5-1$	20	20	$12\zeta_5$	$-12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$12\zeta_5^2$	$12\zeta_5^3$
1	ζ_5^3	20	20	$12\zeta_5^2$	$12\zeta_5^3$	$-12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$12\zeta_5$
1	ζ_5^2	20	20	$12\zeta_5^3$	$12\zeta_5^2$	$12\zeta_5$	$-12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$
1	ζ_5	20	20	$-12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$12\zeta_5$	$12\zeta_5^3$	$12\zeta_5^2$
1	ζ_{15}^2	$-20(\zeta_{15}^5+1)$	$20\zeta_{15}^5$	$12\zeta_{15}^3$	$-12(\zeta_{15}^7+\zeta_{15}^2)$	$12\zeta_{15}^6$	$12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$
1	ζ_{15}^7	$20\zeta_{15}^5$	$-20(\zeta_{15}^5+1)$	$12\zeta_{15}^3$	$-12(\zeta_{15}^7+\zeta_{15}^2)$	$12\zeta_{15}^6$	$12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$
1	$-\zeta_{15}^7+\zeta_{15}^6-\zeta_{15}^4+\zeta_{15}^3-\zeta_{15}^2+1$	$-20(\zeta_{15}^5+1)$	$20\zeta_{15}^5$	$12\zeta_{15}^6$	$12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$	$-12(\zeta_{15}^7+\zeta_{15}^2)$	$12\zeta_{15}^3$
1	ζ_{15}^4	$20\zeta_{15}^5$	$-20(\zeta_{15}^5+1)$	$12\zeta_{15}^6$	$12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$	$-12(\zeta_{15}^7+\zeta_{15}^2)$	$12\zeta_{15}^3$
1	$-\zeta_{15}^6-\zeta_{15}$	$-20(\zeta_{15}^5+1)$	$20\zeta_{15}^5$	$12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$	$12\zeta_{15}^6$	$12\zeta_{15}^3$	$-12\zeta_{15}^7-12\zeta_{15}^2$
1	ζ_{15}	$20\zeta_{15}^5$	$-20(\zeta_{15}^5+1)$	$12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$	$12\zeta_{15}^6$	$12\zeta_{15}^3$	$-12\zeta_{15}^7-12\zeta_{15}^2$
1	$\zeta_{15}^7-\zeta_{15}^5+\zeta_{15}^4-\zeta_{15}^3+\zeta_{15}-1$	$-20(\zeta_{15}^5+1)$	$20\zeta_{15}^5$	$-12(\zeta_{15}^7+\zeta_{15}^2)$	$12\zeta_{15}^3$	$12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$	$12\zeta_{15}^6$
1	$-\zeta_{15}^7+\zeta_{15}^5-\zeta_{15}^4-\zeta_{15}+1$	$20\zeta_{15}^5$	$-20(\zeta_{15}^5+1)$	$-12(\zeta_{15}^7+\zeta_{15}^2)$	$12\zeta_{15}^3$	$12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$	$12\zeta_{15}^6$
2	0	-20	-20	$-12(\zeta_5^3+\zeta_5^2)-12$	$-12(\zeta_5^3+\zeta_5^2)-12$	$12(\zeta_5^3+\zeta_5)^2$	$12(\zeta_5^3+\zeta_5)^2$
2	0	-20	-20	$12(\zeta_5^3+\zeta_5)^2$	$12(\zeta_5^3+\zeta_5)^2$	$-12(\zeta_5^3+\zeta_5^2)-12$	$-12(\zeta_5^3+\zeta_5^2)-12$
2	0	-20	-20	$12\zeta_5^2+12$	$12\zeta_5^3+12$	$-12(\zeta_5^3+\zeta_5^2)-12\zeta_5$	$12\zeta_5+12$
2	0	-20	-20	$-12(\zeta_5^3+\zeta_5^2)-12\zeta_5$	$12\zeta_5+12$	$12\zeta_5^3+12$	$12\zeta_5^2+12$
2	0	-20	-20	$12\zeta_5+12$	$-12(\zeta_5^3+\zeta_5^2)-12\zeta_5$	$12\zeta_5^2+12$	$12\zeta_5^3+12$
2	0	-20	-20	$12\zeta_5^3+12$	$12\zeta_5^2+12$	$12\zeta_5+12$	$-12(\zeta_5^3+\zeta_5^2)-12\zeta_5$
2	0	-20	-20	$-12(\zeta_5^2+\zeta_5)-12$	$12\zeta_5^2+12\zeta_5$	$12(\zeta_5^3+\zeta_5)$	$-12(\zeta_5^3+\zeta_5)-12$
2	0	-20	-20	$12(\zeta_5^3+\zeta_5)$	$-12(\zeta_5^3+\zeta_5)-12$	$12\zeta_5^2+12\zeta_5$	$-12(\zeta_5^2+\zeta_5)-12$
2	0	-20	-20	$-12(\zeta_5^3+\zeta_5)-12$	$12(\zeta_5^3+\zeta_5)$	$-12(\zeta_5^2+\zeta_5)-12$	$12\zeta_5^2+12\zeta_5$
2	0	-20	-20	$12\zeta_5^2+12\zeta_5$	$-12(\zeta_5^2+\zeta_5)-12$	$-12(\zeta_5^3+\zeta_5)-12$	$12(\zeta_5^3+\zeta_5)$
2	0	$20(\zeta_{15}^5+1)$	$-20\zeta_{15}^5$	$-12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2)$	$-12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2)$	$12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2-1)$	$12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2-1)$
2	0	$20(\zeta_{15}^5+1)$	$-20\zeta_{15}^5$	$12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2-1)$	$12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2-1)$	$-12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2)$	$-12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2)$
2	0	$-20\zeta_{15}^5$	$20(\zeta_{15}^5+1)$	$-12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2)$	$-12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2)$	$12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2-1)$	$12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2-1)$
2	0	$-20\zeta_{15}^5$	$20(\zeta_{15}^5+1)$	$12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2-1)$	$12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2-1)$	$-12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2)$	$-12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2)$
2	0	$20(\zeta_{15}^5+1)$	$-20\zeta_{15}^5$	$12(\zeta_{15}^6+1)$	$12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2)$	$-12(\zeta_{15}^7+\zeta_{15}^2)+12$	$12(\zeta_{15}^3+1)$
2	0	$20(\zeta_{15}^5+1)$	$-20\zeta_{15}^5$	$-12(\zeta_{15}^7+\zeta_{15}^2)+12$	$12(\zeta_{15}^3+1)$	$12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2)$	$12(\zeta_{15}^6+1)$
2	0	$20(\zeta_{15}^5+1)$	$-20\zeta_{15}^5$	$12(\zeta_{15}^3+1)$	$-12(\zeta_{15}^7+\zeta_{15}^2-1)$	$12(\zeta_{15}^6+1)$	$12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2)$
2	0	$20(\zeta_{15}^5+1)$	$-20\zeta_{15}^5$	$12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2)$	$12(\zeta_{15}^6+1)$	$12(\zeta_{15}^3+1)$	$-12(\zeta_{15}^7+\zeta_{15}^2-1)$
2	0	$-20\zeta_{15}^5$	$20(\zeta_{15}^5+1)$	$12(\zeta_{15}^6+1)$	$12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2)$	$-12(\zeta_{15}^7+\zeta_{15}^2-1)$	$12(\zeta_{15}^3+1)$
2	0	$-20\zeta_{15}^5$	$20(\zeta_{15}^5+1)$	$-12(\zeta_{15}^7+\zeta_{15}^2-1)$	$12(\zeta_{15}^3+1)$	$12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2)$	$12(\zeta_{15}^6+1)$
2	0	$-20\zeta_{15}^5$	$20(\zeta_{15}^5+1)$	$12(\zeta_{15}^3+1)$	$-12(\zeta_{15}^7+\zeta_{15}^2-1)$	$12(\zeta_{15}^6+1)$	$12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2)$
2	0	$-20\zeta_{15}^5$	$20(\zeta_{15}^5+1)$	$12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2)$	$12(\zeta_{15}^6+1)$	$12(\zeta_{15}^3+1)$	$-12(\zeta_{15}^7+\zeta_{15}^2-1)$
2	0	$20(\zeta_{15}^5+1)$	$-20\zeta_{15}^5$	$-12(\zeta_{15}^6+\zeta_{15}^3+1)$	$12(\zeta_{15}^6+1\zeta_{15}^3)$	$12\zeta_{15}^7-12(\zeta_{15}^6+1\zeta_{15}^2-1)$	$-12(\zeta_{15}^7-12\zeta_{15}^6+\zeta_{15}^2)$
2	0	$20(\zeta_{15}^5+1)$	$-20\zeta_{15}^5$	$12(\zeta_{15}^7-\zeta_{15}^6+1\zeta_{15}^2-1)$	$-12(\zeta_{15}^7-12\zeta_{15}^6+\zeta_{15}^2)$	$12(\zeta_{15}^6+\zeta_{15}^3)$	$-12(\zeta_{15}^6+\zeta_{15}^3+1)$
2	0	$20(\zeta_{15}^5+1)$	$-20\zeta_{15}^5$	$-12(\zeta_{15}^7-12\zeta_{15}^6+\zeta_{15}^2)$	$12(\zeta_{15}^7-\zeta_{15}^6+\zeta_{15}^2-1)$	$-12(\zeta_{15}^6+\zeta_{15}^3+1)$	$12(\zeta_{15}^6+\zeta_{15}^3)$
2	0	$20(\zeta_{15}^5+1)$	$-20\zeta_{15}^5$	$12(\zeta_{15}^6+\zeta_{15}^3)$	$-12(\zeta_{15}^6+\zeta_{15}^3+1)$	$-12(\zeta_{15}^7-12\zeta_{15}^6+\zeta_{15}^2)$	$12(\zeta_{15}^7-\zeta_{15}^6+\zeta_{15}^2-1)$
2	0	$-20\zeta_{15}^5$	$20(\zeta_{15}^5+1)$	$-12(\zeta_{15}^6+\zeta_{15}^3+1)$	$12(\zeta_{15}^6+\zeta_{15}^3)$	$12(\zeta_{15}^7-\zeta_{15}^6+\zeta_{15}^2-1)$	$-12(\zeta_{15}^7-12\zeta_{15}^6+\zeta_{15}^2)$
2	0	$-20\zeta_{15}^5$	$20(\zeta_{15}^5+1)$	$12(\zeta_{15}^7-\zeta_{15}^6+\zeta_{15}^2-1)$	$-12(\zeta_{15}^7-12\zeta_{15}^6+\zeta_{15}^2)$	$12(\zeta_{15}^6+\zeta_{15}^3)$	$-12(\zeta_{15}^6+\zeta_{15}^3+1)$
2	0	$-20\zeta_{15}^5$	$20(\zeta_{15}^5+1)$	$-12(\zeta_{15}^7-12\zeta_{15}^6+\zeta_{15}^2)$	$12(\zeta_{15}^7-\zeta_{15}^6+\zeta_{15}^2-1)$	$-12(\zeta_{15}^6+\zeta_{15}^3+1)$	$12(\zeta_{15}^6+\zeta_{15}^3)$
2	0	$-20\zeta_{15}^5$	$20(\zeta_{15}^5+1)$	$12(\zeta_{15}^6+\zeta_{15}^3)$	$-12(\zeta_{15}^6+\zeta_{15}^3+1)$	$-12(\zeta_{15}^7-12\zeta_{15}^6+\zeta_{15}^2)$	$12(\zeta_{15}^7-\zeta_{15}^6+\zeta_{15}^2-1)$
3	-1	0	0	$-12(\zeta_5^3+\zeta_5^2)$	$-12(\zeta_5^3+\zeta_5^2)$	$12(\zeta_5^3+\zeta_5^2+1)$	$12(\zeta_5^3+\zeta_5^2+1)$
3	-1	0	0	$12(\zeta_5^3+\zeta_5^2+1)$	$12(\zeta_5^3+\zeta_5^2+1)$	$-12(\zeta_5^3+\zeta_5^2)$	$-12(\zeta_5^3+\zeta_5^2)$
3	$\zeta_5^3+\zeta_5^2+\zeta_5+1$	0	0	$-12(\zeta_5^2+1)$	$-12(\zeta_5^3+1)$	$12(\zeta_5^3+\zeta_5^2+\zeta_5)$	$-12\zeta_5-12$
3	$-\zeta_5^3$	0	0	$12(\zeta_5^3+\zeta_5^2+\zeta_5)$	$-12(\zeta_5+1)$	$-12(\zeta_5^3+1)$	$-12(\zeta_5^2+1)$
3	$-\zeta_5^2$	0	0	$-12(\zeta_5+1)$	$12(\zeta_5^3+\zeta_5^2+\zeta_5)$	$-12(\zeta_5^2+1)$	$-12(\zeta_5^3+1)$
3	$-\zeta_5$	0	0	$-12(\zeta_5^3+1)$	$-12(\zeta_5^2+1)$	$-12(\zeta_5+1)$	$12(\zeta_5^3+\zeta_5^2+\zeta_5)$
3	$\zeta_5^3+\zeta_5^2+\zeta_5+1$	0	0	$12(\zeta_5^2+\zeta_5+1)$	$-12(\zeta_5^2+\zeta_5)$	$-12(\zeta_5^3+\zeta_5)$	$12(\zeta_5^3+\zeta_5+1)$
3	$-\zeta_5^3$	0	0	$-12(\zeta_5^3+\zeta_5)$	$12(\zeta_5^3+\zeta_5+1)$	$-12(\zeta_5^2+\zeta_5)$	$12(\zeta_5^2+\zeta_5+1)$
3	$-\zeta_5^2$	0	0	$12(\zeta_5^3+\zeta_5+1)$	$-12(\zeta_5^3+\zeta_5)$	$12(\zeta_5^2+\zeta_5+1)$	$-12(\zeta_5^2+\zeta_5)$
3	$-\zeta_5$	0	0	$-12(\zeta_5^2+\zeta_5)$	$12(\zeta_5^2+\zeta_5+1)$	$12(\zeta_5^3+\zeta_5+1)$	$-12(\zeta_5^3+\zeta_5)$
3	$-\zeta_{15}^5$	0	0	$-12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2+1)$	$-12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2+1)$	$-12(\zeta_{15}^7+\zeta_{15}^3-\zeta_{15}^2)$	$-12(\zeta_{15}^7+\zeta_{15}^3-\zeta_{15}^2)$
3	$-\zeta_{15}^5$	0	0	$-12(\zeta_{15}^7+\zeta_{15}^3-\zeta_{15}^2)$	$-12(\zeta_{15}^7+\zeta_{15}^3-\zeta_{15}^2)$	$-12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2+1)$	$-12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2+1)$
3	ζ_{15}^5+1	0	0	$-12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2+1)$	$-12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2+1)$	$-12(\zeta_{15}^7+\zeta_{15}^3-\zeta_{15}^2)$	$-12(\zeta_{15}^7+\zeta_{15}^3-\zeta_{15}^2)$
3	ζ_{15}^5+1	0	0	$-12(\zeta_{15}^7+\zeta_{15}^3-\zeta_{15}^2)$	$-12(\zeta_{15}^7+\zeta_{15}^3-\zeta_{15}^2)$	$-12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2+1)$	$-12(\zeta_{15}^7-\zeta_{15}^3+\zeta_{15}^2+1)$
3	$-\zeta_{15}^2$	0	0	$-12(\zeta_{15}^6+1)$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2)$	$12(\zeta_{15}^7+\zeta_{15}^2-1)$	$-12(\zeta_{15}^3+1)$
3	$\zeta_{15}^7-\zeta_{15}^6+\zeta_{15}^4-\zeta_{15}^3+\zeta_{15}^2-1$	0	0	$12(\zeta_{15}^7+\zeta_{15}^2-1)$	$-12(\zeta_{15}^3+1)$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2)$	$-12(\zeta_{15}^6+1)$
3	$\zeta_{15}^6+\zeta_{15}$	0	0	$-12(\zeta_{15}^3+1)$	$12(\zeta_{15}^7+\zeta_{15}^2-1)$	$-12(\zeta_{15}^6+1)$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2)$
3	$-\zeta_{15}^7+\zeta_{15}^5-\zeta_{15}^4+\zeta_{15}^3-\zeta_{15}+1$	0	0	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2)$	$-12(\zeta_{15}^6+1)$	$-12(\zeta_{15}^3+1)$	$12(\zeta_{15}^7+\zeta_{15}^2-1)$
3	$-\zeta_{15}^7$	0	0	$-12(\zeta_{15}^6+1)$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2)$	$12(\zeta_{15}^7+\zeta_{15}^2-1)$	$-12(\zeta_{15}^3+1)$
3	$-\zeta_{15}^4$	0	0	$12(\zeta_{15}^7+\zeta_{15}^2-1)$	$-12(\zeta_{15}^3+1)$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2)$	$-12(\zeta_{15}^6+1)$
3	$-\zeta_{15}$	0	0	$-12(\zeta_{15}^3+1)$	$12(\zeta_{15}^7+\zeta_{15}^2-1)$	$-12(\zeta_{15}^6+1)$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2)$
3	$\zeta_{15}^7-\zeta_{15}^5+\zeta_{15}^4+\zeta_{15}-1$	0	0	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2)$	$-12(\zeta_{15}^6+1)$	$-12(\zeta_{15}^3+1)$	$12(\zeta_{15}^7+\zeta_{15}^2-1)$
3	$-\zeta_{15}^2$	0	0	$12(\zeta_{15}^6+\zeta_{15}^3+1)$	$-12(\zeta_{15}^6+\zeta_{15}^3)$	$-12(\zeta_{15}^7-12\zeta_{15}^6+\zeta_{15}^2-1)$	$12(\zeta_{15}^7-\zeta_{15}^6+\zeta_{15}^2)$
3	$\zeta_{15}^7-\zeta_{15}^6+\zeta_{15}^4-\zeta_{15}^3+\zeta_{15}^2-1$	0	0	$-12(\zeta_{15}^7-12\zeta_{15}^6+\zeta_{15}^2-1)$	$12(\zeta_{15}^7-\zeta_{15}^6+\zeta_{15}^2)$	$-12(\zeta_{15}^6+\zeta_{15}^3)$	$12(\zeta_{15}^6+\zeta_{15}^3+1)$
3	$\zeta_{15}^6+\zeta_{15}$	0	0	$12(\zeta_{15}^7-\zeta_{15}^6+\zeta_{15}^2)$	$-12(\zeta_{15}^7-12\zeta_{15}^6+\zeta_{15}^2-1)$	$12(\zeta_{15}^6+\zeta_{15}^3+1)$	$-12(\zeta_{15}^6+\zeta_{15}^3)$
3	$-\zeta_{15}^7+\zeta_{15}^5-\zeta_{15}^4+\zeta_{15}^3-\zeta_{15}+1$	0	0	$-12(\zeta_{15}^6+\zeta_{15}^3)$	$12(\zeta_{15}^6+\zeta_{15}^3+1)$	$12(\zeta_{15}^7-\zeta_{15}^6+\zeta_{15}^2)$	$-12(\zeta_{15}^7-12\zeta_{15}^6+\zeta_{15}^2-1)$
3	$-\zeta_{15}^7$	0	0	$12(\zeta_{15}^6+\zeta_{15}^3+1)$	$-12(\zeta_{15}^6+\zeta_{15}^3)$	$-12(\zeta_{15}^7-12\zeta_{15}^6+\zeta_{15}^2-1)$	$12(\zeta_{15}^7-\zeta_{15}^6+\zeta_{15}^2)$
3	$-\zeta_{15}^4$	0	0	$-12(\zeta_{15}^7-12\zeta_{15}^6+\zeta_{15}^2-1)$	$12(\zeta_{15}^7-\zeta_{15}^6+\zeta_{15}^2)$	$-12(\zeta_{15}^6+\zeta_{15}^3)$	$12(\zeta_{15}^6+\zeta_{15}^3+1)$
3	$-\zeta_{15}$	0	0	$12(\zeta_{15}^7-\zeta_{15}^6+\zeta_{15}^2)$	$-12(\zeta_{15}^7-12\zeta_{15}^6+\zeta_{15}^2-1)$	$12(\zeta_{15}^6+\zeta_{15}^3+1)$	$-12(\zeta_{15}^6+\zeta_{15}^3)$
3	$\zeta_{15}^7-\zeta_{15}^5+\zeta_{15}^4+\zeta_{15}-1$	0	0	$-12(\zeta_{15}^6+\zeta_{15}^3)$	$12(\zeta_{15}^6+\zeta_{15}^3+1)$	$12(\zeta_{15}^7-\zeta_{15}^6+\zeta_{15}^2)$	$-12(\zeta_{15}^7-12\zeta_{15}^6+\zeta_{15}^2-1)$
4	0	20	20	-12	-12	-12	-12
4	0	20	20	-12	-12	-12	-12
4	0	20	20	$-12\zeta_5$	$12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$-12\zeta_5^2$	$-12\zeta_5^3$
4	0	20	20	$-12\zeta_5^2$	$-12\zeta_5^3$	$12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$-12\zeta_5$
4	0	20	20	$-12\zeta_5^3$	$-12\zeta_5^2$	$-12\zeta_5$	$12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$
4	0	20	20	$12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$-12\zeta_5$	$-12\zeta_5^3$	$-12\zeta_5^2$
4	0	$20\zeta_3^2$	$20\zeta_3$	-12	-12	-12	-12
4	0	$20\zeta_3$	$20\zeta_3^2$	-12	-12	-12	-12
4	0	$-20(\zeta_{15}^5+1)$	$20\zeta_{15}^5$	$-12\zeta_{15}^3$	$12(\zeta_{15}^7+\zeta_{15}^2)$	$-12\zeta_{15}^6$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$
4	0	$-20(\zeta_{15}^5+1)$	$20\zeta_{15}^5$	$-12\zeta_{15}^6$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$	$12(\zeta_{15}^7+\zeta_{15}^2)$	$-12\zeta_{15}^3$
4	0	$-20(\zeta_{15}^5+1)$	$20\zeta_{15}^5$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$	$-12\zeta_{15}^6$	$-12\zeta_{15}^3$	$12(\zeta_{15}^7+\zeta_{15}^2)$
4	0	$-20(\zeta_{15}^5+1)$	$20\zeta_{15}^5$	$12(\zeta_{15}^7+\zeta_{15}^2)$	$-12\zeta_{15}^3$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$	$-12\zeta_{15}^6$
4	0	$20\zeta_{15}^5$	$-20(\zeta_{15}^5+1)$	$-12\zeta_{15}^3$	$12(\zeta_{15}^7+\zeta_{15}^2)$	$-12\zeta_{15}^6$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$
4	0	$20\zeta_{15}^5$	$-20(\zeta_{15}^5+1)$	$-12\zeta_{15}^6$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$	$12(\zeta_{15}^7+\zeta_{15}^2)$	$-12\zeta_{15}^3$
4	0	$20\zeta_{15}^5$	$-20(\zeta_{15}^5+1)$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$	$-12\zeta_{15}^6$	$-12\zeta_{15}^3$	$12(\zeta_{15}^7+\zeta_{15}^2)$
4	0	$20\zeta_{15}^5$	$-20(\zeta_{15}^5+1)$	$12(\zeta_{15}^7+\zeta_{15}^2)$	$-12\zeta_{15}^3$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$	$-12\zeta_{15}^6$
4	0	20	20	$-12\zeta_5$	$12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$-12\zeta_5^2$	$-12\zeta_5^3$
4	0	20	20	$-12\zeta_5^2$	$-12\zeta_5^3$	$12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$-12\zeta_5$
4	0	20	20	$-12\zeta_5^3$	$-12\zeta_5^2$	$-12\zeta_5$	$12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$
4	0	20	20	$12(\zeta_5^3+\zeta_5^2+\zeta_5+1)$	$-12\zeta_5$	$-12\zeta_5^3$	$-12\zeta_5^2$
4	0	$20\zeta_3^2$	$20\zeta_3$	-12	-12	-12	-12
4	0	$20\zeta_3$	$20\zeta_3^2$	-12	-12	-12	-12
4	0	$-20(\zeta_{15}^5+1)$	$20\zeta_{15}^5$	$-12\zeta_{15}^3$	$12(\zeta_{15}^7+\zeta_{15}^2)$	$-12\zeta_{15}^6$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$
4	0	$-20(\zeta_{15}^5+1)$	$20\zeta_{15}^5$	$-12\zeta_{15}^6$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$	$12(\zeta_{15}^7+\zeta_{15}^2)$	$-12\zeta_{15}^3$
4	0	$-20(\zeta_{15}^5+1)$	$20\zeta_{15}^5$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$	$-12\zeta_{15}^6$	$-12\zeta_{15}^3$	$12(\zeta_{15}^7+\zeta_{15}^2)$
4	0	$-20(\zeta_{15}^5+1)$	$20\zeta_{15}^5$	$12(\zeta_{15}^7+\zeta_{15}^2)$	$-12\zeta_{15}^3$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$	$-12\zeta_{15}^6$
4	0	$20\zeta_{15}^5$	$-20(\zeta_{15}^5+1)$	$-12\zeta_{15}^3$	$12(\zeta_{15}^7+\zeta_{15}^2)$	$-12\zeta_{15}^6$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$
4	0	$20\zeta_{15}^5$	$-20(\zeta_{15}^5+1)$	$-12\zeta_{15}^6$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$	$12(\zeta_{15}^7+\zeta_{15}^2)$	$-12\zeta_{15}^3$
4	0	$20\zeta_{15}^5$	$-20(\zeta_{15}^5+1)$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$	$-12\zeta_{15}^6$	$-12\zeta_{15}^3$	$12(\zeta_{15}^7+\zeta_{15}^2)$
4	0	$20\zeta_{15}^5$	$-20(\zeta_{15}^5+1)$	$12(\zeta_{15}^7+\zeta_{15}^2)$	$-12\zeta_{15}^3$	$-12(\zeta_{15}^7-\zeta_{15}^6-\zeta_{15}^3+\zeta_{15}^2-1)$	$-12\zeta_{15}^6$

$\deg(\chi)$	$\chi(c^{-1})$	$\chi(R_0)$	$\chi(R_1)$	$\chi(R_2)$	$\chi(R_3)$	$\chi(R_4)$	$\chi(R_5)$
5	1	-20	-20	0	0	0	0
5	$-\zeta_5^3 - \zeta_5^2 - \zeta_5 - 1$	-20	-20	0	0	0	0
5	ζ_5^3	-20	-20	0	0	0	0
5	ζ_5^2	-20	-20	0	0	0	0
5	ζ_5	-20	-20	0	0	0	0
5	ζ_3	$-20\zeta_3^2$	$-20\zeta_3$	0	0	0	0
5	$-\zeta_3 - 1$	$-20\zeta_3$	$-20\zeta_3^2$	0	0	0	0
5	ζ_{15}^2	$20(\zeta_{15}^5 + 1)$	$-20\zeta_{15}^5$	0	0	0	0
5	$-\zeta_{15}^7 + \zeta_{15}^6 - \zeta_{15}^4 + \zeta_{15}^3 - \zeta_{15}^2 + 1$	$20(\zeta_{15}^5 + 1)$	$-20\zeta_{15}^5$	0	0	0	0
5	$-\zeta_{15}^6 - \zeta_{15}$	$20(\zeta_{15}^5 + 1)$	$-20\zeta_{15}^5$	0	0	0	0
5	$\zeta_{15}^7 - \zeta_{15}^5 + \zeta_{15}^4 - \zeta_{15}^3 + \zeta_{15} - 1$	$20(\zeta_{15}^5 + 1)$	$-20\zeta_{15}^5$	0	0	0	0
5	ζ_{15}^7	$-20\zeta_{15}^5$	$20(\zeta_{15}^5 + 1)$	0	0	0	0
5	ζ_{15}^4	$-20\zeta_{15}^5$	$20(\zeta_{15}^5 + 1)$	0	0	0	0
5	ζ_{15}	$-20\zeta_{15}^5$	$20(\zeta_{15}^5 + 1)$	0	0	0	0
5	$-\zeta_{15}^7 + \zeta_{15}^5 - \zeta_{15}^4 - \zeta_{15} + 1$	$-20\zeta_{15}^5$	$20(\zeta_{15}^5 + 1)$	0	0	0	0
6	0	0	0	12	12	12	12
6	0	0	0	$12\zeta_5$	$-12(\zeta_5^3 + \zeta_5^2 + \zeta_5 + 1)$	$12\zeta_5^2$	$12\zeta_5^3$
6	0	0	0	$12\zeta_5^2$	$12\zeta_5^3$	$-12(\zeta_5^3 + \zeta_5^2 + \zeta_5 + 1)$	$12\zeta_5$
6	0	0	0	$12\zeta_5^3$	$12\zeta_5^2$	$12\zeta_5$	$-12(\zeta_5^3 + \zeta_5^2 + \zeta_5 + 1)$
6	0	0	0	$-12(\zeta_5^3 + \zeta_5^2 + \zeta_5 + 1)$	$12\zeta_5$	$12\zeta_5^3$	$12\zeta_5^2$
6	0	0	0	12	12	12	12
6	0	0	0	12	12	12	12
6	0	0	0	$12\zeta_{15}^3$	$-12(\zeta_{15}^7 + \zeta_{15}^2)$	$12\zeta_{15}^6$	$12(\zeta_{15}^7 - \zeta_{15}^6 - \zeta_{15}^3 + \zeta_{15}^2 - 1)$
6	0	0	0	$12\zeta_{15}^6$	$12(\zeta_{15}^7 - \zeta_{15}^6 - \zeta_{15}^3 + \zeta_{15}^2 - 1)$	$-12(\zeta_{15}^7 + \zeta_{15}^2)$	$12\zeta_{15}^3$
6	0	0	0	$12(\zeta_{15}^7 - \zeta_{15}^6 - \zeta_{15}^3 + \zeta_{15}^2 - 1)$	$12\zeta_{15}^6$	$12\zeta_{15}^3$	$-12\zeta_{15}^7 - 12\zeta_{15}^2$
6	0	0	0	$-12(\zeta_{15}^7 + \zeta_{15}^2)$	$12\zeta_{15}^3$	$12(\zeta_{15}^7 - \zeta_{15}^6 - \zeta_{15}^3 + \zeta_{15}^2 - 1)$	$12\zeta_{15}^6$
6	0	0	0	$12\zeta_{15}^3$	$-12(\zeta_{15}^7 + \zeta_{15}^2)$	$12\zeta_{15}^6$	$12(\zeta_{15}^7 - \zeta_{15}^6 - \zeta_{15}^3 + \zeta_{15}^2 - 1)$
6	0	0	0	$12\zeta_{15}^6$	$12(\zeta_{15}^7 - \zeta_{15}^6 - \zeta_{15}^3 + \zeta_{15}^2 - 1)$	$-12(\zeta_{15}^7 + \zeta_{15}^2)$	$12\zeta_{15}^3$
6	0	0	0	$12(\zeta_{15}^7 - \zeta_{15}^6 - \zeta_{15}^3 + \zeta_{15}^2 - 1)$	$12\zeta_{15}^6$	$12\zeta_{15}^3$	$-12(\zeta_{15}^7 + \zeta_{15}^2)$
6	0	0	0	$-12(\zeta_{15}^7 + \zeta_{15}^2)$	$12\zeta_{15}^3$	$12(\zeta_{15}^7 - \zeta_{15}^6 - \zeta_{15}^3 + \zeta_{15}^2 - 1)$	$12\zeta_{15}^6$

Diagram: 

$$\mathrm{FAC}_{G_{18}}(u,t) = \frac{1}{|G_{18}|}\left(e^{40u} - e^{-20u}\right)\left(e^{48t} - e^{-12t}\right)$$

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