PROOF OF HAN'S HOOK EXPANSION CONJECTURE

KEVIN CARDE, JOE LOUBERT, AARON POTECHIN, AND ADRIAN SANBORN

Abstract. We primarily prove a conjecture by Guo-Niu Han which interpolates between two classical hook expansion formulas. We give an equivalent formulation of this conjecture and expand one side in terms of Schur functions of skew tableaux. From looking at low-degree terms, we derive several identities involving the numbers of hooktypes and border strips. Finally, we prove a theorem which calculates the alternating sum of hooktypes over standard Young tableaux with n boxes.

Contents

1. Introduction

A *partition* λ is a sequence of integers $\lambda = (\lambda_1, ..., \lambda_\ell)$ such that $\lambda_1 \geq ... \geq \lambda_\ell \geq 1$. The integers $\lambda_1, ..., \lambda_\ell$ are called the *parts* of λ, and the number of parts ℓ of λ is called the *length* of λ. Define $|\lambda| = \lambda_1 + ... + \lambda_\ell$. For some positive integer n, we say that λ *is a partition of* n if $n = |\lambda|$, and we write $\lambda \vdash n$.

With each partition λ with length ℓ we associate a *Young diagram*, a lattice of left-justified and top-justified boxes with λ_i boxes in the ith row, for $i = 1, ..., \ell$. The Young diagram for the partition $(5, 3, 2)$ is shown in Figure 1(a). We think of these boxes as ordered pairs (i, j) such that $i = 1, ..., \ell$ and $1 \leq j \leq \lambda_i$. If $x = (i, j)$ is a box in the Young diagram of λ, then we can write $x \in \lambda$. With each $x \in \lambda$ we associate a number called the *hook length* of x in λ, denoted $h_\lambda(x)$, which is the

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(a) Young diagram

7	6	4	2	1
4	3	1		
2	1			

(b) Hook lengths

1	2	4	5	7
3	8	10		
6	9			

(c) standard Young tableau

FIGURE 1

number of squares y in the Young diagram of λ such that $y = x$, or y appears below x in the same column, or y appears to the right of x in the same row. Often the partition λ is assumed and we simply write $h(x)$. An example of the hook lengths for a partition are shown in Figure 1(b).

A *standard Young tableau (SYT)* of shape $\lambda \vdash n$ is a filling of the boxes in the Young diagram of λ with the numbers $1, ..., n$ such that the numbers increase along each row and each column. Figure 1(c) shows one SYT for $(5, 3, 2)$. Let f^λ denote the number of SYT of shape λ and $\mathrm{SYT}(n)$ denote the set of SYT of all shapes $\lambda \vdash n$.

Two classical results in the study of integer partitions relate hooks of a shape and the number of SYT of the shape. The first is the Hook Formula, due to Frame, Robinson, and Thrall [1], which states

$$f^\lambda = \frac{n!}{\prod_{x \in \lambda} h(x)}. \tag{1.1}$$

The second result follows from the Robinson-Schensted-Knuth Algorithm (RSK), which establishes a bijection between elements of S_n (permutations of the integers $\{1, ..., n\}$) and pairs of SYT with the same shape $\lambda \vdash n$. Thus, we have

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2.$$

For an explanation of the RSK algorithm, see Sagan §3.3 [5]. Combining these two identities gives us the hook expansion formula

$$e^t = \sum_{n=0}^{\infty} t^n \sum_{\lambda \vdash n} \prod_{x \in \lambda} \frac{1}{h(x)^2}. \tag{1.2}$$

Let $\mathrm{Inv}(n) = \{\pi \in S_n \mid \pi = \pi^{-1}\}$ denote the set of involutions in S_n; $\mathrm{Inv}(0) = \{1\}$. The RSK algorithm also has the following property: for $\pi \in S_n$, if $\pi \overset{RSK}{\longleftrightarrow} (P, Q)$, then $\pi^{-1} \overset{RSK}{\longleftrightarrow} (Q, P)$. Thus, we have a bijection between involutions of n elements and SYT with n boxes; i.e.,

$$|\,\mathrm{Inv}(n)| = |\,\mathrm{SYT}(n)| = \sum_{\lambda \vdash n} f^\lambda.$$

This gives a second hook expansion formula

$$
\begin{aligned}
e^{t+t^2/2} &= \sum_{n=0}^{\infty} |\operatorname{Inv}(n)| \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} t^n \sum_{\lambda \vdash n} \prod_{x \in \lambda} \frac{1}{h(x)}.
\end{aligned} \tag{1.3}
$$

This paper focuses on proving the following equation conjectured by Han (Conjecture 2.1 in [3] and Conjecture 1.4 in [2]) which interpolates between the two hook expansions given above—setting $z = 0$ yields (1.2) and setting $z = 1$ yields (1.3).

Theorem 1.1.

$$
e^{t+zt^2/2} = \sum_{n=0}^{\infty} t^n \sum_{\lambda \vdash n} \prod_{x \in \lambda} \rho(h(x), z) \tag{1.4}
$$

where

$$
\rho(n, z) = \frac{\displaystyle\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} z^k}{n \displaystyle\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} z^k}.
$$

The left hand side of (1.4) is interesting because it has a combinatorial interpretation in terms of involutions

$$
e^{t+zt^2/2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\pi \in \operatorname{Inv}(n)} z^{\alpha(\pi)}
$$

where $\alpha(\pi)$ is the number of 2-cycles of the permutation π. To prove Theorem 1.1, we first formulate an equivalent claim through two change-of-variable substitutions:

Theorem 1.1$'$ (Reformulation of Theorem 1.1)**.** *For all $n \geq 0$,*

$$
\sum_{\pi \in \operatorname{Inv}(n)} \left(\frac{1+q}{1-q} \right)^{\beta(\pi)} = \sum_{\lambda \vdash n} f^{\lambda} \prod_{x \in \lambda} \frac{1 + q^{h(x)}}{1 - q^{h(x)}} \tag{1.5}
$$

where $\beta(\pi)$ is the number of fixed points of the permutation π.

Accordingly, we define

$$
\begin{aligned}
w(h) &= \frac{1+q^h}{1-q^h} \\
w(\lambda) &= \prod_{x \in \lambda} w(h(x)) = \prod_{x \in \lambda} \frac{1+q^{h(x)}}{1-q^{h(x)}}.
\end{aligned}
$$

In the proof of Theorem 1.1$'$, we primarily use the following result

Lemma 1.2 (Extension-Retraction)**.** *Fix $\lambda \vdash n$. Then*

$$
\sum_{\lambda^+ \gtrdot \lambda} w(\lambda^+) = w(1) w(\lambda) + \sum_{\lambda^- \lessdot \lambda} w(\lambda^-)
$$

where $\lambda^+ \gtrdot \lambda$ (resp. $\lambda^- \lessdot \lambda$) indicates that λ^+ (resp. λ^-) is obtained by adding (resp. removing) a square to λ.

Summing the Lemma over $\mathrm{SYT}(n)$ yields a recursion for $w(\lambda)$ similar to a recursion on involutions counting fixed points. This recursion inductively proves Theorem 1.1′, completing the proof of the main result; see Section 2.3 below.

After proving the main result, we give a quick review of Schur functions in Section 3. In Sections 4 and 5, we deduce several identities by equating coefficients of q^m on both sides of Theorem 1.1′. For small values of m, the identities are fairly simple and easy to verify independently of the main result. The cubic term in particular suggests an elegant generalization, the Alternating Hook Sum Theorem, which we prove and apply to a corollary in Section 6.

2. Proof of Theorem 1.1

Before we begin the various parts involved in the proof of the main result, we present a combinatorial interpretation of the left hand side of the main result using a recursion on involutions that counts fixed points and 2-cycles. The result will be used several times in the proof of the main result.

We then begin the proof of Theorem 1.1 in Section 2.2 by showing the equivalence of Theorem 1.1 and Theorem 1.1′. In Section 2.3, we prove that the Extension-Retraction Lemma (Lemma 1.2) implies Theorem 1.1′. Finally, in Section 2.4, we prove the Extension-Retraction Lemma in two stages: first, by translating the statement into an equation relating contents of certain squares in a partition λ, and then proving that the equation is true with the contents replaced by arbitrary variables.

2.1. **Recursion on Involutions.**

Proposition 2.1. *Let $\alpha(\pi)$ and $\beta(\pi)$ denote the number of 2-cycles and fixed points of a permutation π respectively. Then*

$$e^{ut+vt^2/2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\pi \in \mathrm{Inv}(n)} u^{\beta(\pi)} v^{\alpha(\pi)}.$$

Proof. Expand

$$e^{ut+vt^2/2} = \sum_{n=0}^{\infty} f_n(u,v) \frac{t^n}{n!}.$$

Let $g_n(u,v) = \sum_{\pi \in \mathrm{Inv}(n)} u^{\beta(\pi)} v^{\alpha(\pi)}$. Then we wish to show that $f_n(u,v) = g_n(u,v)$. By taking the coefficient of $t^n/n!$ in the equation

$$\frac{\partial}{\partial t} e^{ut+vt^2/2} = (u+vt)e^{ut+vt^2/2},$$

we have the relation

$$\frac{f_{n+1}(u,v)}{n!} = u\frac{f_n(u,v)}{n!} + v\frac{f_{n-1}(u,v)}{(n-1)!}$$
$$\Rightarrow f_{n+1}(u,v) = u \cdot f_n(u,v) + nv \cdot f_{n-1}(u,v)$$

for all $n \geq 1$. Now consider $g_n(u,v)$: given $\pi \in \mathrm{Inv}(n+1)$, either $\pi(n+1) = n+1$ or $\pi(n+1) = i$ for some $i = 1, ..., n$.

In the first case, restricting π to the numbers 1 through n produces an element of $\mathrm{Inv}(n)$ with the same number of 2-cycles but one more fixed point, so the term $u^{\beta(\pi)}v^{\alpha(\pi)}$ in $g_{n+1}(u,v)$ cancels with a unique term in $u \cdot g_n(u,v)$.

In the second case, if $\pi(n+1) = i$, then restricting π to the numbers $\{1,2,...,n\} - \{i\}$ produces an element of $\mathrm{Inv}(n-1)$ with one fewer 2-cycle, so the term $u^{\beta(\pi)}v^{\alpha(\pi)}$ in $g_{n+1}(u,v)$ cancels with a unique term in $v \cdot g_{n-1}(u,v)$. Summing over $i = 1,...,n$, we have shown that g_n satisfies the relation

$$g_{n+1}(u,v) = u \cdot g_n(u,v) + nv \cdot g_{n-1}(u,v)$$

by showing that all the terms cancel. Finally, it is easy to see that $f_0 = 1 = g_0$ and $f_1 = u = g_1$, so $f_n = g_n$ for all $n \geq 0$, as desired. □

2.2. **Equivalence of Theorem 1.1 and Theorem 1.1′.**

Proof. Using the Binomial Theorem, we can rewrite Han's weight function as

$$\rho(n,z) = \frac{(1+\sqrt{z})^n + (1-\sqrt{z})^n}{(1+\sqrt{z})^n - (1-\sqrt{z})^n} \cdot \frac{\sqrt{z}}{n} = \frac{1 + \left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^n}{1 - \left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^n} \cdot \frac{\sqrt{z}}{n}.$$

Substitute

$$q = \frac{1-\sqrt{z}}{1+\sqrt{z}}$$
$$T = t\sqrt{z}$$

into Theorem 1.1, then simplify and use the Hook Formula (1.1). This gives

$$\begin{aligned} e^{T\frac{1+q}{1-q} + \frac{T^2}{2}} &= \sum_{n=0}^{\infty} \frac{T^n}{z^{n/2}} \sum_{\lambda \vdash n} \prod_{x \in \lambda} \left(\frac{1+q^{h(x)}}{1-q^{h(x)}} \frac{\sqrt{z}}{h(x)} \right) \\ &= \sum_{n=0}^{\infty} \frac{T^n}{n!} \sum_{\lambda \vdash n} f^\lambda \prod_{x \in \lambda} \frac{1+q^{h(x)}}{1-q^{h(x)}}. \end{aligned} \tag{2.1}$$

On the other hand, setting $u = (1+q)/(1-q)$ and $v = 1$ in Proposition 2.1, we express the exponential in (2.1) as an infinite sum which counts involutions by their fixed points. Explicitly, we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{T^n}{n!} \sum_{\pi \in \mathrm{Inv}(n)} \left(\frac{1+q}{1-q}\right)^{\beta(\pi)} &= e^{T\frac{1+q}{1-q} + T^2/2} \\ &= \sum_{n=0}^{\infty} \frac{T^n}{n!} \sum_{\lambda \vdash n} f^\lambda \prod_{x \in \lambda} \frac{1+q^{h(x)}}{1-q^{h(x)}}. \end{aligned}$$

Equating coefficients of $T^n/n!$ gives the desired formulation. □

2.3. **Extension-Retraction Lemma Implies Theorem 1.1.**

Proof. Define

$$\phi_n = \sum_{\lambda \vdash n} f^\lambda w(\lambda)$$
$$\psi_n = \sum_{\pi \in \mathrm{Inv}(n)} w(1)^{\beta(\pi)}.$$

Thus, we wish to show that $\phi_n = \psi_n$ for all $n \geq 0$. By Proposition 2.1, ψ_n satisfies the recursion

$$\psi_{n+1} = w(1)\psi_n + n \cdot \psi_{n-1}.$$

Since $\psi_0 = 1 = \phi_0$ and $\psi_1 = (1+q)/(1-q) = \phi_1$, it suffices to show that ϕ_n satisfies the same recursion; namely,

$$\phi_{n+1} = w(1)\phi_n + n \cdot \phi_{n-1}.$$

For $P \in \mathrm{SYT}(n)$, let $\lambda(P)$ be the partition corresponding to the shape of P. Notice that we can alternatively write

$$\phi_n = \sum_{P \in \mathrm{SYT}(n)} w(\lambda(P)).$$

Suppose the Extension-Retraction Lemma (Lemma 1.2) is true; i.e., for all $\lambda \vdash n$,

$$\sum_{\lambda^+ \gtrdot \lambda} w(\lambda^+) = w(1)w(\lambda) + \sum_{\lambda^- \lessdot \lambda} w(\lambda^-) \tag{2.2}$$

where $\lambda^+ \gtrdot \lambda$ indicates that λ^+ is a partition such that $\lambda^+ > \lambda$ by the inclusion ordering and $|\lambda^+| = |\lambda| + 1$. If $\lambda^+ \gtrdot \lambda$, then λ^+ is simply obtained from λ by adding a square, and we call λ^+ an *extension of* λ *(by one square)*. Similarly, $\lambda^- \lessdot \lambda$, then λ^- is obtained from λ by removing a corner square, and we call λ^- a *retraction of* λ *(by one square)*.

Summing (2.2) over all SYT P of shape λ for shapes $\lambda \vdash n$, one obtains

$$\sum_{P \in \mathrm{SYT}(n)} \sum_{\lambda^+ \gtrdot \lambda(P)} w(\lambda^+) = w(1) \sum_{P \in \mathrm{SYT}(n)} w(\lambda(P)) + \sum_{P \in \mathrm{SYT}(n)} \sum_{\lambda^- \lessdot \lambda(P)} w(\lambda^-). \tag{2.3}$$

In the sum on the left hand side, we can lift a SYT P of λ to a SYT P^+ of λ^+ by labeling the new square in λ^+ with the number $n+1$. Indeed, every such P^+ is clearly obtained exactly once in this way. Thus, (2.3) is equivalent to

$$\sum_{P^+ \in \mathrm{SYT}(n+1)} w(\lambda(P^+)) = w(1) \sum_{P \in \mathrm{SYT}(n)} w(\lambda(P)) + \sum_{P \in \mathrm{SYT}(n)} \sum_{x \text{ corner}} w(\lambda(P) - x) \tag{2.4}$$

where the last sum is over corner cells $x \in \lambda(P)$. To simplify the second term on the right hand side, note that the RSK algorithm establishes a bijection

$$\begin{aligned}&\{(P^-, i) \mid P^- \in \mathrm{SYT}(n-1), i \in \{1, \cdots, n\}\} \overset{RSK}{\longleftrightarrow} \\ &\{(P, x) \mid P \in \mathrm{SYT}(n), x \text{ is a corner square of } P\}.\end{aligned}$$

The algorithm proceeds as follows: given $P^- \in \mathrm{SYT}(n-1)$ and some $i \in \{1, 2, \cdots, n\}$, increment each number j in P^- such that $j \geq i$. This produces a SYT which is filled with the numbers $1, \cdots, i-1, i+1, \cdots n$. Finally, insert i into this SYT using the RSK method to give a SYT $P \in \mathrm{SYT}(n)$ and a corner square x of P which is the single square in P but not in P^-. Clearly, this process is reversible—given a SYT P and a corner square x, simply eject the square using the RSK method and decrement the numbers appropriately.

Thus, we have

$$\sum_{P \in \mathrm{SYT}(n)} \sum_{x \text{ corner}} w(\lambda(P) - x) = \sum_{P^- \in \mathrm{SYT}(n-1)} \sum_{i=1}^{n} w(\lambda(P^-))$$

Finally, substituting this into (2.4), we obtain

$$\sum_{P^+\in \mathrm{SYT}(n+1)} w(\lambda(P^+)) = w(1) \sum_{P\in \mathrm{SYT}(n)} w(\lambda(P)) + n \sum_{P^-\in \mathrm{SYT}(n-1)} w(\lambda(P^-))$$
$$\Rightarrow \phi_{n+1} = w(1)\phi_n + n\cdot \phi_{n-1}$$

as desired. □

2.4. **Proof of the Extension-Retraction Lemma.** [1] Let λ be a partition. Label the outer corners of λ (i.e., squares outside λ which are directly below and to the right of squares in λ) as $M_1(a_1,b_1),...,M_d(a_d,b_d)$ and label the inner corners (i.e. 1-hooks) as $N_1(\alpha_1,\beta_1),...,N_{d-1}(\alpha_{d-1},\beta_{d-1})$. Define the *content* of the square (i,j) to be $c(i,j) = j-i$. To prove the Extension-Retraction Lemma for λ, we will reduce it to an equation relating the contents of the inner and outer corners of λ. Then we will prove that this equation is in fact true with the contents replaced by arbitrary variables.

If λ^+ is obtained from λ by adding an outer corner M_k, then we can find an explicit formula for $w(\lambda^+)$ in terms of $w(\lambda)$ and the contents of the outer and inner corners. The terms of $w(\lambda^+)$ mostly agree with the terms of $w(\lambda)$ because the hook length of a square will only change if it is in the same row or column as M_k. Moreover, when the hook length changes, it must increase by one. Finally, we introduce the 1-hook M_k as an extra term in $w(\lambda^+)$.

$$w(\lambda^+) = w(\lambda)w(1)\left(\prod_{j=1}^{a_k-1}\frac{w(h_{\lambda^+}(j,b_k))}{w(h_\lambda(j,b_k))}\right)\left(\prod_{j=1}^{b_k-1}\frac{w(h_{\lambda^+}(a_k,j))}{w(h_\lambda(a_k,j))}\right)$$

Within these products, more terms cancel. If there is no inner corner in row j, then $h_\lambda(j,b_k) = h_\lambda(j+1,b_k)+1$. Also, row j has an inner corner if and only if row $j+1$ has an outer corner. Hence, if row $j+1$ has no inner corner, then the term $w(h_{\lambda^+}(j,b_k))$ in the numerator cancels with the term $w(h_\lambda(j+1,b_k))$ in the denominator, and only the terms in rows or columns with inner or outer corners remain. (See Figure 2 for an example.)

This allows us to write

$$\prod_{j=1}^{a_k-1}\frac{w(h_{\lambda^+}(j,b_k))}{w(h_\lambda(j,b_k))} = \frac{w(h_{\lambda^+}(a_1,b_k))}{w(h_\lambda(\alpha_1,b_k))}\cdot\frac{w(h_{\lambda^+}(a_2,b_k))}{w(h_\lambda(\alpha_2,b_k))}\cdots\frac{w(h_{\lambda^+}(a_{k-1},b_k))}{w(h_\lambda(\alpha_{k-1},b_k))}.$$

Similarly, swapping rows and columns above,

$$\prod_{j=1}^{b_k-1}\frac{w(h_{\lambda^+}(a_k,j))}{w(h_\lambda(a_k,j))} = \frac{w(h_{\lambda^+}(a_k,b_d))}{w(h_\lambda(a_k,\beta_{d-1}))}\cdot\frac{w(h_{\lambda^+}(a_k,b_{d-1}))}{w(h_\lambda(a_k,\beta_{d-2}))}\cdots\frac{w(h_{\lambda^+}(a_k,b_{k+1}))}{w(h_\lambda(a_k,\beta_k))}.$$

For $i<k$, the hook of (a_i,b_k) in λ starts one square to the left of M_i and ends one square above M_k. Since the content decreases by 1 when going to the left or down, the contents of the squares in the hook of (a_i,b_k) are consecutive integers. It follows that the length of the hook is just $c(a_i,b_i-1)-c(a_k-1,b_k)+1 = c(M_i)-c(M_k)-1$.

[1]Ideas used in this proof go back to a proof of the Hook Forumla by Kerov. (See Lemma 3.1 in [4])

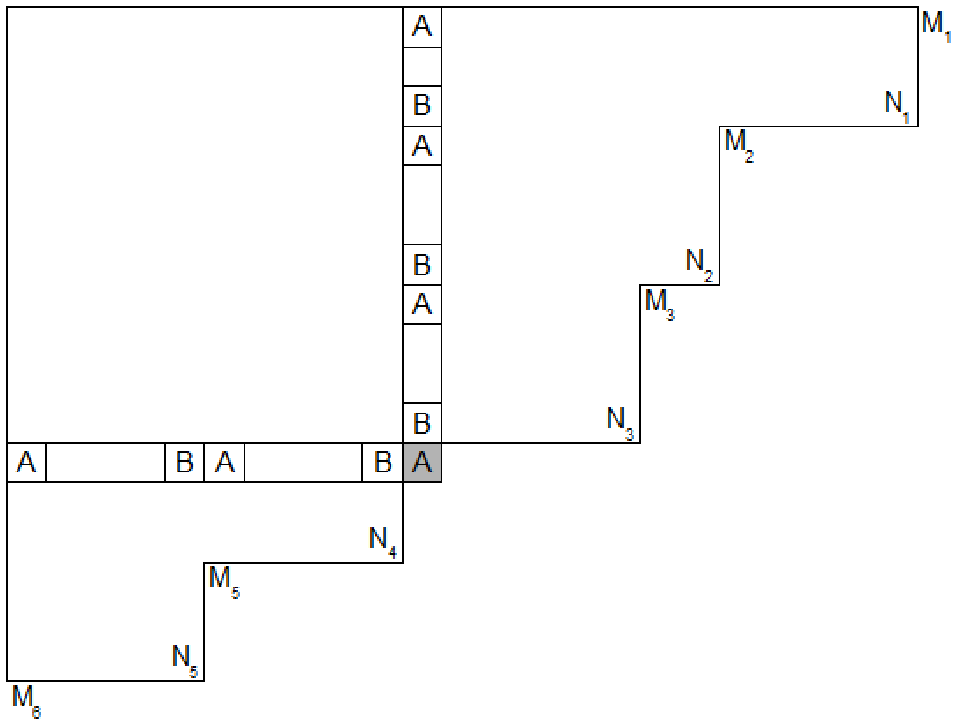


FIGURE 2. Adding an outer corner at M_4. Hooks at squares labeled 'A' (resp. 'B') remain uncanceled in the numerator (resp. denominator).

Set $x_i = c(M_i)$ and $y_i = c(N_i)$. Then

$$\begin{aligned} h_{\lambda^+}(a_i, b_k) &= x_i - x_k \quad \text{for } i \in \{1, ..., k-1\} \\ h_{\lambda}(\alpha_i, b_k) &= y_i - x_k \quad \text{for } i \in \{1, ..., k-1\} \\ h_{\lambda^+}(a_k, b_i) &= x_k - x_i \quad \text{for } i \in \{k+1, ..., d\} \\ h_{\lambda}(a_k, \beta_i) &= x_k - y_i \quad \text{for } i \in \{k, ..., d-1\}. \end{aligned} \tag{2.5}$$

Then

$$w(\lambda^+) = w(1)w(\lambda) \frac{\prod_{i=1}^{k-1} w(x_i - x_k)}{\prod_{i=1}^{k-1} w(y_i - x_k)} \cdot \frac{\prod_{i=k+1}^{d} w(x_k - x_i)}{\prod_{i=k}^{d-1} w(x_k - y_i)}.$$

Summing over $k \in \{1, ..., d\}$ we obtain

$$\sum_{\lambda^+ \gtrdot \lambda} w(\lambda^+) = w(1)w(\lambda) \sum_{k=1}^{d} \left(\frac{\prod_{i=1}^{k-1} w(x_i - x_k)}{\prod_{i=1}^{k-1} w(y_i - x_k)} \cdot \frac{\prod_{i=k+1}^{d} w(x_k - x_i)}{\prod_{i=k}^{d-1} w(x_k - y_i)} \right). \tag{2.6}$$

Now if $k \in \{1, ..., d-1\}$, let λ^- be the partition obtained by removing the corner N_k from λ. A similar formula holds for $w(\lambda^-)$. Again, the only hooks affected by deleting N_k come from squares in the same row or column as N_k, giving the equality

$$w(\lambda^-) = \frac{w(\lambda)}{w(1)} \prod_{j=1}^{\alpha_k - 1} \frac{w(h_{\lambda^-}(j, \beta_k))}{w(h_\lambda(j, \beta_k))} \prod_{j=1}^{\beta_k - 1} \frac{w(h_{\lambda^-}(\alpha_k, j))}{w(h_\lambda(\alpha_k, j))}.$$

Again, many of these terms cancel, reducing to

$$\prod_{j=1}^{\alpha_k - 1} \frac{w(h_{\lambda^-}(j, \beta_k))}{w(h_\lambda(j, \beta_k))} = \frac{w(h_{\lambda^-}(\alpha_1, \beta_k)}{w(h_\lambda(a_1, \beta_k))} \cdots \frac{w(h_{\lambda^-}(\alpha_{k-1}, \beta_k)}{w(h_\lambda(a_{k-1}, \beta_k))} \cdot \frac{w(1)}{w(h_\lambda(a_k, \beta_k))}.$$

and

$$\prod_{j=1}^{\beta_k - 1} \frac{w(h_{\lambda^-}(\alpha_k, j))}{w(h_\lambda(\alpha_k, j))} = \frac{w(h_{\lambda^-}(\alpha_k, \beta_{d-1}))}{w(h_\lambda(\alpha_k, b_d))} \cdots \frac{w(h_{\lambda^-}(\alpha_k, \beta_{k+1})}{w(h_\lambda(\alpha_k, b_{k+2}))} \cdot \frac{w(1)}{w(h_\lambda(\alpha_k, b_{k+1}))}.$$

Analogous to equations (2.5), we have

$$\begin{aligned} h_{\lambda^-}(\alpha_i, \beta_k) &= y_i - y_k \quad \text{for } i \in \{1, ..., k-1\} \\ h_\lambda(a_i, \beta_k) &= x_i - y_k \quad \text{for } i \in \{1, ..., k\} \\ h_{\lambda^-}(\alpha_k, \beta_i) &= y_k - y_i \quad \text{for } i \in \{k+1, ..., d-1\} \\ h_\lambda(\alpha_k, b_i) &= y_k - x_i \quad \text{for } i \in \{k+1, ..., d\}. \end{aligned}$$

These allow us to write

$$w(\lambda^-) = w(1) w(\lambda) \frac{\prod_{i=1}^{k-1} w(y_i - y_k)}{\prod_{i=1}^{k} w(x_i - y_k)} \cdot \frac{\prod_{i=k+1}^{d-1} w(y_k - y_i)}{\prod_{i=k+1}^{d} w(y_k - x_i)}.$$

Summing this over $k \in \{1, ..., d-1\}$ we have

$$\sum_{\lambda^- \lessdot \lambda} w(\lambda^-) = w(1) w(\lambda) \sum_{k=1}^{d-1} \left(\frac{\prod_{i=1}^{k-1} w(y_i - y_k)}{\prod_{i=1}^{k} w(x_i - y_k)} \cdot \frac{\prod_{i=k+1}^{d-1} w(y_k - y_i)}{\prod_{i=k+1}^{d} w(y_k - x_i)} \right). \tag{2.7}$$

Plugging (2.6) and (2.7) into Lemma 1.2 and employing the fact that $w(-x) = -w(x)$, we are reduced to proving

Proposition 2.2. *For distinct complex-valued $x_1, x_2, ..., x_d$ and $y_1, y_2, ..., y_{d-1}$, we have:*

$$\sum_{k=1}^{d} \frac{\prod_{i=1, i \neq k}^{d} w(x_k - x_i)}{\prod_{i=1}^{d-1} w(x_k - y_i)} + \sum_{k=1}^{d-1} \frac{\prod_{i=1, i \neq k}^{d-1} w(y_k - y_i)}{\prod_{i=1}^{d} w(y_k - x_i)} = 1. \tag{2.8}$$

(Note that this equation is true for arbitrary variables x_i and y_i, not just when $x_i = c(M_i)$ and $y_i = c(N_i)$.)

Proof. This result is a special case of the following lemma:

Proposition 2.3. *For distinct complex-valued $a_1, a_2, ..., a_n$, we have:*

$$\sum_{k=1}^{n} \prod_{i=1, i\neq k}^{n} \frac{a_k + a_i}{a_k - a_i} = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}. \tag{2.9}$$

We present two proofs of Proposition 2.3.
Proof 1: Set

$$b_k := \prod_{i=1, i\neq k}^{n} \frac{a_k + a_i}{a_k - a_i}.$$

We wish to show that the sum of the b_k is 0 or 1 depending on the parity of n.

Consider the partial fraction decomposition

$$\prod_{i=1}^{n} \frac{t + a_i}{t - a_i} = c_0 + \sum_{k=1}^{n} \frac{c_k}{t - a_k}. \tag{2.10}$$

We can retrieve c_0 by taking the limit $t \to \infty$ on both sides; this yields $c_0 = 1$. We can find c_k by multiplying both sides by $t - a_k$ and setting $t = a_k$. All the terms except one vanish on the right hand side, so this yields:

$$2a_k \prod_{i=1, i\neq k}^{n} \frac{a_k + a_i}{a_k - a_i} = c_k.$$

So $c_k = 2a_k b_k$. Setting $t = 0$ in (2.10), we obtain

$$(-1)^n = c_0 - \sum_{k=1}^{n} \frac{c_k}{a_k}.$$

Plugging in $c_0 = 1$ and $c_k = 2a_k b_k$ yields

$$1 - (-1)^n = 2\sum_{k=1}^{n} b_k.$$

The left hand side is 0 if n is even and 2 if n is odd, so dividing through by 2 gets us the desired result. □

Proof 2: Multiply through by the denominator in (2.9), so that the equation to be proved is

$$\sum_{k=1}^{n} (-1)^{k-1} \prod_{\substack{i=1 \\ i\neq k}}^{n} (a_k + a_i) \prod_{\substack{i<j \\ i\neq k \\ j\neq k}} (a_i - a_j) = \delta_n \cdot \prod_{i<j} (a_i - a_j) \tag{2.11}$$

where δ_n is 0 if n is even and 1 if n is odd.

Denote the polynomial on the left hand side of (2.11) by $F(a_1, \cdots, a_n)$. We first show that F is an alternating function. Consider

$$F(a_1, \cdots, a_{r-1}, a_{r+1}, a_r, a_{r+2}, \cdots, a_n).$$

For $k \neq r, r+1$, the only change in the summand is that the $a_r - a_{r+1}$ in the second product is replaced with $a_{r+1} - a_r$, with the net effect of just changing the sign. For $k = r, r+1$, the summand itself stays the same, but the $(-1)^{k-1}$

factor is off by one on each summand, again having the net effect of changing the sign. So $F(a_1, \cdots, a_{r-1}, a_{r+1}, a_r, a_{r+2}, \cdots, a_n) = -F(a_1, \cdots, a_n)$, showing that F is alternating.

Since F is alternating, setting $a_i = a_j$ gives $F = 0$; i.e., $(a_i - a_j) \mid F$ for all $i < j$. However, F has degree at most $n(n-1)/2$ so we must have

$$\sum_{k=1}^{n}(-1)^{k-1}\prod_{\substack{i=1\\ i\neq k}}^{n}(a_k+a_i)\prod_{\substack{i<j\\ i\neq k\\ j\neq k}}(a_i-a_j) = F = \delta_n \cdot \prod_{i<j}(a_i-a_j) \tag{2.12}$$

for some constant δ_n. The coefficient of $a_1^{n-1}a_2^{n-2}\cdots a_{n-1}$ in the right hand side of (2.12) is δ_n and the same coefficient in the left hand side is $\sum_{k=1}^{n}(-1)^{k-1}$. Thus, δ_n is 0 if n is even and 1 if n is odd, as desired. □

We are now ready to prove Proposition 2.2.

For $d = 1$ the statement is trivial, so assume $d \geq 2$. Rewrite Proposition 2.3 with $n = 2d - 1$ as:

$$\sum_{k=1}^{d}\frac{\displaystyle\prod_{i=1, i\neq k}^{d}\frac{a_k+a_i}{a_k-a_i}}{\displaystyle\prod_{i=d+1}^{2d-1}\frac{a_k-a_i}{a_k+a_i}} + \sum_{k=d+1}^{2d-1}\frac{\displaystyle\prod_{i=d+1, i\neq k}^{2d-1}\frac{a_k+a_i}{a_k-a_i}}{\displaystyle\prod_{i=1}^{d}\frac{a_k-a_i}{a_k+a_i}} = 1.$$

Now we plug in $a_i = q^{-x_i}$ for $1 \leq i \leq d$ and $a_i = -q^{-y_{i-d}}$ for $d+1 \leq i \leq 2d-1$:

$$\sum_{k=1}^{d}\frac{\displaystyle\prod_{i=1, i\neq k}^{d}\frac{q^{-x_k}+q^{-x_i}}{q^{-x_k}-q^{-x_i}}}{\displaystyle\prod_{i=1}^{d-1}\frac{q^{-x_k}+q^{-y_i}}{q^{-x_k}-q^{-y_i}}} + \sum_{k=1}^{d-1}\frac{\displaystyle\prod_{i=1, i\neq k}^{d-1}\frac{-q^{-y_k}-q^{-y_i}}{-q^{-y_k}+q^{-y_i}}}{\displaystyle\prod_{i=1}^{d}\frac{-q^{-y_k}-q^{-x_i}}{-q^{-y_k}+q^{-x_i}}} = 1.$$

Multiplying each factor by q^{x_k}/q^{x_k} or q^{y_k}/q^{y_k} as appropriate and simplifying signs,

$$\sum_{k=1}^{d}\frac{\displaystyle\prod_{i=1, i\neq k}^{d}\frac{1+q^{x_k-x_i}}{1-q^{x_k-x_i}}}{\displaystyle\prod_{i=1}^{d-1}\frac{1+q^{x_k-y_i}}{1-q^{x_k-y_i}}} + \sum_{k=1}^{d-1}\frac{\displaystyle\prod_{i=1, i\neq k}^{d-1}\frac{1+q^{y_k-y_i}}{1-q^{y_k-y_i}}}{\displaystyle\prod_{i=1}^{d}\frac{1+q^{y_k-x_i}}{1-q^{y_k-x_i}}} = 1.$$

Recalling that $w(z) = \dfrac{1+q^z}{1-q^z}$,

$$\sum_{k=1}^{d}\frac{\displaystyle\prod_{i=1, i\neq k}^{d} w(x_k-x_i)}{\displaystyle\prod_{i=1}^{d-1} w(x_k-y_i)} + \sum_{k=1}^{d-1}\frac{\displaystyle\prod_{i=1, i\neq k}^{d-1} w(y_k-y_i)}{\displaystyle\prod_{i=1}^{d} w(y_k-x_i)} = 1$$

which is precisely (2.8). □

This establishes the Extension-Retraction Lemma, which proves Theorem 1.1′ and Theorem 1.1.

3. Schur Functions

Recall that a standard Young tableau of $\lambda \vdash n$ is a filling of the Young diagram of λ using each of the numbers $1, ..., n$ exactly once such that rows and columns are increasing. A *semistandard Young tableau* (SSYT) is a filling of λ using any positive integers any number of times such that rows are weakly increasing and columns are strictly increasing.

To each partition λ, there is an important symmetric function called the *Schur function* $s_\lambda(x_1, x_2, \cdots)$. The Schur functions can be defined combinatorially:

Definition 3.1. *For any partition λ,*

$$s_\lambda(x_1, x_2, \cdots) := \sum_T x^T$$

where the sum is over all SSYT T of shape λ, and

$$x^T = x_1^{\#1\text{s in } T} x_2^{\#2\text{s in } T} \cdots .$$

We will often write $s_\lambda(x)$ when we mean $s_\lambda(x_1, x_2, \cdots)$.

We will also need skew versions of these concepts.

Definition 3.2. *Let μ be a partition contained in λ. The skew shape λ/μ consists of all the cells in λ that are not in μ.*

We can have standard and semistandard Young tableaux on skew shapes exactly as in the non-skew case. We can thus define *skew Schur functions*:

Definition 3.3. *For any skew shape λ/μ,*

$$s_{\lambda/\mu}(x_1, x_2, \cdots) := \sum_T x^T$$

where the sum is over all SSYT T of shape λ/μ.

Our main concern with Schur functions will be the following fact:

Fact 3.4. *If $\lambda \vdash n$, then the coefficient of $x_1 x_2 \cdots x_n$ in $s_\lambda(x)$ is f^λ.*

This is easy to see, as any SSYT using all the numbers $1, \cdots, n$ exactly once must be a SYT.

Similarly, for skew shapes:

Fact 3.5. *If $\lambda \vdash n$ and $\mu \vdash k < n$, then the coefficient of $x_1 x_2 \cdots x_{n-k}$ in $s_{\lambda/\mu}(x)$ is $f^{\lambda/\mu}$.*

We will cite other results about Schur functions as needed. For an overview, see Chapter 17 of Stanley [6].

4. Expanding Coefficients

Since both sides of (1.5) in Theorem 1.1′ are power series in q, the equation holds if and only if the coefficient of q^m is the same on both sides. We would like to be able to compare these coefficients, so our goal in this section will be to provide a tool for translating the left hand side of (1.5) from a statement about fixed points of involutions to a statement about Young tableaux. The following section will be devoted to applying this to the constant, linear, quadratic, cubic, and quartic terms of (1.5).

Proposition 4.1.

$$\sum_{\pi\in\mathrm{Inv}(n)} (\beta(\pi))_k = \sum_{\lambda\vdash n} f^\lambda \left(\sum_{\mu} f^{\lambda/\mu}\right)$$

where μ ranges over all partitions of $n-k$ contained in λ and $(n)_k = n(n-1)(n-2)\cdots(n-k+1)$.

Proof. Note that the left hand side is a sum over all involutions π of the number of ways to permute k fixed points of π. We can thus think of it as an involution in $\mathrm{Inv}(n-k)$ with a permutation of k numbers from the set $\{1,2,\cdots,n\}$. Therefore:

$$\sum_{\pi\in\mathrm{Inv}(n)} (\beta(\pi))_k = (n)_k|\,\mathrm{Inv}(n-k)|.$$

It remains to show that the right hand side is also equal to this quantity. We will do this in two different ways:

1. Row insertion.

We interpret the right hand side as pairs (P,Q) of SYT with n boxes generated through RSK such that P and Q agree for the first $n-k$ insertions. Such a pair is precisely determined by the final tableau P and the standardization of the skew tableau Q/μ resulting from the final k insertions, so the right hand side does indeed count this.

If we let T be the tableau of size $n-k$ after the first $n-k$ insertions, then T with a permutation of the remaining k numbers to be inserted precisely determine (P,Q). If we standardize T (a reversible map, since the remaining k numbers tell us which numbers were originally not in T), then we have a bijection between such pairs (P,Q) and ordered pairs (T,ω), where T is a SYT of size $n-k$ and ω is a permutation of k numbers from $\{1,2,\cdots,n\}$. Since the number of such ordered pairs is clearly $(n)_k\sum_{\lambda\vdash n-k} f^\lambda = (n)_k|\,\mathrm{Inv}(n-k)|$, we conclude

$$\sum_{\lambda\vdash n} f^\lambda \left(\sum_{\mu} f^{\lambda/\mu}\right) = (n)_k|\,\mathrm{Inv}(n-k)|.$$

2. Schur functions.

We consider

$$\sum_{\lambda} s_\lambda(x)\left(\sum_{\mu} s_{\lambda/\mu}(y)\right)$$

so that we are now summing over all partitions λ, and the desired quantity is just the coefficient of $x_1x_2\cdots x_ny_1y_2\cdots y_k$. Rearranging and using Littlewood-Richardson coefficients $c^{\lambda}_{\mu\nu}$ (see Stanley [6] 7.15-16),

$$\begin{aligned}\sum_{\lambda} s_\lambda(x)\left(\sum_{\mu} s_{\lambda/\mu}(y)\right) &= \sum_{\mu}\sum_{\lambda} s_\lambda(x)s_{\lambda/\mu}(y)\\ &= \sum_{\mu}\sum_{\lambda}\sum_{\nu} c^{\lambda}_{\mu\nu}s_\nu(y)s_\lambda(x)\\ &= \sum_{\mu}\sum_{\nu} s_\nu(y)s_\mu(x)s_\nu(x).\end{aligned}$$

To get the coefficient on $y_1y_2\cdots y_k$, we merely replace $s_\nu(y)$ with f^ν. To then get the coefficient on $x_1x_2\cdots x_n$, we choose k variables to get from $s_\nu(x)$ and we get the other $n-k$ from $s_\mu(x)$. Therefore,

$$\begin{aligned}\sum_{\lambda\vdash n} f^\lambda\left(\sum_{\mu} f^{\lambda/\mu}\right) &= \sum_{\mu\vdash n-k}\sum_{\nu\vdash k}\binom{n}{k} f^\mu (f^\nu)^2\\ &= \binom{n}{k}\sum_{\nu\vdash k}(f^\nu)^2\sum_{\mu\vdash n-k} f^\mu\\ &= \binom{n}{k}k!|\,\mathrm{SYT}(n-k)|\\ &= (n)_k|\,\mathrm{Inv}(n-k)|.\end{aligned}$$

□

5. Constant Through Quartic Terms

Equality of the constant terms, obtained by setting $q=0$ in Theorem 1.1′, corresponds to the equation

$$\sum_{\pi\in\mathrm{Inv}(n)} 1 = \sum_{\lambda\vdash n} f^\lambda,$$

which is exactly the identity derived from the RSK algorithm on involutions.

For higher order terms, it simplifies matters to write

$$\left(\frac{1+q}{1-q}\right)^{\beta(\pi)} = 1+\sum_{k\geq 1} q^k\sum_{i=1}^{k} 2^i\binom{\beta(\pi)}{i}\binom{k-1}{i-1}$$

from which the coefficient of q^k can be easily determined for fixed k, and

$$\frac{1+q^{h(x)}}{1-q^{h(x)}} = 1+2q^{h(x)}+2q^{2h(x)}+\cdots.$$

Also, for $k\geq 1$ define the number of k-hooks in λ to be

$$h_k(\lambda) := |\{x\in\lambda \mid h_\lambda(x)=k\}|.$$

For the linear terms, the left hand side is easy to compute using the above formula. For fixed $\lambda\vdash n$, the right hand side gets a contribution of 2 for every

1-hook in λ. So, equality of the linear terms is equivalent to the identity

$$\sum_{\pi\in\mathrm{Inv}(n)} 2\beta(\pi) = \sum_{\lambda\vdash n} f^{\lambda}(2h_1(\lambda)).$$

The validity of this follows immediately from Proposition 4.1 of Section 4, as the only skew shapes with only one element are 1-hooks.

For the quadratic term, the right hand side gets a contribution of 2 from every 1-hook, a contribution of 4 from every pair of 1-hooks, and a contribution of 2 from every 2-hook. The identity is then (suppressing the variables of β and h_k)

$$\sum_{\pi\in\mathrm{Inv}(n)} 2\beta + 2\beta(\beta-1) = \sum_{\lambda\vdash n} f^{\lambda}\left(2h_1 + 4\binom{h_1}{2} + 2h_2\right).$$

Or, upon dividing by 2 and subtracting the linear term

$$\sum_{\pi\in\mathrm{Inv}(n)} \beta(\beta-1) = \sum_{\lambda\vdash n} f^{\lambda}(h_1(h_1-1) + h_2).$$

This is also a consequence of Proposition 4.1 in Section 4, as any two element skew shape is either a 2-hook (with just one possible tableau) or a pair of disjoint 1-hooks (with two possible tableaux).

The cubic term introduces some new complications. We define a *border strip* to be a skew shape μ/ν containing no 2×2 box. We define $b_{\mu/\nu}(\lambda)$ as the number of times the border strip μ/ν appears along the border of λ. In general, if $M = \{\mu_1/\nu_1, \mu_2/\nu_2, \cdots, \mu_r/\nu_r\}$ is a multiset of connected skew shapes, let b_M be the number of $\eta \subset \lambda$ such that M is the multiset of the connected skew shapes of λ/η.

For the cubic term of the right hand side, we get the following contributions: 8 from each triple of 1-hooks, 8 from each pair of 1-hooks, 2 from each 1-hook, 4 from each unordered pair consisting of a 1-hook and a 2-hook, and 2 from each 3-hook. So, the cubic term identity is

$$\sum_{\pi\in\mathrm{Inv}(n)} 2\beta + 4\beta(\beta-1) + \frac{4}{3}\beta(\beta-1)(\beta-2)$$
$$= \sum_{\lambda\vdash n} f^{\lambda}\left(8\binom{h_1}{3} + 8\binom{h_1}{2} + 2h_1 + 4h_1h_2 + 2h_3\right).$$

Equivalently, upon subtracting suitable factors of the linear and quadratic terms and multiplying by a constant,

$$\sum_{\pi\in\mathrm{Inv}(n)} \beta(\beta-1)(\beta-2) = \sum_{\lambda\vdash n} f^{\lambda}\left(h_1(h_1-1)(h_1-2) + 3h_2(h_1-1) + \frac{3}{2}h_3\right).$$

Again employing Proposition 4.1 of Section 4, the left hand side becomes

$$\sum_{\lambda\vdash n} f^{\lambda}(6b_{\{1,1,1\}} + 3b_{\{11,1\}} + 3b_{\{2,1\}} + b_{111} + b_3 + 2b_{21} + 2b_{22/1}).$$

Since

$$\begin{aligned} b_{\{1,1,1\}} &= h_1(h_1-1)(h_1-2)/6 \\ b_{\{11,1\}} + b_{\{2,1\}} &= h_2(h_1-1) \\ h_3 &= b_{111} + b_3 + b_{21} + b_{22/1}, \end{aligned}$$

the above equation simplifies to

$$\sum_{\lambda \vdash n} f^\lambda (b_{111} + b_3) = \sum_{\lambda \vdash n} f^\lambda (b_{21} + b_{22/1}). \tag{5.1}$$

This is the special case $k = 3$ of the Alternating Hook Sum Theorem (Section 6).

In a similar way, the quartic term on the right hand side gets the following contributions: 16 from four 1-hooks; 24 from three 1-hooks; 12 from two 1-hooks; 2 from one 1-hook; 4 from two 2-hooks; 8 from one 2-hook and two 1-hooks; 4 from one 2-hook and one 1-hook; 2 from one 2-hook; 4 from one 3-hook and one 1-hook; and 2 from one 4-hook. Subtracting off multiples of the linear, quadratic and cubic terms from the quartic, and grouping in a suggestive manner, we get that the equality of the quartic terms is equivalent to

$$\begin{aligned} &\sum_{\pi \in \mathrm{Inv}(n)} \beta(\beta-1)(\beta-2)(\beta-3) \\ &\quad = \sum_{\lambda \vdash n} f^\lambda \left(24\binom{h_1}{4} + 12h_2\binom{h_1-1}{2} + 6\binom{h_2}{2} + 6h_3(h_1-1) - 3h_3 + 3h_4 \right). \end{aligned}$$

Again employing Proposition 4.1 of Section 4 and simplifying, the equality of the quartic terms is equivalent to the identity

$$\begin{aligned} &\sum_{\lambda \vdash n} f^\lambda (b_{32/1} + b_{221/1} + b_{22} + b_{21}(b_1-2) + b_{22/1}(b_1-1)) \\ &\qquad = \sum_{\lambda \vdash n} f^\lambda (b_{1111} + b_4 + b_{111}(b_1-1) + b_3(b_1-1)). \end{aligned}$$

Similar techniques can be applied to higher degree terms, but the identities they yield seem to become more and more arcane. We have been unable to find a pattern in these identities.

6. The Alternating Hook Sum Theorem

Define the arm length of a hook h to be the number of boxes to the right of the corner box, and the leg length to be the number of boxes below the corner box. We will refer to a hook h by its *hooktype* $(a(h), \ell(h))$. Finally, we define $h_{(a,\ell)}(\lambda)$ to be the number of hooks of type (a, ℓ) in λ. We will from now on drop the explicit dependence on λ, though it should always be assumed.

Recall the cubic term (5.1) when expanding coefficients in the q-formulation of the main result. In this equation, b_{111},b_3, and $b_{21} + b_{22/1}$ correspond to the number of vertical, horizontal, and L-shaped 3-hooks respectively, so we can replace them with $h_{(2,0)}$, $h_{(0,2)}$, and $h_{(1,1)}$ respectively. Therefore, the cubic term asserts that

$$\sum_{\lambda \vdash n} f^\lambda \left(h_{(2,0)} + h_{(0,2)}\right) = \sum_{\lambda \vdash n} f^\lambda \left(h_{(1,1)}\right). \tag{6.1}$$

This is a special case of the following theorem:

Theorem 6.1 (Alternating Hook Sum)**.** *For $n \geq 0$ and $k \geq 2$,*

$$\sum_{\lambda \vdash n} f^\lambda \left(\sum_{a=0}^{k-1} (-1)^a h_{(a,k-a-1)} \right) = 0. \tag{6.2}$$

Remark. (6.1) above is, of course, the $k = 3$ special case of this theorem with terms rearranged. Also, note that for even k, every pair of conjugate hooks cancel, so the sum is trivially zero.

Proof. We will use the following precursor to the Murnaghan-Nakayama rule (see Stanley [6], 7.17):

Theorem 6.2. *For $n \geq k \geq 0$, $\mu \vdash n - k$,*

$$s_\mu p_k = \sum_\lambda (-1)^{\mathrm{ht}(\lambda/\mu)} s_\lambda$$

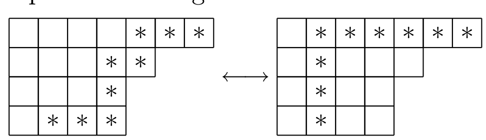
where the sum is over all $\lambda \supset \mu$ such that λ/μ is a border strip of size k, and where the height ht *of a border strip is its number of rows minus one.*

If we sum over all $\mu \vdash n - k$, then λ/μ will range over all border strips of size k in partitions of size n, so we have:

$$\sum_{\mu \vdash n-k} s_\mu p_k = \sum_{\lambda \vdash n} \left(\sum_B (-1)^{\mathrm{ht} B} \right) s_\lambda \tag{6.3}$$

where B ranges through all border strips of size k in λ, and λ ranges through all partitions of n.

Now to each border strip B, there is a unique hook h of precisely the same size whose leg length is equal to the height of B:



Therefore,

$$\sum_B (-1)^{\mathrm{ht} B} = \sum_h (-1)^{\ell(h)} = \sum_{\ell=0}^{k-1} (-1)^\ell h_{(k-\ell-1,\ell)} = \sum_{a=0}^{k-1} (-1)^a h_{(a,k-a-1)}$$

since transposition allows us to switch indexing by leg length to arm length.

Plugging this into (6.3),

$$\sum_{\mu \vdash n-k} s_\mu p_k = \sum_{\lambda \vdash n} \left(\sum_{a=0}^{k-1} (-1)^a h_{(a,k-a-1)} \right) s_\lambda.$$

As we did in Fact 3.4, we retrieve $\sum_{\lambda \vdash n} f^\lambda \left(\sum_{a=0}^{k-1} (-1)^a h_{(a,k-a-1)} \right)$ as the coefficient of $x_1 x_2 \cdots x_n$ in the right hand side. But now look at the coefficient of $x_1 x_2 \cdots x_n$ in the left hand side. Since $p_k = x_1^k + x_2^k + \cdots$ divides it, our assumption that $k \geq 2$ implies that there is no way to get any term of the form $x_1 x_2 \cdots x_n$. The coefficient is thus 0. Therefore, since of course the coefficient of $x_1 x_2 \cdots x_n$ in the left hand side and right hand side must be the same, we conclude

$$\sum_{\lambda \vdash n} f^\lambda \left(\sum_{a=0}^{k-1} (-1)^a h_{(a,k-a-1)} \right) = 0.$$

□

Though not relevant to the main result, we have the following corollary:

Corollary 6.3. *Let $o(\lambda)$ be the number of odd parts of a partition λ, and $d(\lambda)$ be the number of different parts of λ. Then*

$$\sum_{\lambda \vdash n} f^{\lambda}\left(o(\lambda)\right) = \sum_{\lambda \vdash n} f^{\lambda}\left(d(\lambda)\right).$$

Proof. We sum (6.2) for all $k \geq 1$. By the Alternating Hook Sum Theorem, the summands $k \geq 2$ will all contribute 0:

$$\sum_{\lambda \vdash n} f^{\lambda} \left(\sum_{k \geq 1} \sum_{a=0}^{k-1} (-1)^a h_{(a,k-a-1)} \right) = \sum_{\lambda \vdash n} f^{\lambda}\left(h_{(0,0)}\right) \tag{6.4}$$

Since there is a hook corresponding to each box in every tableau, we can interpret the left hand side as a sum over each box, where each box contributes $+1$ or -1 depending on the arm length of its hook. Therefore, each shape λ contributes overall a $+1$ for every odd row in λ, and a 0 for every even row. The following example makes clear that the odd rows - the first and second - contribute $+1$, while the even rows - the third and fourth - contribute 0:

$+1$	-1	$+1$	-1	$+1$	-1	$+1$
$+1$	-1	$+1$	-1	$+1$		
-1	$+1$	-1	$+1$			
-1	$+1$	-1	$+1$			

.

We thus have

$$\sum_{\lambda \vdash n} f^{\lambda} \left(\sum_{k \geq 1} \sum_{a=0}^{k-1} (-1)^a h_{(a,k-a-1)} \right) = \sum_{\lambda \vdash n} f^{\lambda}\left(o(\lambda)\right). \tag{6.5}$$

For the right hand side, note that we are merely counting up all hooks of size 1, which are just the corners of our partition. Since there is a corner for every different part, we have

$$\sum_{\lambda \vdash n} f^{\lambda}\left(h_{(0,0)}\right) = \sum_{\lambda \vdash n} f^{\lambda}\left(d(\lambda)\right). \tag{6.6}$$

Plugging (6.5) and (6.6) into (6.4) yields

$$\sum_{\lambda \vdash n} f^{\lambda}\left(o(\lambda)\right) = \sum_{\lambda \vdash n} f^{\lambda}\left(d(\lambda)\right).$$

□

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E-mail address: kcarde@fas.harvard.edu

E-mail address: loub0007@umn.edu

E-mail address: potechin@princeton.edu

E-mail address: asanborn@fas.harvard.edu