REU REPORT: MONOTONE PATHS ON ZONOTOPES

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Contents

1. Introduction

A zonotope $Z \subseteq \mathbb{R}^k$ is a linear projection of the unit n-dimensional cube $[0,1]^n$ onto $\mathbb{R}^k$ when $n \geqslant k \geqslant 2$. A generic functional $f \in (\mathbb{R}^k)^*$ distinguishes the minimum and the maximum vertices, $-v$ and v, in Z.

Consider a projection $\pi : \mathbb{R}^k \to \mathbb{R}^2$ such that $\pi(Z)$ is a polygon in $\mathbb{R}^2$ with the property that $\pi(v)$ and $\pi(-v)$ are two distinct vertices in $\pi(Z)$. For each such π, there are two boundary paths in $\mathbb{R}^2$ from $\pi(-v)$ to $\pi(v)$. Call them γ_1^π and γ_2^π. We say that an f-monotone path on $Z \subset \mathbb{R}^k$ along the edges from $-v$ to v is *coherent* when there exists such a projection $\pi : \mathbb{R}^k \to \mathbb{R}^2$ for which either $\pi^{-1}(\gamma_1^\pi)$ or $\pi^{-1}(\gamma_2^\pi)$ coincides with the path on Z. We describe a zonotope by its corresponding linear projection $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^k$, and think of $\mathcal{A}$ as a matrix in $\mathbb{R}^{k\times n}$ with a certain set of non-degeneracy conditions. When $\mathcal{A}$ generates Z, we write $Z = Z(\mathcal{A})$.

When all the monotone paths in $(Z(\mathcal{A}), f)$ are coherent, we say that the pair $(Z(\mathcal{A}), f)$ is *all-coherent.* If there is no confusion, it is also convenient to say that $(\mathcal{A}, f)$ is all-coherent. This report concerns the classification of all $(\mathcal{A}, f)$ which are all-coherent when the corank $n - k$ is three and when the Gale dual of $\mathcal{A}$ contains a regular tetrahedron subconfiguration.

We thank Rob Edman and Vic Reiner for their contributions to this project. We are especially grateful to Vic Reiner for introducing us to the problem. This research was part of the 2015 summer REU program at the University of Minnesota, Twin Cities, and was supported by RTG grant NSF/DMS-1148634. PJ's undergraduate studies at Harvard University are supported by King's Scholarship (Thailand).

The analogs of this problem for lower coranks (0, 1, and 2) have been studied in 2015 by Robert Edman [1]. The main tools he uses in the classification are deletion and contraction of Gale duals. We adapt the method for one corank higher. The main result of this report is given in Theorem 2.5.

In the following, we shall introduce a few terminologies used throughout this work.

Definition 1.1. *The matrix*

$$\mathcal{A} = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{bmatrix} \in \mathbb{R}^{k\times n}$$

is said to be acyclic *if for any $c_1, \dots, c_n \geqslant 0$, the equality*

$$c_1 a_1 + \dots + c_n a_n = \underline{0}$$

holds only if $c_1 = \dots = c_n = 0$.

Definition 1.2. *We shall say that the matrix*

$$\mathcal{A} = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{bmatrix} \in \mathbb{R}^{k\times n}$$

strongly captures 0*, or simply* capture 0*, if $\underline{0} \in \mathbb{R}^k$ is in the interior of the convex hull $\mathcal{H} \subseteq \mathbb{R}^k$ of $a_1, \dots, a_n$. In other words, there is a k-dimensional open ball $B \subseteq \mathbb{R}^k$ such that $\underline{0} \in B \subseteq \mathcal{H}$.*

Example 1.3. As an example, the matrix

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

does not capture 0.

Lemma 1.4. *The column vectors $a_1, \dots, a_n$ of $\mathcal{A} \in \mathbb{R}^{k\times n}$ are all on a same closed half-space in $\mathbb{R}^k$ if and only if $\mathcal{A}$ does* not *capture* 0*.*

Proof. ($\Rightarrow$) Suppose $a_1, \dots, a_n$ are all on a same closed half-space in $\mathbb{R}^k$. Then, the convex hull $\mathcal{H}$ is contained in that closed half-space $C \simeq \mathbb{R}_{\geqslant 0} \times \mathbb{R}^{k-1}$. The interior $\mathcal{H}^\circ$ must then be contained in $C^\circ \simeq \mathbb{R}_{>0} \times \mathbb{R}^{k-1}$ which does not contain $\underline{0} \in \mathbb{R}^k$. Thus, $\underline{0} \notin \mathcal{H}^\circ$.

($\Leftarrow$) Suppose that $\underline{0}$ is outside $\mathcal{H}^\circ$. Take a hyperplane $\mathcal{P}$ through $\underline{0}$ not intersecting $\mathcal{H}^\circ$. All points in $\mathcal{H}^\circ$ must be on the same side of $\mathcal{P}$. Thus, all the vectors $a_1, \dots, a_n$ must be on a same closed half-space defined by $\mathcal{P}$. ☐

Definition 1.5. *We shall say that the matrix*

$$\mathcal{A} = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{bmatrix} \in \mathbb{R}^{k\times n}$$

weakly captures 0 *if* $\underline{0} \in \mathbb{R}^k$ *is in the convex hull* $\mathcal{H} \subseteq \mathbb{R}^k$ *of* $a_1, \dots, a_n$.

Lemma 1.6. *Let* $\mathcal{A} \in \mathbb{R}^{k \times n}$ *be a matrix with column vectors* $a_1, \dots, a_n$. *The following statements are equivalent:*

(1) $\mathcal{A}$ *is acyclic.*
(2) $\mathcal{A}$ *does* not *weakly capture* 0.
(3) *The vectors* $a_1, \dots, a_n$ *are all on a same open half-space in* $\mathbb{R}^k$.
(4) *There exists a linear functional* $f \in (\mathbb{R}^k)^*$ *such that* $f(a_i) > 0$ *for every* $i = 1, \dots, n$.

Proof. ($1 \Rightarrow 2$) Suppose, for sake of contradiction, that $\mathcal{A}$ is acyclic but $\mathcal{A}$ weakly captures 0. Having $\underline{0}$ in the convex hull of $a_1, \dots, a_n$ means that there are $c_1, \dots, c_n \geqslant 0$ with $c_1 + \dots + c_n = 1$ such that $c_1 a_1 + \dots + c_n a_n = 0$. However, acyclicity implies that $c_1 = \dots = c_n = 0$. This gives a contradiction.

($2 \Rightarrow 3$) Suppose that $\underline{0} \notin \mathcal{H}$. Take a hyperplane $\mathcal{P}$ through $\underline{0}$ not intersecting $\mathcal{H}$. All points in $\mathcal{H}$ must be on the same side of $\mathcal{P}$. Thus, all the points $a_1, \dots, a_n$ must be on a same open half-space determined by $\mathcal{P}$.

($3 \Rightarrow 4$) Let $\mathcal{P}$ be a hyperplane which defines an open half-space the vectors $a_1, \dots, a_n$ belong to. Let u be a normal vector to $\mathcal{P}$ pointing into the same open half-space as $a_1, \dots, a_n$. Define a linear functional $f \in (\mathbb{R}^k)^*$ by $f(x) := u \cdot x$, where $\cdot$ is the standard inner product in the Euclidean space $\mathbb{R}^k$. Since $u, a_1, \dots, a_n$ are all in a same open half-space, $f(a_i) > 0$ for every $i = 1, \dots, n$.

($4 \Rightarrow 1$) Suppose there are $c_1, \dots, c_n \geqslant 0$ such that $c_1 a_1 + \dots + c_n a_n = \underline{0}$. Then, by linearity, $\sum_i c_i f(a_i) = 0$. Therefore, $c_1 = \dots = c_n = 0$. □

For convenience, when a linear functional $f \in (\mathbb{R}^k)^*$ satisfies the condition (4) in the lemma above, we shall say that f is *positive on the column vectors of* $\mathcal{A}$. Schematically, we have the following relation for a matrix $\mathcal{A} \in \mathbb{R}^{k \times n}$

$$\left\{ \begin{array}{c} \text{(acyclic)} \equiv \\ \text{(not weakly capture 0)} \equiv \\ \text{(all column vectors} \\ \text{on a same open half-space)} \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} \text{(not capture 0)} \equiv \\ \text{(all column vectors} \\ \text{on a same closed half-space)} \end{array} \right\}.$$

Lemma 1.7. *If* $\mathcal{A}$ *has full rank and is acyclic, then the Gale dual* $\mathcal{A}^*$ *strongly captures* 0.

Proof. Suppose, for sake of contradiction, that $\mathcal{A}^*$ does not capture 0. Thus, $a_1^*, \dots, a_n^*$ are all on a same closed half-space. This means there is a nonzero linear functional $\alpha \in (\mathbb{R}^{n-k})^*$ such that $\alpha(a_i^*) \geqslant 0$ for all i. Let $f_i := \alpha(a_i^*) \geqslant 0$. Note that $\mathcal{A}^* \cdot \mathcal{A}^T = \underline{0}$. Therefore,

$$[f_1 \dots f_n] \cdot \mathcal{A}^T = \alpha \cdot \mathcal{A}^* \cdot \mathcal{A}^T = \underline{0}.$$

This shows that $f_1 a_1 + \dots + f_n a_n = 0$. Since $\mathcal{A}$ is acyclic, $f_1 = \dots = f_n = 0$. Thus, $a_1^*, \dots, a_n^*$ are all on the hyperplane $\{\mathbf{x} : \alpha(\mathbf{x}) = 0\}$ which implies that $\mathcal{A}^*$ dose not have full rank, so $\mathcal{A}$ does not have full rank. This gives a contradiction. □

Lemma 1.8. *If no two vectors a_i and a_j in $\mathcal{A}$ are parallel, then there is no hyperplane containing at least $n-2$ dual vectors in $\mathcal{A}^*$.*

Proof. Suppose that there is a hyperplane containing at least $n-2$ dual vectors in $\mathcal{A}^* \in \mathbb{R}^{(n-k)\times n}$. Without loss of generality, let the hyperplane be $x_{n-k} = 0$ and the $n-2$ dual vectors are $a_1^*, \dots, a_{n-2}^*$. We have

$$\mathcal{A}^* = \left[\begin{array}{ccc|cc} & * & & & * \\ \hline 0 & 0 & \dots \; 0 & a & b \end{array}\right]$$

for some $a, b \in \mathbb{R}$. Let $\mathcal{A} = [a_{ij}]$. Because $\begin{bmatrix} 0 & 0 & \dots & 0 & a & b \end{bmatrix} \cdot [a_{ij}]_{j=1}^n = 0$ for every i, we must have $a \cdot a_{i(n-1)} + b \cdot a_{in} = 0$ for every i. This means that a_{n-1} and a_n are parallel. □

Another degeneracy property of the matrix $\mathcal{A}$ that can occur is reducibility. Because we like to study $\mathcal{A}^*$, we avoid the case in which $\mathcal{A}^*$ contains a zero vector. We know that $\mathcal{A}^*$ contains a zero vector if and ony if $\mathcal{A}$ can be decomposed as a direct sum $\mathcal{A}_0 \oplus \mathcal{A}_1$, where $\mathcal{A}_0$ is a 1×1-matrix. For example, consider

$$\mathcal{A} = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

In this case,

$$\mathcal{A}^* = \begin{bmatrix} 0 & -1 & -1 & 1 \end{bmatrix}.$$

For any generic function $f \in (\mathbb{R}^3)^*$ that is positive on all the column vectors of $\mathcal{A}$, it is actually a valid question to ask whether $(Z(\mathcal{A}), f)$ is all-coherent. In fact, the answer for this case is it is not all-coherent by a lemma of Edman's [1], which states that if $\mathcal{A}$ is reducible, then $(\mathcal{A}, f)$ is incoherent. Therefore, we may only consider irreducible matrices in our classification. In doing so, we also avoid the cases in which there exist zero vectors in $\mathcal{A}^*$.

2. Classification of Corank 3

In this section, we shall assume the matrix $\mathcal{A}$ to have the following non-degeneracy properties.

- $\mathcal{A}$ has full rank.
- No two a_i and a_j are parallel.
- $\mathcal{A}$ is acyclic.
- $\mathcal{A}$ is irreducible.

The conditions above are not surprising. Given a non-degenerate zonotope $Z \subset \mathbb{R}^k$ with a specified minimum and maximum vertices $-v$ and v, we can always find such a matrix $\mathcal{A} \in \mathbb{R}^{k\times n}$ with $Z(\mathcal{A}) = Z$ so that the monotone paths on Z are compatible with those on $Z(\mathcal{A})$ and so that the first three conditions are satisfied. The fourth condition, as described in the previous section, always an incoherent $(\mathcal{A}, f)$.

We shall make extensive use of the dual $\mathcal{A}^* \in \mathbb{R}^{(n-k)\times n}$. Let $e_1, \ldots, e_{n-k}$ be the standard basis for $\mathbb{R}^{n-k}$. For most of the matrices $\mathcal{A}$ arising from a zonotope Z we shall consider, the vector configuration of the dual matrix $\mathcal{A}^*$ has the vector configuration of $e_1, e_2, \ldots, e_{n-k}, -(e_1 + \cdots + e_{n-k})$ (called the $(n-k)$-simplex vector configuration) as a subconfiguration. Consider the following example.

Example 2.1. Consider

$$\mathcal{A} = \begin{bmatrix} 5 & 0 & 0 & -2 & -1 & 0 & 2 \\ -1 & 0 & 0 & 2 & 2 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and its dual

$$\mathcal{A}^* = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 2 & -2 & 0 & 1 \\ 1 & 1 & 1 & 2 & -1 & -1 & -1 \end{bmatrix}.$$

The matrix $\mathcal{A}$ satisfies all three conditions listed above. Furthermore, the dual $\mathcal{A}^*$ has the following four vectors as a subcollection of vectors

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

which has the 3-simplex (tetrahedron) vector configuration.

Example 2.2. Consider

$$\mathcal{A} = \begin{bmatrix} -1 & & & & & & 1 & & \\ & -1 & & & & & & 1 & \\ & & -1 & & & & & & 1 \\ 1 & & & 1 & & & & & \\ & 1 & & & 1 & & & & \\ & & 1 & & & 1 & & & \end{bmatrix} \in \mathbb{R}^{6\times 9}$$

and its dual

$$\mathcal{A}^* = \begin{bmatrix} 1 & & & 1 & & & -1 & & \\ & 1 & & & 1 & & & -1 & \\ & & 1 & & & 1 & & & -1 \end{bmatrix} \in \mathbb{R}^{3\times 9}.$$

The matrix $\mathcal{A}$ still satisfies the three conditions. However, the dual $\mathcal{A}^*$ does not have the 3-simplex vector configuration as a subconfiguration. This is one of the special cases, as we note that $\mathcal{A}^*$ only contains 3 parallelism classes.

The matrix in Example 2.2 is rather special. It only occurs because many vectors are packed together in 3 parallelism classes. Throughout the rest of this report, we will additionally impose this extra condition on $\mathcal{A}^*$: the vector configuration of the dual $\mathcal{A}^*$ must contain the $(n-k)$-simplex vector configuration.

Recall the definition of coherency for a monotone path.

Definition 2.3. *A monotone path γ in $(Z(\mathcal{A}), f)$ is said to be* coherent *if there exists a linear functional $f \in (\mathbb{R}^k)^*$ such that*

$$\frac{g(a_{\gamma(1)})}{f(a_{\gamma(1)})} < \cdots < \frac{g(a_{\gamma(n)})}{f(a_{\gamma(n)})}.$$

The reason Gale duals $\mathcal{A}^*$ serve as a convenient tool for classifying by all-coherency property of $(\mathcal{A}, f)$ is because we have the following lemma, proved in [1]. This lemma says that the geometry of $\mathcal{A}^*$ is sufficient to decide the all-coherency property of $(\mathcal{A}, f)$, even as f varies.

Lemma 2.4. *(R. Edman [1]) Suppose that two pairs $(\mathcal{A}, f)$ and $(\mathcal{A}', f')$ satisfy the following properties:*

- *$\mathcal{A}$ and $\mathcal{A}'$ satisfy the three non-degeneracy conditions above,*
- *$f(a_i) > 0$ and $f'(a_i') > 0$ for every i, and*
- *$\mathcal{A}^*$ is dual to both $\mathcal{A}$ and $\mathcal{A}'$.*

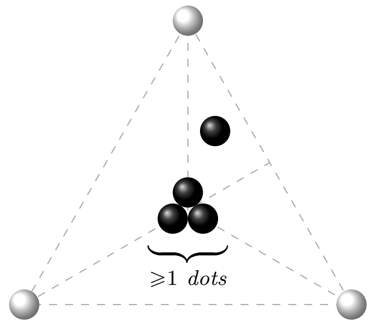
Then, $(\mathcal{A}, f)$ is all-coherent if and only if $(\mathcal{A}', f')$ is. Moreover, the all-coherency is determined by the vector configuration of $\mathcal{A}^*$.

It therefore makes sense to talk about the all-coherency of the dual $\mathcal{A}^*$. For convenience, we shall say that the dual $\mathcal{A}^*$ is *all-coherent* if $(\mathcal{A}, f)$ is all-coherent. Furthermore, this lemma is very useful for doing computations. To see whether a pair $(\mathcal{A}, f)$ is all-coherent, we may just check for a certain pair $(\mathcal{A}, f)$ realizing $\mathcal{A}^*$. To do this, Edman has developed a Mathematica code which has been immensely useful for this project [2].

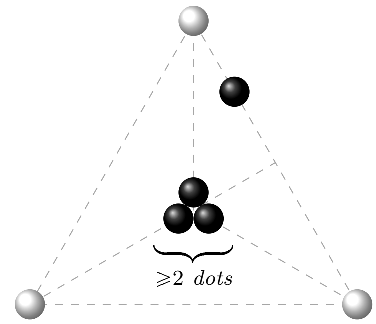
We now classify the all-coherent duals $\mathcal{A}^*$. We split into cases by the number of parallelism classes $\mathcal{A}^*$ has. Recall that in all the following discussion, $\mathcal{A}^*$ always has the 3-simplex "tetrahedron" vector configuration as a subconfiguration. In our classification, we start with the vertices of a tetrahedron, and add more vertices to the dual in all possible ways. The main result of this report is the following.

Theorem 2.5. *Let the matrix $\mathcal{A} \in \mathbb{R}^{k\times n}$ satisfy the non-degeneracy properties above and let $f \in (\mathbb{R}^k)^*$ be positive on the column vectors of $\mathcal{A}$. In corank $n - k = 3$, if $\mathcal{A}$ contains the 3-simplex vector configuration as a subconfiguration, then the pair $(\mathcal{A}, f)$ is all-coherent if and only if at least one of the following three conditions on $\mathcal{A}^*$ is satisfied:*

1. *$n = 5$; in other words, $\mathcal{A}^*$ contains exactly five vectors.*
2. *the Gale diagram of $\mathcal{A}^*$ belongs to the families $\mathcal{F}_1$ or $\mathcal{F}_2$, as shown in the diagram below. Here, in each diagram, multiple black dots in the center denote vectors pointing out of the page at the same location with multiplicities as labeled.*

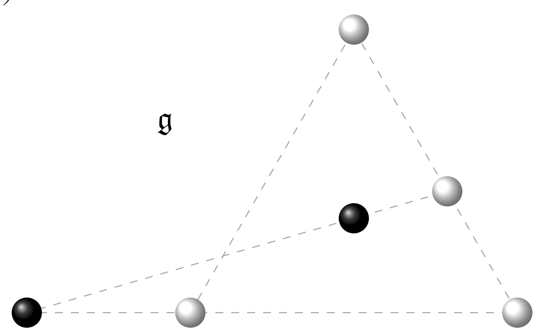


(Family $\mathcal{F}_1$)



(Family $\mathcal{F}_2$)

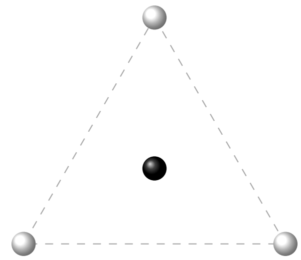
(3) *the Gale diagram of $\mathcal{A}^*$ is $\mathfrak{g}$ as shown below. (Note the collinearities in the diagram.)*



Theorem 2.5 follows immediately after we finish (i) the casework and (ii) the proof that the Gale diagrams described above are actually all-coherent. This report will present the first part consisting of the casework. We hope to give the second part in later reports. Note that the all-coherent duals in (1) in the theorem must necessarily contain exactly 5 parallelism classes in order to satsify the non-degeneracy conditions. A dual in (1) cannot contain a point in the Gale diagram with multiplicity greater than 1, because Lemma 1.8 tells that no three column vectors in $\mathcal{A}^*$ are on the same plane. Indeed, duals in $\mathcal{F}_1$ or $\mathcal{F}_2$ also have exactly 5 parallelism classes. In the special case, the dual in (3) is the only all-coherent dual with 6 parallelism classes. The following corollary is thus immediate from the theorem.

Corollary 2.6. *In the situation of the above theorem, if $(\mathcal{A}, f)$ is all-coherent, then $\mathcal{A}^*$ has either* 5 *or* 6 *parallelism classes.*

Let's begin the casework. We first note that the whole casework will involve only a few set of techniques applied again and again. For incoherence, we either remove or contract (project) vectors until the diagram becomes an incoherent diagram of lower dimension. If a special case occurs, we investigate it separately. For coherence, we prove that the diagrams are all-coherent directly. From the assumption, we may assume that the dual $\mathcal{A}^*$ contains the following subconfiguration.

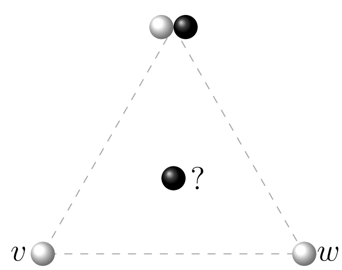


Currently, the four vertices are symmetric. Without loss of generality, we may assume throughout this section that the parallelism class of the black dot in the middle contains the highest number of vectors.

Note that the dual diagram with only four points as shown cannot yet be a dual of a non-degenerate $\mathcal{A}$, because there is a plane in $\mathbb{R}^3$ containing two vectors (say the black dot and a white dot), which means that it contains $n-2=2$ vectors. This would violate Lemma 1.8. We then have to add more vectors. We begin with the first case.

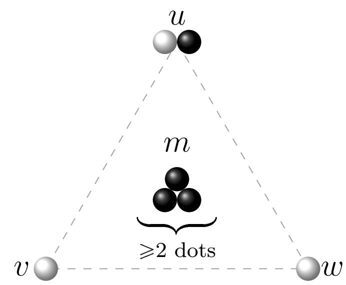
2.1. **Case 1. $\mathcal{A}^*$ has exactly 4 parallelism classes.** If $\mathcal{A}^*$ contains exactly 4 parallelism classes, then any vector to be added to the diagram must be added to one of the existing parallelism classes. By assumption, there must be more vectors to the middle black dot parallelism class. Note that we also need to add at least a vector to the white dot classes as well, because of Lemma 1.8. For simplicity, we call the black dot parallelism class "inside" and the white dot classes "outside".

Case 1.1 Suppose there is at least one more black dot outside.

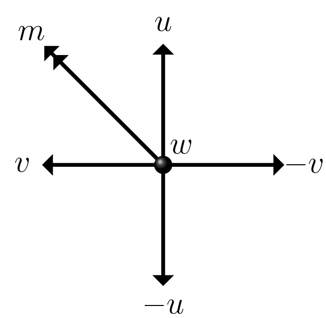


By assumption, there must be at least one more dots inside (labeled with a question mark). If any of the additional dots at ? is a white dot, then consider the projection via the vector v, and then at w, as labeled in the diagram. The resulting Gale diagram in $\mathbb{R}^1$ has the following diagram as a subconfiguration

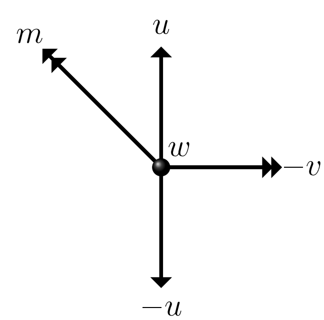
which is an incoherent Gale diagram in $\mathbb{R}^1$. By the contraction and deletion lemma, the resulting Gale diagram when there is at least one white dot to the center in Case 1.1 is incoherent. Thus, we are left with the following case for Case 1.1.



For convenience, call the parallelism class of the middle black dot, m, and call the other class u, as shown in the diagram. In fact, we note that the diagram below is not a complete Gale dual of a non-degenerate $(\mathcal{A}, f)$ yet, because there exists a hyperplane generated by u and m which contains $n-2$ vectors in $\mathcal{A}^*$. This indicates that there must be at least one more vectors at either v or w. Without loss of generality, suppose that there is another vector at v. If the vector is black, the projection at w gives the following 2-dimensional Gale diagram.



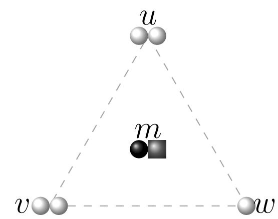
This belongs to an incoherent family in 2-dimension. (For example, project through m to see the inconherence.) On the other hand, if the vector is v is white, consider the projection via w again. This gives:



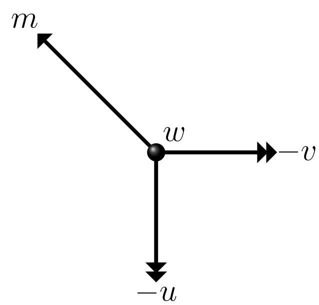
which belongs to the incoherent family in two dimension. (For example, project through u to see the incoherence.) Therefore, every Gale dual $\mathcal{A}^*$ which occurs in Case 1.1 is incoherent.

Case 1.2 Suppose all the dots outside are white.

By the same argument, there must be at least two parallelism classes outside which have multiplicities at least two. We are in the situation of the following diagram.

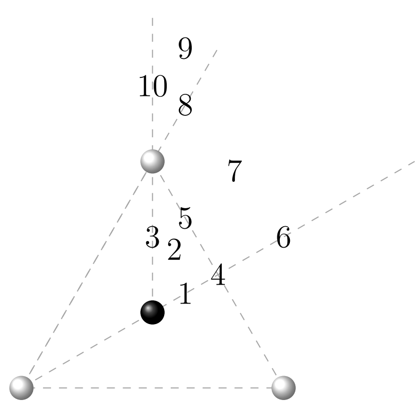


In the diagram, each of u and v has at least two white dots. The middle parallelism class m has one original dot, and at least one other dot. (The gray square indicates a dot whose color has not been determined.) We claim that this is always incoherent. Project via w. We have the following Gale diagram.

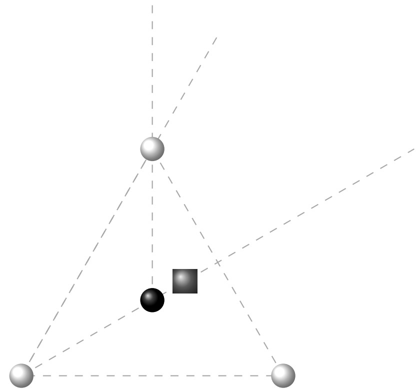


This belongs to the incoherent family. Therefore, all Gale duals in Case 1.2 are incoherent. We have also shown that when the number of parallelism classes is 4, the Gale dual is incoherent.

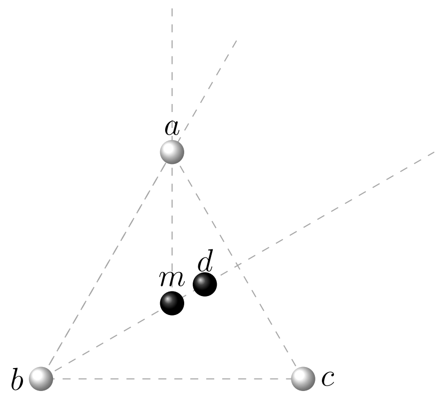
2.2. **Case 2.** $\mathcal{A}^*$ **has exactly** 5 **parallelism classes.** Start with the tetrahedron subconfiguration. We still make the assumption that the middle black dot parallelism class contains the highest number of vectors. The other three parallelism classes are symmetric. Therefore, we may without loss of generality assume that the fifth parallelism class occur in one of the ten following locations:



<u>Case 2.1</u> The fifth parallelism class occurs at location 1. We are in the situation of the following diagram.

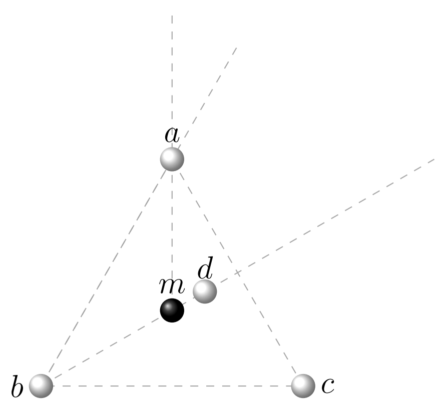


<u>Case 2.1b</u> The fifth dot is black.



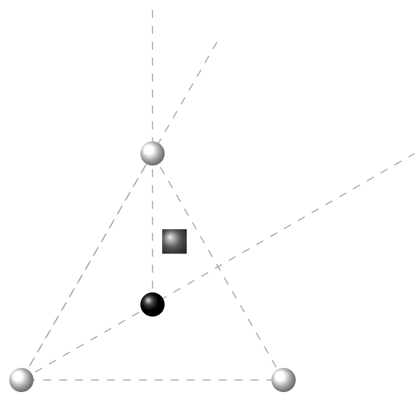
Consider the plane through b, m, and d. This shows that either a or c must have a multiplicity greater than 1. Without loss of generality, suppose that a has another dot. If the dot is white, projection via b, and then via c shows that the diagram is incoherent. If the dot is black, protection via c, and then via d shows that the diagram is incoherent. Therefore, in Case 2.1b, all Gale diagrams that occur are incoherent.

<u>Case 2.1w</u> The fifth dot is white.



As above, assume a has another dot. If the dot is black, project via c and then b to see that the diagram is incoherent. If the dot is white, by the assumption, m must contain another dot as well. If there is another black dot at m, project via b and then c to see that the diagram is incoherent. If there is another white dot at m, project via c and then d to see that the diagram is incoherent. Therefore, all Gale diagrams that occur in Case 2.1w are incoherent.

Case 2.2 The fifth parallelism class occurs at location 2. We are in the situation of the following diagram.



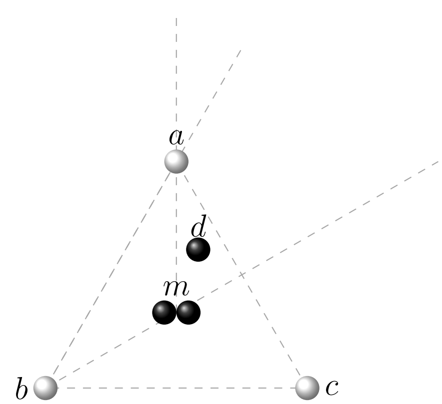
As claimed in the theorem, we expect to see an infinite family of all-coherent Gale diagrams in this case.

Case 2.2b The fifth dot is black.

Case 2.2b.1 There are exactly five vectors in $\mathcal{A}^*$. If this happens, the Gale diagram belongs to (1) in the theorem. We will check later that it is all-coherent.

Case 2.2b.2 There are more points. That is, there is at least one parallelism class with multiplicity greater than 1. In this case, by assumption, the class of the middle black dot must also contain another point.

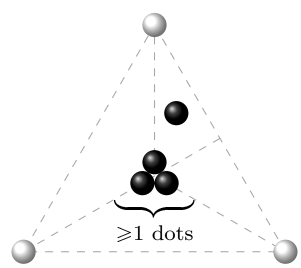
Case 2.2b.2b There is at least one more black dot at the middle class.



We claim in this case that if either a, b, c, or d has multiplicity greater than 1 (regardless of what color the dots are), then the Gale diagram is automatically incoherent.

First, if b has an additional white dot, then project via a and then via c to obtain a 1-dimensional incoherent diagram. Second, if b has an additional black dot, then project via a and then via m to obtain an incoherent diagram. Third, if a has an additional white dot, then project via b and then c. Fourth, if a has an additional black dot, then project via b and then d. Fifth, if c has an additional white dot, then project via a and then b. Sixth, if c has an additional black dot, then project via b and then m. Seventh, if d has an additional white dot, then project via a and then c. Finally, eighth, if d has an additional black dot, then project via a, and we will obtain a 2-dimensional Gale diagram that was decribed in Edman's thesis as the exceptional incoherent case of corank 2. Thus, we still have an incoherent diagram.

Similarly, if the middle class m has a white dot, then projection via b and then via c gives an incoherent diagram as well. Therefore, in this case 2.2b.2b, all Gale duals are incoherent, except for a family of diagrams shown below.



(Family $\mathcal{F}_1$)

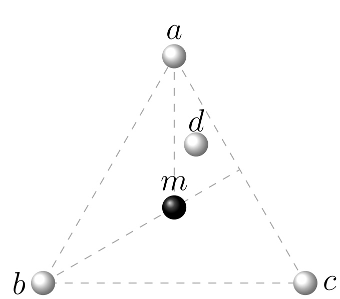
This family is Family $\mathcal{F}_1$ in the theorem. For the converse that this family is indeed all-coherent, we will give a proof later, after all incoherent cases have been investigated.

Case 2.2b.2w There is at least one more white dot at the middle class.

In this case, projection via b and then via c gives an incoherent diagram.

Therefore, we finish Case 2.2b.

Case 2.2w The fifth dot is white.



Case 2.2w.1 There are exactly five vectors in $\mathcal{A}^*$. In other words, the diagram above has all the dots. Then, this belongs to (1) in the theorem. We will show later that this diagram is indeed all-coherent.

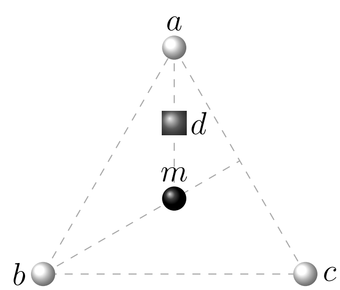
Case 2.2w.2 There are more points. By assumption, there must be more points to m.

Case 2.2w.2b There is at least one additional black point to m. In this case, project via b and then via c to obtain an incoherent diagram.

Case 2.2w.2w There is at least one additional white point to m. Then, project via c and then via d to obtain an incoherent diagram.

Therefore, we finish Case 2.2w, and hence finish Case 2.2.

Case 2.3 The fifth parallelism class occurs at location 3.



There is a plane containing the parallelism classes a, d, and m. To prevent this plane from containing $n-2$ vectors in the diagram, there must be at least one more vector at b or c. Without loss of generality, suppose that b has multiplicity greater than 1.

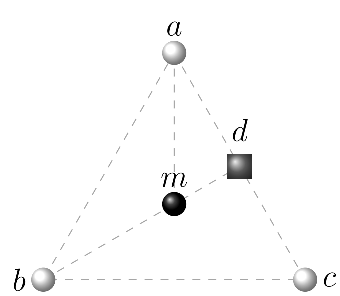
Case 2.3b The class b has at least one more black dot. Consider what color the dot at d is.

Case 2.3bb There is at least one black dot at d. Then, projection via m and then via c gives an incoherent diagram. Case 2.3bw There is at least one white dot at d. Then, projection via a and then via c gives an incoherent diagram.

Case 2.3w The class b has at least one more white dot. Then, projection via d and then via c gives an incoherent 1-dimensional diagram.

Thus, we see that Case 2.3b is an easy case. We quickly see that if a non-degenerate Gale dual occurs in this case then it is incoherent.

Case 2.4 The fifth parallelism class occurs at location 4.



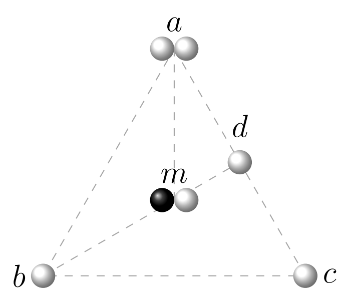
Case 2.4b There is at least one black dot at d. Consider the plane passing through b, m, and d. To prevent this plane from containing $n-2$ points, there

must be more points to a or c. Without loss of generality, suppose there are more points at a. By assumption, m must have multiplicity greater than 1.

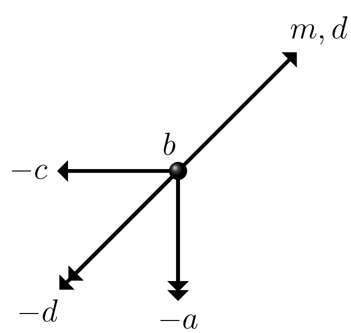
Case 2.4bb There is at least one more black dot at m. Now, there is at least an additional dot at a as well. Project via c to obtain a 2-dimensional incoherent Gale diagram.

Case 2.4bw There is at least one more white dot at m. Project via b and then via c to obtain an incoherent Gale diagram.

Case 2.4w There is at least one white dot at d. Again, we know there must be more points at m. If there is an additional black point at m, then projection via b and then via c gives an incoherent diagram. Consider when an additional point at m is white. There must be more points at a or c. Without loss of generality, suppose there are more points at a. Again, if there is at least one more black point at a, then projection via b and then via c gives an incoherent diagram. Consider when an addition point at a is white. We are left with the following diagram.

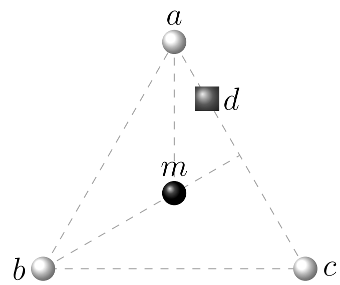


Projection via b gives the following.



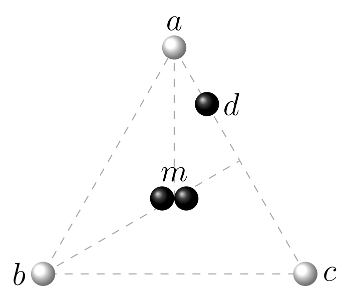
which can be verified to be an incoherent Gale diagram in two dimension. Therefore, all Gale diagrams in Case 2.4 are incoherent.

Case 2.5 The fifth parallelism class occurs at location 5. Note that we expect to see an infinite family of all-coherent Gale diagrams in this case.



Case 2.5b There is a black dot at d. To prevent the hyperplane through a, c, and d from containing $n-2$ points, there must be some other points in the diagram. Thus, m has multiplicity greater than 1.

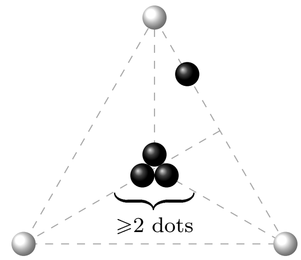
Case 2.5bb Suppose that there are at least two black dots at m. Then, we are in the following situation.



We claim that if any more points are added to a, b, c, or d, then the diagram is incoherent. We analyze each case as follows.

If a black dot is added to a, then project via b and then via d to obtain a 1-dimensional incoherent Gale diagram. If a white dot is added to a, then projection via b and then via c gives an incoherent diagram. If a black dot is added to b, then project via m and then via d. If a white dot is added to b, then project via a and then via c. If a black dot is added to c, then project via b and then m. If a white dot is addd to c, then project via a and then via c. If a black dot is added to d, then project via b to obtain the exceptional incoherent diagram of corank 2, described by Edman. If a white dot is added to d, then project via b and then via c.

Moreover, if a white dot is added to m, then projection via b and then c also gives an incoherent diagram. Therefore, we are only left with the case in the following diagram, which was labeled in the theorem as Family $\mathcal{F}_2$. Note that we need at least two dots at m so that no plane contains $n-2$ vectors in $\mathcal{A}^*$.



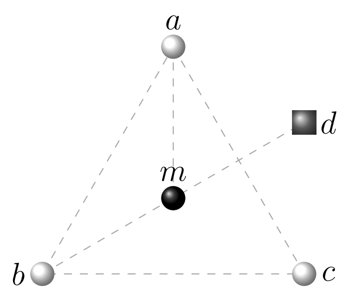
(Family $\mathcal{F}_2$)

Case 2.5bw Suppose there there is at least a white dot at m. Then, projection via b and then via c gives an incoherent diagram.

Case 2.5w There is a white dot at d. Again, m must have multiplicity greater than 1. If there is an additional black dot at m, projection via b and then via c gives an incoherent diagram. If there is an additional white dot at m, then projection via b and then via d gives an incoherent diagram.

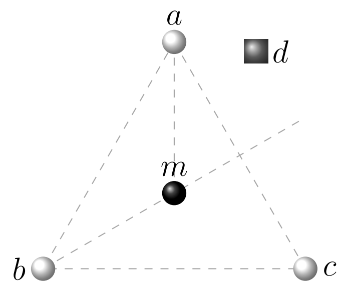
We finish Case 2.5

Case 2.6 The fifth parallelism class occurs at location 6.



We claim that all non-degenerate Gale duals that occur in this case is incoherent. Because the hyperplane through b, m, and d passes every parallelism class except a and c, we know that there must be more points to the diagram. Thus, m must have multiplicity greater than 1. If there is an additional black dot at m, then projection via a and then via d gives an incoherent diagram. If there is an additional white dot at m, then consider what color a dot at d is. If a dot at d is black, then projection via a and then via b gives an incoherent diagram. If a dot at d is white, then projection via a and then via c gives an incoherent diagram. This finishes Case 2.6.

Case 2.7 The fifth parallelism class occurs at location 7.

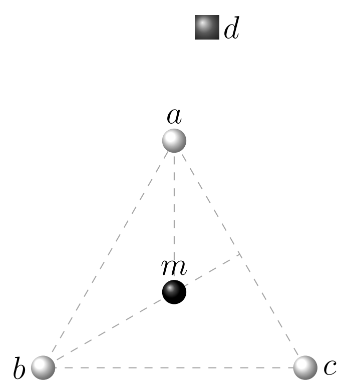


Case 2.7.1 If there are no more points, then the diagram contains exactly five vectors. This belongs to (1) in the theorem. We will check later that it is all-coherent.

Case 2.7.2 Suppose there are more points. We claim that all non-degenerate Gale diagrams in Case 2.7.2 are incoherent. Then, m must have multiplicity greater than 1. If there is an additional black dot at m, then projection via a and then via d gives an incoherent diagram. If there is an additional white dot at m, then consider what color a dot at d is. If a dot at d is black, then projection via a and then via b gives an incoherent diagram. If a dot at d is white, then projection via b and then via d gives an incoherent diagram.

Next, we will treat Cases 2.8, 2.9. and 2.10 together with the same argument.

Cases 2.8, 2.9, 2.10 The fifth parallelism class occurs at location 8, 9, or 10.



If there are no other points to the diagram, then we must be in Case 2.9 (because in 2.8 and 2.10 there is a hyperplane containing three parallelism classes). In that case, the diagram belongs to (1) in the theorem.

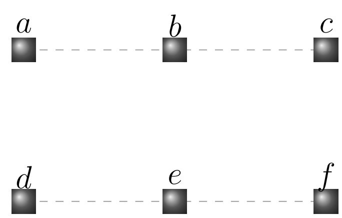
Suppose that there are more points to the diagram. Then, m must be multiplicity greater than 1. If there is an additional black dot at m, then projection via c and then via d gives an incoherent diagram. Suppose there is a white dot at m. If there is a white dot at d, then projection via a and then via c gives an incoherent diagram. Suppose there is a black dot at d. Then, projection via b and then via c gives an incoherent diagram. This shows that unless there are exactly five vectors, all non-degenerate Gale diagrams in these three cases are incoherent.

2.3. **Case 3. $\mathcal{A}^*$ has exactly 6 parallelism classes.** For this case, we use the online catalog of isomorphism classes of oriented matroids by Lukas Finschi [3]. The Gale diagram $\mathcal{A}^*$ consists of six different points on the plane. Suppose we ignore the colors of the points. Then, the six points, up to isomorphism, belong to one of the seventeen types, labeled IC(6,3,i), for $i = 1, 2, \ldots, 17$.

Recall from corank 2 classification of all-coherent Gale diagrams in [1] that if a Gale diagram is all-coherent then it has at most four parallelism classes. Therefore, if a non-degenerate 2-dimensional Gale diagram has at least 5 parallelism classes, it is incoherent. With 6 parallelism classes in $\mathcal{A}^*$, suppose there is a class which is not on any line joining any two of the other five parallelism classes. Then, if we project via that parallelism class, the resulting Gale diagram in two dimension must contain at least five parallelism classes, in which case $\mathcal{A}^*$ is immediately incoherent.

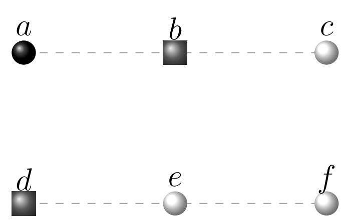
Using the argument above with Finschi's catalog, we conclude that for the case $i = 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 15, 17$, any Gale diagram corresponding to IC(6,3,i) is incoherent. Thus, we only need to consider the cases in which $i = 8, 12, 13, 14, 16$.

Case 3.8 The six parallelism classes follow IC(6,3,8).



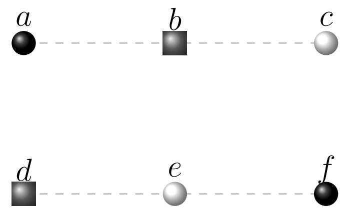
We claim that all non-degenerate Gale diagrams in Case 3.8 are incoherent. Without loss of generality, suppose there is a black dot at a. Consider the projection via b and then via d. If among c, e, f, there are two black dots and a white dot, then the diagram is incoherent. Otherwise, there are 5 cases. For convenience, let -1 denote the color white, and $+1$ denote the color black.

Case 3.8.1 (c, e, f) has color $(-1, -1, -1)$.



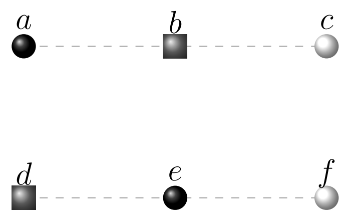
If there is a white dot at d, then projection via b and then via f gives an incoherent diagram. Thus, suppose there is a black dot at d. If there is a black dot at b, then projection via a and then via f gives an incoherent diagram. Thus, suppose there is a white dot at b. Now, the projection via c and then via f gives an incoherent diagram. Therefore, in Case 3.8.1, all non-degerate Gal diagrams are incoherent.

Case 3.8.2 (c, e, f) has color $(-1, -1, 1)$.



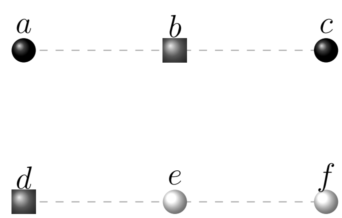
If there is a white dot at d, then projection via b and then e gives incoherence. Suppose there is a black dot at d. If there is a white dot at b, then projection via c and then via d gives incoherence. Suppose there is a black dot at b. Then, projection via a and then via f gives incoherence.

Case 3.8.3 (c, e, f) has color $(-1, 1, -1)$.



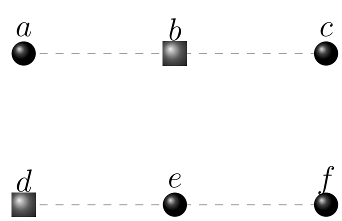
If there is a white dot at b, projection via c and then via d gives incoherence. Suppose there is a black dot at b. If there is a white dot at d, then projection via a and then via e gives incoherence. Thus, suppose there is a black dot at d. Then, projection via a and then via d gives incoherence.

Case 3.8.4 (c, e, f) has color $(1, -1, -1)$.



If there is a white dot at d, then projection via b and then via e gives incoherence. Thus, suppose there is a black dot at d. Then, projection via b and then via f gives incoherence.

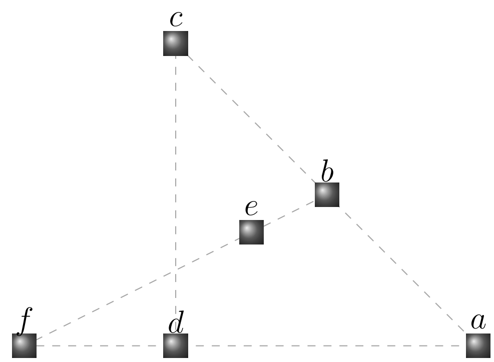
Case 3.8.5 (c, e, f) has color $(1, 1, 1)$.



If d has a black dot, then projection via b and then via e gives incoherence. Thus, suppose d has a white dot. Then, projection via b and then via f gives incoherence.

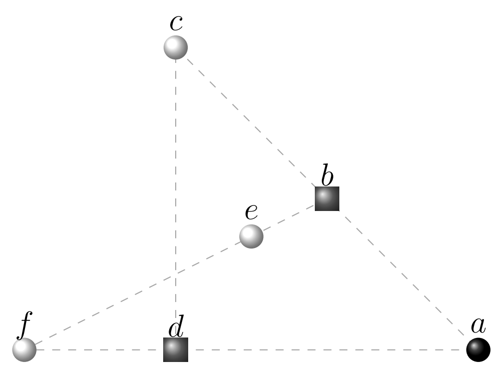
Therefore, we have shown that all non-degenerate Gale diagrams in Case 3.8 are incoherent.

Case 3.12 The six parallelism classes follow IC(6,3,12).



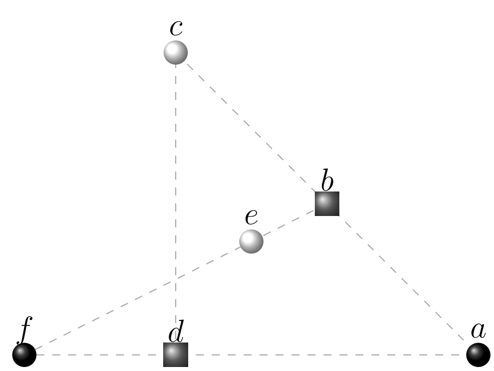
Without loss of generality, assume that there is a black dot at a. Like in the previous case, consider the projection via b and then via d. If among c, e, f, there are two black dots and a white dot, then the diagram is incoherent. Otherwise, there are 5 cases depending on the colors of dots at c, e, and f.

Case 3.12.1 (c, e, f) has color $(-1, -1, -1)$.



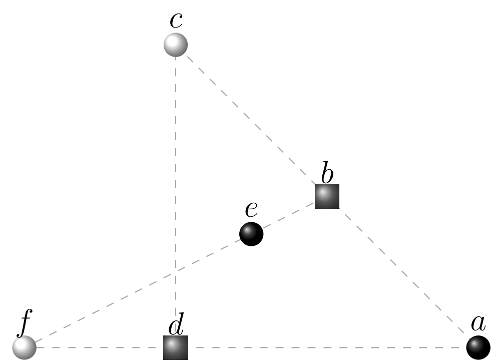
If b has a white dot, then projection via c and then via d gives incoherence. Suppose b has a black dot. If d also has a black dot, then projection via a and then via e gives incoherence. Suppose d has a white dot. Then, projection via c and then via f gives incoherence.

Case 3.12.2 (c, e, f) has color $(-1, -1, +1)$.



If b has a white dot, then projection via d and then via e gives incoherence. Suppose b has a black dot. Then, projection via d and then via c gives incoherence.

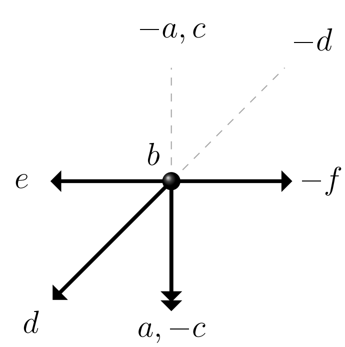
Case 3.12.3 (c, e, f) has color $(-1, +1, -1)$.



If b and d have dots of the same color, then projection via a and then via e gives incoherence. Otherwise, we have two cases.

Case 3.12.3.1 b has a black dot and d has a white dot. In this case, we notice that a, b, and e all have black dots, and c, d, and f all have white dots. Currently, the six vectors in the diagram are all in the same open half-space. Therefore, they cannot form a non-degenerate Gale diagram for $\mathcal{A}^*$. Thus, we have multiplicity. If we project the current diagram via b, the resulting Gale diagram in two dimension has two vectors in the direction $+a$, one in the direction $+e$, one in $-e$, one in $-d$. For this to be a coherent Gale diagram, it must belong to Type II all-coherent family, described in [1]. This means that multiplicity in the three-dimensional Gale diagram can only occur at a or c. If it occurs at a, the additional points must be black. If it occurs at c, the additional points must be white. We see that this does not solve the problem of all vectors being in the same open half-space. Therefore, there cannot be all-coherent Gale diagrams in this case.

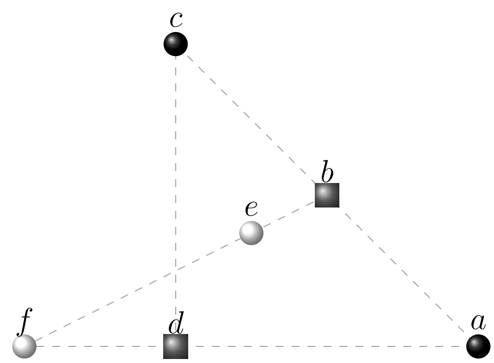
Case 3.12.3.2 Suppose b has a white dot and d has a black dot. Consider projection via b. The resulting Gale diagram is as follows.



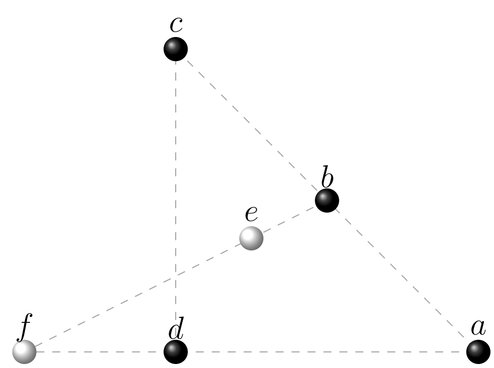
Indeed all the six vectors are still in the same open half-space, and therefore, there must be some multiplicities in the 3-dimensional Gale diagram. In particular, there must be multiplicities in the direction $-a$, c, or $-d$. However, we see that if any of the vectors $-a$, c, or $-d$ in two dimension is added, the Gale diagram is immediately incoherent. That is, the current diagram cannot be extended to a coherent diagram. Therefore, if a non-degenerate Gale diagram is in Case 3.12.3.2, it must be incoherent.

This finishes Case 3.12.3.

<u>Case 3.12.4</u> (c, e, f) has color $(+1, -1, -1)$.



If there is a white dot at b, then projection via d and then via e gives incoherence. Thus, suppose there is a black dot at b. If there is a white dot at d, then projection via c and then via f gives incoherence. Thus, suppose there is a black dot at d. Now, we are in the following situation.

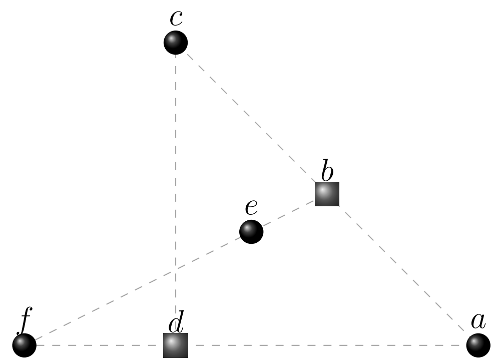


This is actually an equivalent Gale diagram to type $\mathfrak{g}$ described in the theorem. We claim that if any additional point is added to the diagram, then the diagram will be incoherent.

If a black dot is added at a, then projection via b gives incoherence. If a black dot is added at b, then projection via f gives incoherence. If a black dot is added at d, then projection via a and then via e gives incoherence. If a black dot is added at c, e, or f, then projection via b and then via d gives incoherence. If a white dot is added at a, b, d, or e, then projection via c and then f gives incoherence. If a white dot is added at c, then projection via a and then via d gives incoherence. If a white dot is added at f, then projection via a gives incoherence.

Therefore, we have shown that the diagram above is *maximal* in the sense that if any dot is added to any existing parallelism class, then the diagram will be incoherent. We will save showing that the diagram is all-coherent for later. For now, we have them in the list in the theorem.

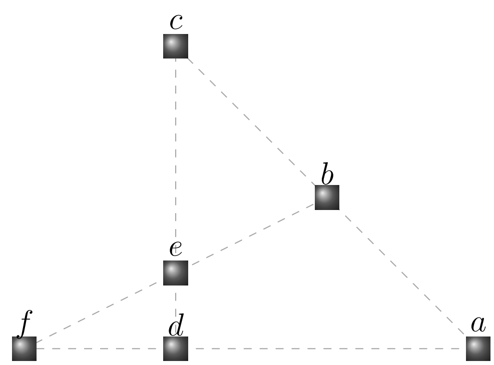
Case 3.12.5 (c, e, f) has color $(+1, +1, +1)$.



If there is a black dot at b, then projection via d and then via e gives incoherence. Thus, suppose there is a white dot at b. If there is a white dot at d, then projection via c and then via e gives incoherence. Thus, suppose there is a black dot at d. Then, projection via c and then via d gives incoherence.

We have finished Case 3.12.

Case 3.13 The six parallelism classes follow IC(6,3,13).



We claim that all non-degenerate Gale diagrams that occur in this case in incoherent. Suppose otherwise that the diagram is coherent. Without loss of generality, assume there is a black dot at a. Consider the projection via f. In the resulting two-dimensional Gale diagram, there are at least two vectors on the line feb, at least two vectors on the line fda, and at least a vector on the line fc. By corank 2 classification (cf. [1]), if the diagram is coherent, then we know that the multiplicity at c is 1 and that either

- all vectors on feb are in the same direction and there are exactly two vectors on fda on opposite direction, or
- all vectors on fda are in the same direction and there are exactly two vectors on feb on opposite direction.

We exploit the collinearity in the diagram, and use the same analysis with projections via any point. This is possible because every point on the diagram is on two of the four dotted lines as shown above.

If we use the analysis through projection at f, we conclude that its "dual" point c has multiplicity 1. On the other hand, if we do at c, we conclude that f has multiplicity 1. Since all points have their own duals ((a, e), (b, d), and (c, f)), we conclude that all points in the diagram have multiplicity exactly 1.

Case 3.13.1 (b, d) has color $(-1, -1)$.

Then, by projection via f and the analysis above, the point at e must be white. Now, consider the plane generated by a, d, and f. In order for the Gale diagram to strongly capture zero, the point at c must be black. Then, by projection via e and the analysis above, the point at f must be white. However, we note that all the vectors are in the same closed half-space generated by the plane through a, b, and c. This gives a contradiction.

Case 3.13.2 (b, d) has color $(-1, +1)$.

Then, by projection via f and the analysis above, e must be black. The plane through a, b, and c forces f to be white. Projection at e implies that c is white. However, all the vectors now lie on the same side of the plane through b, e, and f. A contradiction.

Case 3.13.3 (b, d) has color $(+1, -1)$.

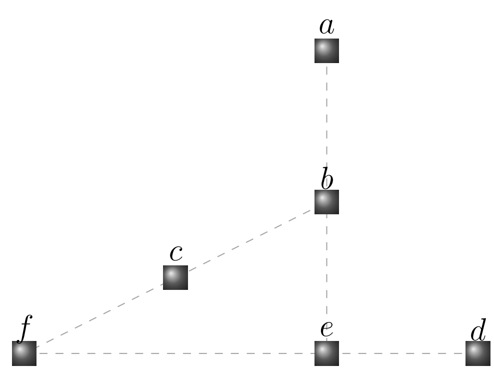
The same argument as in Case 3.13.2 applies due to symmetry.

Case 3.13.4 (b, d) has color $(+1, +1)$.

By the same argument as above, e must be white. The plane through b, e, and f forces the dot at c to be black. Similarly, the dot at f is black. However, the projection via a and then via e gives an incoherent diagram.

Therefore, all non-degenerate Gale diagrams in Case 3.13 are incoherent.

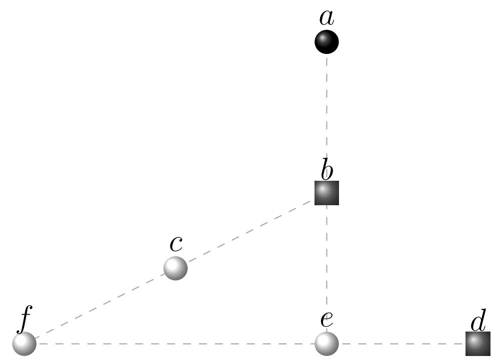
Case 3.14 The six parallelism classes follow IC(6,3,14).



Assume without loss of generality that there is a black dot at a. As before, if among c, e, and f, there are two black dots and a white dot, then the diagram is incoherent. Otherwise, there are 5 cases to consider.

We will use the analysis of corank 2 classification that we used in Case 3.13 again. Unlike Case 3.13, we cannot use the technique at any point we wish. Instead, we can only use it at b, e, and f. An immediate result is that if the diagram is coherent, then the parallelism classes a, c, and d all have multiplicity 1.

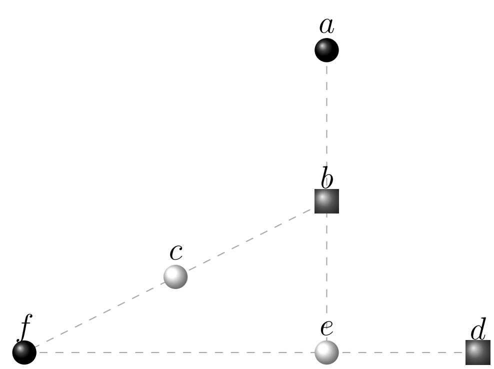
Case 3.14.1 (c, e, f) has color $(-1, -1, -1)$.



Recall that a, c, and d all have multiplicity 1. If the point at d is black, then consider the plane through a, b, and e. The side of the plane containing d has only one black point, while there are white points and c and f. Since the multiplicity of c is 1, there cannot be any more points to c. In order for the diagram to strongly capture zero, there must be a black point at f. Then, projection via c and then via e gives incoherence.

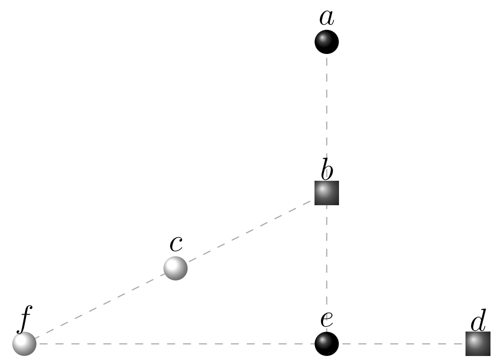
Suppose the point at d is white. If there is a white dot at b, then projection via c and then via e gives incoherence. Suppose all dots at b are black. Now consider the plane generated by b, c, and f. There is only one black dot at a, and only one white dot at d. For the diagram to capture zero strongly, there must be a black dot at e. Note that now b and e both have black dots, while each of c and d has a white dot. Thus, projection via a and then via f gives incoherence.

Case 3.14.2 (c, e, f) has color $(-1, -1, +1)$.



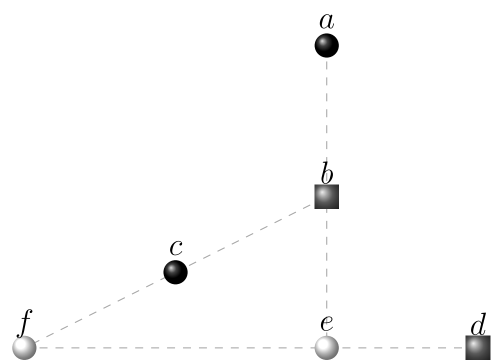
If there is a white dot at b, then projection via c and then via d gives incoherence. Thus, suppose all the dots at b are black. If there is a black dot at d, then projection via a and then via f gives incoherence. Thus, suppose all the dots at d are white. Then, the projection via a and then via d gives incoherence.

Case 3.14.3 (c, e, f) has color $(-1, +1, -1)$.



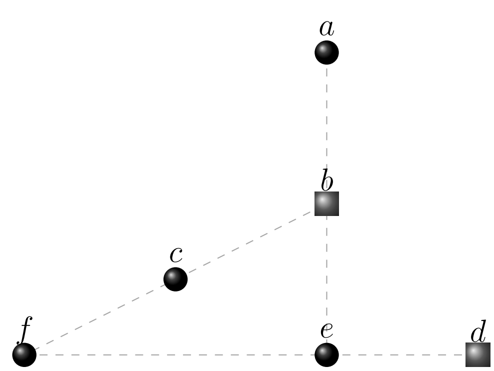
If there is a black dot at b, then projection via a and then via d gives incoherence. If there is a white dot at b, then projection via c and then via d gives incoherence.

Case 3.14.4 (c, e, f) has color $(+1, -1, -1)$.



If there is a black dot at b, then projection via a and then via d gives incoherence. Thus, suppose all dots at b are white. If the dot at d is black, then projection via a and then via f gives incoherence. Thus, suppose that the dot at d is white. Now, consider the plane through b, c, and f. For the Gale diagram to capture zero strongly, either there must be more white dots on the a-side, or there must be more black dots on the d, e-side. Since a and d both have multiplicity 1, this forces e to contain a black dot. Then, projection via a and then via f gives incoherence.

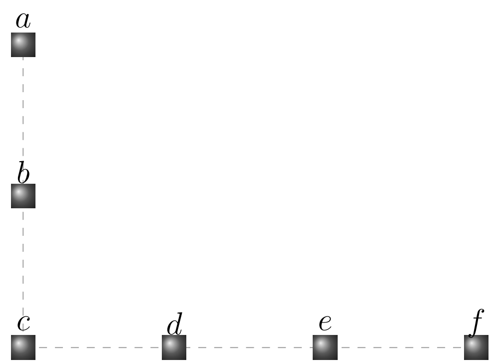
Case 3.14.5 (c, e, f) has color $(+1, +1, +1)$.



If there is an additional white dot at e or f, then projection via b and then via d gives incoherence. Thus, suppose that all dots at e and at f are black. Consider the plane through a, b, and e. For the diagram to capture zero strongly, either there must be more white dots on the c, f-side, or the dot at d must be black. Since all the points at c, e, and f are black, the dot at d must be black. Now, we see that all the dots at a, c, d, e, and f are black, but there must be at least a white dot so that the diagram captures zero strongly. Thus, there must be a white dot at b. Then, projection via c and then via e gives incoherence.

Therefore, all non-degenerate Gale diagrams that occur in Case 3.14 are incoherent.

Case 3.16 The six parallelism classes follow IC(6,3,16).



In this case, the Gale diagram cannot strongly capture zero. Therefore, there is no non-degenerate Gale diagram for $\mathcal{A}^*$.

Therefore, we have finished Case 3. All non-degenerate Gale diagrams that occur are incoherent, except one case in Case 3.12.4 which is labeled as $\mathfrak{g}$ in the theorem.

2.4. **Case 4. $\mathcal{A}^*$ has at least 7 parallelism classes.** We claim that all non-degenerate Gale diagrams with at least 7 parallelism classes are incoherent.

Recall that from corank 2 classification (cf. [1]), we have the following result.

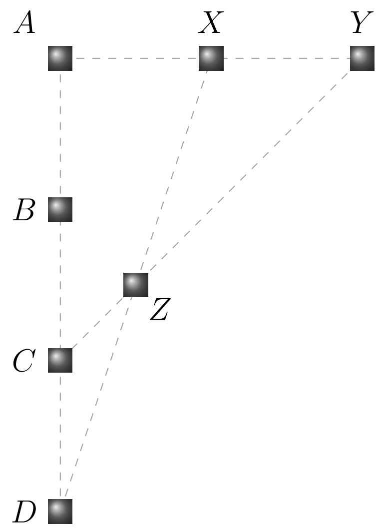
Fact 2.7. *If a non-degenerate Gale diagram of $\mathcal{A}^*$ is coherent, then at least one of the following holds.*

- *$\mathcal{A}^*$ has exactly four parallelism classes with at most one class having multiplicity greater than 1.*
- *$\mathcal{A}^*$ has exactly three parallelism classes with at most two classes having multiplicity greater than 1. Furthermore, if two classes have multiplicity greater than 1, then one of them has multiplicity exactly 2.*

Consider a non-degenerate Gale diagram of $\mathcal{A}^*$ with at least 7 parallelism classes. Consider any seven classes from the diagram and ignore the dot colors. As a result, we obtain seven distinct points on the plane. Call the set of these seven points P.

If at least six points of P are collinear, then there cannot be a tetrahedron subconfiguration in the diagram. We will not consider this case.

If five points of P are collinear, then consider a point outside X the line of collinearity. The projection via the parallelism class corresponding to the point X gives a two-dimensional Gale diagram of at least five parallelism classes (from the five points on the line). Thus, by Fact 2.7, the diagram is incoherent.



If four points of P are collinear, say the points are A, B, C, and D, then consider two points outside of this line, say X and Y. If Y is not on the lines XA, XB, XC, or XD, then projection via X gives incoherence as before. Thus, suppose without loss of generality, that Y is on XA. Let the other point be Z. Note that by the same argument, if the diagram is coherent, then Z must be in both $XA \cup XB \cup XC \cup XD$ and $YA \cup YB \cup YC \cup YD$. If Z is on the line XY, then projection via A yeilds a 2-dimensional Gale diagram with at least two classes of multiplicity at least three, which cannot be coherent. Suppose, then, that Z is on the intersection $X\square \cap Y\square'$ where $\square$ and $\square'$ denote some letters in $\{B, C, D\}$. Then, projection via Z gives a 2-dimensional Gale diagram with four parallelism classes and with two classes having multiplicity 2. This cannot be coherent either.

Thus, suppose that at most three points of P are collinear. Consider any point X in P. Draw lines joining X to all other size points. Since no four points are collinear, there must be at least three distinct lines. If there are at least five distinct lines, then projection via X gives incoherence. If there are exactly three distinct lines, then on each line there are X and two other points, and then projection via X gives a 2-dimensional Gale diagram with 3 parallelism classes having multiplicity 2. By Fact 2.7, this diagram is incoherent. Thus, suppose there are exactly four distinct lines, l_1, l_2, l_3, and l_4. Let each of l_1 and l_2 contains two points different from X, and each of l_3 and l_4 contains a point different from X. However, projection via X gives a 2-dimensional Gale diagram with four parallelism classes and with two classes having multiplicity 2. This cannot be coherent either.

Therefore, if the Gale diagram for $\mathcal{A}^*$ contains at least 7 parallelism classes, then $(\mathcal{A}, f)$ is incoherent.

3. Conclusion

In the previous section, we did the casework which shows that if a non-degenerate Gale diagram of corank 3 is all-coherent, then it must belong to one of the three families described in Theorem 2.5. The work in this REU report successfully generalizes Robert Edman's work [1] into zonotopes of corank 3. The proof of the converse, which we currently omit, is more algebraic but less tedious than the casework we did in this REU report. We hope to publish the proof of the converse soon.

References

[1] Robert Edman, *Diameter and Coherence of Monotone Path Graphs in Low Corank*, Ph.D. thesis, University of Minnesota, 2015. `http://www.math.umn.edu/~reiner/edman-thesis.pdf`

[2] Robert Edman, Mathematica code for checking whether a pair $(\mathcal{A}, f)$ is all-coherent, private communication.

[3] Lukas Finschi, *Catalog of Isomorphism Classes of Oriented Matroids.* `http://www.om.math.ethz.ch/?p=catom`