CRITERIA FOR A SIMPLICIAL COMPLEX BEING VIRTUALLY COHEN-MACAULAY

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1. Introduction

In [BES17], Berkesch, Erman, and Smith introduced *virtual resolutions* of graded modules to take into account the freedom given by nontrivial irrelevant ideals in certain ambient spaces, such as a product of projective spaces. Meanwhile, there is a nice class of modules with the *Cohen-Macaulay* property, which is characterized by having a minimal free resolution of length equal to their codimenions. While a lot of research has been done to characterize such modules, there is still little understanding of when modules are virtually Cohen-Macaulay, which are modules with virtual resolutions of length equal to their codimensions.

In this project, we restrict our attention to a specific class of S-modules of the form S/I_Δ, where I_Δ is a squarefree monomial ideal defined by a simplicial complex Δ and S is the appropriate polynomial ring. A simplicial complex Δ is Cohen-Macaulay if S/I_Δ is Cohen-Macaulay. A Cohen-Macaulay simplicial complex has many nice combinatorial properties and characterizations, such as Reisner's criterion and connections to purity, shellability, and gallery-connectedness. Naturally, we want to ask whether similar statements can be made about virtually Cohen-Macaulay simplicial complexes.

In §2, we define necessary concepts such as the Stanley-Reisner correspondence between squarefree monomial ideals and simplicial complexes, Cohen-Macaulay modules and virtual resolutions to orient the reader for the rest of the report. We then show in §3 that virtually Cohen-Macaulay complexes are pure after removing irrelevant facets. We give a partial characterization in §4 of virtually Cohen-Macaulay complexes for which we can obtain a short virtual resolution by intersecting I_Δ with the irrelevant ideal. In fact, we prove the more general statement in Theorem 4.8. Finally, we restrict ourselves to the special case of balanced complexes in §5 and show that every balanced complex is in fact virtually Cohen-Macaulay.

2. Definitions and Background

2.1. Stanley-Reisner Theory. Fix $n > 0$ and let $X = \{x_1, x_2, \ldots, x_n\}$. Let $S = \Bbbk[X] = \Bbbk[x_1, \ldots, x_n]$, where $\Bbbk$ is a field. A free S-module of finite rank is a direct sum $F = S^k$ for some nonnegative integer k. A sequence of linear maps between free S-modules,

$$\mathcal{F}_\bullet : 0 \leftarrow F_0 \xleftarrow{\phi_1} F_1 \xleftarrow{\phi_2} \cdots \xleftarrow{\phi_n} F_n,$$

is a *complex* if $\phi_i \circ \phi_{i+1} = 0$ for $i = 1, \ldots, n-1$. In the case when S is a Cox ring of a toric variety, S is endowed with a $\mathbb{Z}^r$-grading. In this context, we require all ϕ_i's to be degree-preserving, i.e. $\deg \phi_i(\vec{v}) = \deg \vec{v}$ for all $\vec{v} \in F_i$. This can be accomplished by shifting the degrees of elements in S by a constant degree. We write $S(-\vec{a})$ to denote a rank-one free S-module in which $\deg 1 = \vec{a} \in \mathbb{Z}^d$.

A complex $\mathcal{F}_\bullet$ is a *free resolution of S-module M* if $\widetilde{H}_i(\mathcal{F}_\bullet) = 0$ for $i \geq 1$ and $\widetilde{H}_0(\mathcal{F}_\bullet) = F_0/\operatorname{im}\phi_1 = M$. The *length* of a free resolution is the largest positive integer l such that $F_l \neq 0$. A free resolution is *minimal* if for all $\phi_k = [\lambda_{ij}] : \oplus_{j=1}^{r} S(-\vec{a_j}) \to \oplus_{i=1}^{r'} S(-\vec{b_i})$, $\lambda_{ij} = 0$ if $\vec{a_j} = \vec{b_i}$.

The Stanley-Reisner correspondence given in Definition 2.3 between monomial ideals and simplicial complexes allows us to relate combinatorial properties of simplicial complexes to algebraic properties of the corresponding squarefree monomial ideals.

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Definition 2.1. An *abstract simplicial complex* Δ on vertex set X is a collection of subsets of X such that $A \in \Delta$ whenever $A \subseteq B \in \Delta$. An element of Δ is called a *face* or a *simplex*, and a simplex of Δ not properly contained in another simplex of Δ is called a *facet*. The *dimension* of a simplex $A \in \Delta$ is $\dim A := |A| - 1$. We say that Δ is *pure* if all of its facets have the same dimension.

Definition 2.2. Let $A \subset X$. Then the *monomial supported* on A is the squarefree monomial $m_A = \prod_{x_i \in A} x_i$. Conversely, if m is a squarefree monomial, then it has support $\operatorname{supp} m = \{x_i : x_i \text{ divides } m\}$. Notice that $\operatorname{supp} m_A = A$ and $m_{\operatorname{supp} m} = m$.

Definition 2.3. Given a squarefree monomial ideal $I \subseteq S$, the *Stanley-Reisner complex* Δ_I is the simplicial complex consisting of the monomials not in I,

$$\Delta_I = \{m \subset X : m \notin I\}.$$

For a simplicial complex Δ on X, the *Stanley-Reisner ideal of* Δ is the squarefree monomial ideal in $\Bbbk[X]$ generated by the non-faces of Δ,

$$I_\Delta = (m_A : A \notin \Delta).$$

Equivalently,

$$I_\Delta = \bigcap_{A \in \Delta} (x_i : x_i \notin A).$$

The *Stanley-Reisner ring* of Δ is the quotient ring $\Bbbk[\Delta] := S/I_\Delta$.

Note that the above correspondence is a bijection, i.e. $I_{\Delta_I} = I$ and $\Delta_{I_\Delta} = \Delta$.

Example 2.4. The simplicial complex Δ on $X = \{a, b, c, d, e, f\}$ given by all subsets of the sets

$$\{c, d, e, f\}, \{a\}, \{b, c\}, \{b, e\},$$

as shown in Figure 1.

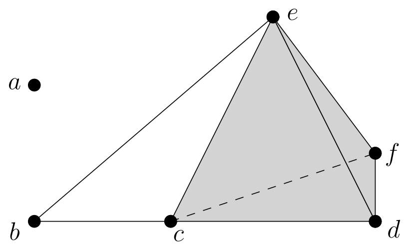

FIGURE 1. Simplicial complex Δ.

The Stanley-Reisner ideal is

$$\begin{aligned} I_\Delta &= \langle a, b\rangle \cap \langle a, d, e, f\rangle \cap \langle a, c, d, f\rangle \cap \langle b, c, d, e, f\rangle \\ &= \langle ab, ac, ad, bd, ae, bce, af, bf\rangle. \end{aligned}$$

Given a minimal free resolution $\mathcal{F}_\bullet$ of S/I_Δ, Hochster's theorem provides a combinatorial description of the degree of each summand of the i-th free module. If the complex $\mathcal{F}_\bullet$ is a minimal free resolution of a finitely generated $\mathbb{N}^n$-graded module M and

$$F_i = \bigoplus_{\vec{a} \in \mathbb{N}^n} S(-\vec{a})^{\beta_{i,\vec{a}}},$$

the *i-th Betti number of M in degree $\vec{a}$* is the invariant $\beta_{i,\vec{a}}$.

Theorem 2.5. [FMS14, Corollary 7.13] Let I be a squarefree monomial ideal and σ be a squarefree multidegree. Then

$$\beta_{i,\sigma}(S/I) = \dim_\Bbbk \tilde{H}^{|\sigma|-i-1}(\Delta_I)$$

and

$$\beta_{i,\sigma}(I) = \dim_\Bbbk \tilde{H}^{|\sigma|-i-2}(\Delta_I)$$

Note that there is a formulation to Hochster's theorem concerning the Alexander dual of a simplicial complex defined in the following.

Definition 2.6. If Δ is a simplicial complex on vertex set X, then the *Alexander dual* of Δ is defined to be

$$\Delta^* := \{\sigma \subseteq X \mid X \setminus \sigma \notin \Delta\}.$$

In other words, Δ^* consists of the complements of the nonfaces of Δ.

If Δ is on vertex set X and $\sigma \in \Delta$, then let $\overline{\sigma}$ be the complement of σ in the vertex set: $\overline{\sigma} := X \setminus \sigma$. We can now state the dual version of Hochster's theorem.

Theorem 2.7 (Hochster's formula, dual version [MS05], Corollary 1.40)**.** All nonzero Betti numbers of I_Δ and S/I_Δ lie in squarefree degrees σ, where

$$\beta_{i,\sigma}(I_\Delta) = \beta_{i+1,\sigma}(S/I_\Delta) = \dim_{\Bbbk} \tilde{H}_{i-1}(\operatorname{link}_{\Delta^*}(\overline{\sigma}; \Bbbk)).$$

2.2. **Cohen-Macaulay Simplicial Complexes.**

Definition 2.8. [MS05, Theorem 13.37] Given an S-module M, M is *Cohen-Macaulay*, if the minimal free resolution $\mathcal{F}_{\bullet}$ of M by free S-modules has length $n - \dim M$.

For equivalent conditions for when a module is Cohen-Macaulay, readers are encouraged to read the referenced theorem in [MS05].

Definition 2.9. We say that an ideal Δ is *Cohen-Macaulay* if its Stanley-Reisner ring $\Bbbk[\Delta]$ is Cohen-Macaulay. Define the codimension of I_Δ as $\operatorname{codim} I_\Delta = n - \dim V(I_\Delta)$.

Reisner's criterion (Theorem 2.11) provides an exact combinatorial description of when a simplicial complex is Cohen-Macaulay using the links of the simplicial complex.

Definition 2.10. The *link* of face σ inside the simplicial complex Δ is

$$\operatorname{link}_\Delta(\sigma) := \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta \text{ and } \tau \cap \sigma = \varnothing\},$$

or the set of faces of Δ that are disjoint from σ but whose unions with σ are faces of Δ.

Theorem 2.11. [MS05, Theorem 5.53] Δ is Cohen-Macaulay if and only if for all $\sigma \in \Delta$,

$$\widetilde{H}_i(\operatorname{link}_\Delta(\sigma)) = 0 \text{ for all } i < \dim(\operatorname{link}_\Delta(\sigma)).$$

2.3. **Virtual Resolutions and Virtually Cohen-Macaulay Complexes.** From now on, we will always be working in the product projective spaces $\mathbb{P}^{\vec{n}} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ with $\vec{n} = (n_1, \ldots, n_r)$. Let $S := \Bbbk[x_{i,j} : 1 \le i \le r, 0 \le j \le n_i]$ be the Cox ring of $\mathbb{P}^{\vec{n}}$ over a field $\Bbbk$. Let $B := \bigcap_{i=1}^{r} \langle x_{i,0}, x_{i,1}, \ldots, x_{i,n_i}\rangle$ be the *irrelevant ideal of* S. We consider S as having the $\mathbb{Z}^r$-grading given by $\deg(x_{i,j}) = \mathbf{e_i}$. For convenience, we write $|\vec{n}| := n_1 + \cdots + n_r$.

A simplicial complex Δ on $\mathbb{P}^{\vec{n}}$ has vertices $X_{\vec{n}} = \{x_{i,j} : 1 \le i \le r, 1 \le j \le n_i+1\}$. From now on, we use X as a shorthand for $X_{\vec{n}}$ to denote the vertex set defined by $\mathbb{P}^{\vec{n}}$. Let $\operatorname{comp} : X \to [n]$ be the map defined by $\operatorname{comp}(x_{i,j}) = i$, i.e. we use $\operatorname{comp}(v)$ to denote the component that the vertex v belongs to.

The ideal quotient $(I : J) := \{s \in S : sJ \subseteq I\}$. A simpicial complex Δ is *saturated* if $I_\Delta : B^\infty = I_\Delta$. The saturation of I by B is the ideal $I : B^\infty := \cup_{k>0}(I : B^k)$. The *annihilator* of M is $\operatorname{ann} M = \{s \in S : sM = 0\}$. Given an S-module M, $\Gamma_B(M) := \{r \in M \mid rB^k = 0\}$. If $M = S/I$, this specializes to

$$\Gamma_B(S/I) = \{r \in S/I \mid rB^k \subseteq I\}.$$

We define a virtual resolution of S/I for an ideal $I \subset S$ as follows:

Definition 2.12. A graded free complex

$$\mathcal{F}_{\bullet} : [F_0 \leftarrow F_1 \cdots \leftarrow F_k \leftarrow 0]$$

is a *virtual resolution of* S/I if the following conditions are satisfied:

(1) $\operatorname{rad}\operatorname{ann} H_i\mathcal{F}_{\bullet} \supseteq B$ for all $i > 0$.
(2) $\operatorname{ann} H_0\mathcal{F}_{\bullet}/\Gamma_B(H_0\mathcal{F}_{\bullet}) = (S/I)/\Gamma_B(S/I)$.

In the special case where $F_0 = S^1$, an equivalent formulation of the second condition is:

$$\operatorname{ann} H_0\mathcal{F}_{\bullet} : B^{\infty} = I : B^{\infty}.$$

Definition 2.13. Let $\operatorname{codim} I_\Delta := |\vec{n}| - \dim V(I_\Delta)$. A B-saturated simplicial complex Δ in a product projective spaces $\mathbb{P}^{\vec{n}}$ supported on vertex set X is *virtually Cohen-Macaulay* if there exists a virtual resolution of $\Bbbk[\Delta]$ of length $\operatorname{codim} I_\Delta$.

Notice that when Δ is B-saturated, $n - \dim V(I_\Delta) = n + r - \dim_{\mathbb{A}^{n+r}} I_\Delta$, which means that the geometric definition of codimension coincides with the definition in terms of the affine Krull dimension of I_Δ in the Cox ring.

Note that all Cohen-Macaulay simplicial complexes are virtually Cohen-Macaulay since a minimal free resolution of a module is also a virtual resolution of the module.

[BES17] describes several constructive methods for obtaining virtual resolutions of some S/I. In particular, we have the following:

Lemma 2.14. If we have two ideals $I, J \subset S$ satisfying $V(I) = V(J)$, then any free resolution $\mathcal{F}_{\bullet}$ of S/J is a virtual resolution of S/I.

Proof. We verify that r satisfies the two conditions of Definition 2.12. Since $\mathcal{F}_{\bullet}$ is a free resolution, it satisfies the first condition of the definition. Furthermore, since $\mathcal{F}_{\bullet}$ is a minimal free resolution, we have that $F_0 = S^1$, letting us use the second form of the second condition. For this condition, note that $\operatorname{ann} H_0\mathcal{F}_{\bullet} = \operatorname{ann}(S/J) = J$. Then $J : B^{\infty} = I : B^{\infty}$ since $V(J) = V(I)$ by hypothesis. Thus, the second condition is satisfied and $\mathcal{F}_{\bullet}$ is a virtual resolution of S/I. □

By the Stanley-Reisner correspondence, we can similarly define the notion of irrelevant and relevant faces.

Definition 2.15. A face of a simplicial complex Δ is *relevant* if it contains at least one vertex of every component; otherwise we say it is *irrelevant*.

With the above definition, we state the following lemma that relates to the main strategy we use to obtain virtually cohen-Macaulay simplicial complexes.

Lemma 2.16. Let Δ, Δ' be two simplicial complexes in $\mathbb{P}^{\vec{n}}$ such that $\Delta \subseteq \Delta'$ and $\Delta' \setminus \Delta$ contains only irrelevant faces. Then the free resolution of $I_{\Delta'}$ is a virtual resolution of I_Δ.

Proof. By the Stanley-Reisner correspondence, I_Δ and $I_{\Delta'}$ are ideals in the ring $S = \Bbbk[x_{i,j} : 1 \leq i \leq r, 1 \leq j \leq n_i + 1]$. Adding a facet $F = \{x_{i_1,j_1}, \ldots, x_{i_k,j_k}\}$ to a simplicial complex Δ is equivalent to intersecting I_Δ with the ideal J generated by the variables $\{x_{i,j} : 1 \leq i \leq r, 1 \leq j \leq n_i + 1\} \setminus \{F\}$. If F is an irrelevant facet, $B \subseteq J$ where B is the irrelevant ideal of S and J has empty variety. Therefore $V(I_\Delta) = V(I_{\Delta'})$ and by Lemma 2.14 we are done. □

3. Purity and Gallery-Connectedness

In this section, we prove that all virtually Cohen-Macaulay complexes are pure up to irrelevant facets and we give an example of a virtually Cohen-Macaulay complex that is not gallery-connected. This section is motivated by the properties of Cohen-Macaulay simplicial complexes. In particular,

Theorem 3.1. If Δ is Cohen-Macaulay, then Δ is pure and gallery-connected.

The first part of this theorem on purity is stated in [FMS14, Proposition 5.12]. The second part on gallery-connectedness is a well-known result and it is not difficult to prove using Reisner's criterion and induction on the dimension of a Cohen-Macaulay simplicial complex.

3.1. **Purity.** First note that there is a bijection between facets of a simplicial complex and the associated primes of its Stanley-Reisner ideal, as explained below.

Theorem 3.2. [FMS14, Theorem 2.8] Let I be a squarefree monomial ideal, then the following are equivalent for a monomial prime ideal P.

(1) P contains I and is minimal among the prime ideals that do so.

(2) I can be written as an irredundant intersection of primary ideals $I = \bigcap Q_j$, and P is the radical of one of the Q_j in this intersection.
(3) There is a monomial $m \notin I$ with the property that $mx \in I \iff x \in P$.

Definition 3.3. Let I be a squarefree monomial ideal. If P is a monomial prime ideal satisfying the equivalent conditions in Theorem 3.2, then P is an *associated prime of* I. We denote the set of all associated primes of I as $\operatorname{Ass} I$.

For Δ supported on X and $S \subseteq X$, by definition of Stanley-Reisner ideals, we have that $Q = (x_i : x_i \in S) \in \operatorname{Ass} I_\Delta$ if and only if $X \setminus S$ is a facet of Δ. Moreover, $\operatorname{codim} Q = |\vec{n}| + r - 1 - \dim(X \setminus S)$. So Δ being pure up to irrelevant facets is equivalent to the following proposition:

Proposition 3.4. Let Δ be a simplicial complex on the product projective space $\mathbb{P}^{\vec{a}} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, where $\vec{n} = (n_1, \ldots, n_r)$. Let $F := [F_0 \leftarrow F_1 \cdots \leftarrow F_p \leftarrow 0]$ be a virtual resolution of S/I_Δ such that $p = |\vec{n}| - \dim V(I_\Delta)$. Then for all $Q \in \operatorname{Ass}(I_\Delta)$ that doesn't contain the irrelevant ideal B, $\operatorname{codim} Q = p$.

Proof. For any associated prime Q, $V(I_\Delta) \supseteq V(Q)$ and $\dim V(I_\Delta) \geq \dim V(Q)$, so $\operatorname{codim} Q = |\vec{n}| - \dim V(Q) \geq |\vec{n}| - \dim V(I_\Delta) = p$. But by Proposition 2.5(i) of [BES17], an associated prime Q of I_Δ that does not contain the irrelevant ideal B satisfies $\operatorname{codim} Q \leq p$. So $\operatorname{codim} Q = p$. □

3.2. **Counterexample to Gallery-Connectedness.** First we define gallery-connectedness.

Definition 3.5. Let Δ be a pure simplicial complex. We say Δ is *gallery-connected* if for any two facets $F, F' \in \Delta$, there exists a path of facets $F = F_1, \ldots, F_{n-1}, F_n = F'$ such that for all $1 \leq i \leq n-1$, $F_i \cap F_{i+1}$ has codimension 1 in Δ.

Because of Theorem 3.1, one would hope that a virtually Cohen-Macaulay complex is gallery-connected up to adding irrelevant faces. A counterexample is the simplicial complex in Figure 2. It is virtually Cohen-Macaulay, and a short virtual resolution is given below in Equation 1.

$$S^2 \xleftarrow{\begin{pmatrix} 0 & f & 0 & a \\ -c & -f & b & -a \end{pmatrix}} S^4 \xleftarrow{\begin{pmatrix} 0 & -b \\ a & 0 \\ 0 & -c \\ -f & 0 \end{pmatrix}} S^2 \tag{1}$$

Note that the degrees of 1 in the two components of the first module are both $\vec{0}$. But the simplicial complex is not gallery-connected because there is no path between the facets $\{a, d, e, f\}$ and $\{b, c, d, e\}$ that satisfy Definition 3.5. Moreover, one cannot modify the complex by adding or removing irrelevant faces to obtain a gallery-connected complex since irrelevant faces are at most two-dimensional, and therefore would not affect whether the complex is gallery-connected.

4. Intersections with the Irrelevant Ideal

Throughout this section, let I be an an ideal in $\mathbb{P}^{\vec{n}} = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r}$ with irrelevant ideal B. For $\vec{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r$, write $B^{\vec{a}} = \cap_{i=1}^r \langle x_{i,0}, \ldots, x_{i,n_i} \rangle^{a_i}$. Let $r_{\vec{a}}$ denote the minimal free resolution of $S/(I \cap B^{\vec{a}})$.

The formal Unit Box Conjecture is described in Conjecture 4.2. However, Conjecture 4.1 is a stronger statement which seemed easier to prove. Both of these conjectures are false.

Conjecture 4.1. If for all vectors $\vec{u} \in \{0,1\}^r$ we have that $\operatorname{len} r_{\vec{u}} > \operatorname{codim} I_\Delta$, then there exists such a vector $\vec{u}$ with $\operatorname{len} r_{\vec{u}} = l > \operatorname{codim} I_\Delta$ with ideal J generated by the maximal minors of ϕ_l satisfying $\dim V(J) \geq 0$.

Conjecture 4.2. If a simplicial complex Δ in $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r}$ is virtually Cohen-Macaulay, then there exists a vector $\vec{u} \in \{0,1\}^r$ such that $r_{\vec{u}}$ is a virtual resolution of S/I_Δ of length $\operatorname{codim} I_\Delta$.

Remark 4.3. Conjecture 4.1 implies Conjecture 4.2.

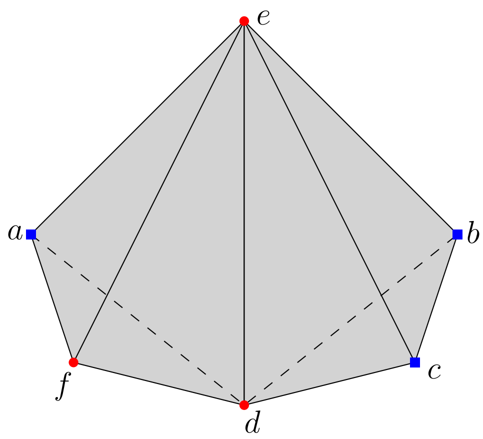

FIGURE 2. Two tetrahedra glued along an edge.

Before proving this, we need a brief lemma:

Lemma 4.4. [Ber] Given I an ideal of S, let $G = \mathrm{Hom}_S(r, S)$ where r is a virtual resolution of S/I. Then $(H.G) : B^\infty$ is independent of the choice of r.

Proof of Remark 4.3. We will use Conjecture 4.1 to prove the contrapositive of 4.2. Note that Remark 4.4 implies that an ideal I is not virtually Cohen-Macaulay if there exists a virtual resolution of I of length $l > \operatorname{codim} I$ such that the ideal I_{ϕ_l} generated by the maximal minors of the map ϕ_l satisfies $\dim V(I_{\phi_l}) \geq 0$, where $V(I_{\phi_l})$ denotes the variety of the ideal J.

Now suppose that for all $\vec{u} \in \{0,1\}^r$, $\operatorname{len} r_{\vec{u}} > \operatorname{codim} I_\Delta$. Then 4.1 implies the existence of a resolution $r_{\vec{u}}$ of I_Δ with $\dim V(J) \geq 0$. Then the aforementioned fact implies that I_Δ is not virtually Cohen-Macaulay, proving the contrapositive of Conjecture 4.2. □

These conjectures are false, as shown by the following counterexample.

Counterexample 4.5. Let $\Delta = \{bdef, acef, bcdf, acdf, abdf, bcde, acde, abce\}$ be a simplicial complex in $\mathbb{P}^2 \times \mathbb{P}^2$ such that the vertices labeled a, b, c correspond to one component of the product and those labeled d, e, f correspond to the other component. Then we have that the irrelevant ideal is given by $B = \langle a, b, c\rangle \cap \langle d, e, f\rangle$. Notably, $I_\Delta = I_\Delta \cap B^{\vec{u}}$ for all $\vec{u} \in \{0,1\}^2$. In order to be virtually Cohen-Macaulay, there must exist a virtual resolution r of I_Δ of length 2. However, we have that the free resolution r of S/I_Δ (which is the same for all resolutions $r_{\vec{u}}$ with $\vec{u} \in \{0,1\}^r$) has length 3, computed using *Macaulay2* [M2]. Also, this resolution r has minor ideal J of its last map φ_3 satisfying $V(J) = \varnothing$. Furthermore, Δ is virtually Cohen-Macaulay because a mapping cone construction done using *Macaulay2* [M2] yields a length $\operatorname{codim} I_\Delta$ virtual resolution.

Although Conjecture 4.2 is false, we can develop a weaker version which essentially states that if there exists some $\vec{a} \in \mathbb{Z}^r$ such that the resolution $r_{\vec{a}}$ provides a short-enough virtual resolution, then there exists $\vec{u} \in \{0,1\}^r$ such that the resolution $r_{\vec{u}}$ also gives a short enough virtual resolution. To prove this, we introduce some lemmas.

Lemma 4.6. Let I be an ideal of the Cox ring $S = \Bbbk[x_{i,j} : 1 \leq i \leq r, 1 \leq j \leq n_i + 1]$ of the product projective space $\mathbb{P}^{\vec{n}} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. If there exists an ideal $J \subseteq S$ with $V(J) = \varnothing$ such that $S/(I \cap J)$ is Cohen-Macaulay, then S/I is virtually Cohen-Macaulay.

Proof. Since $S/(I \cap J)$ has a free resolution of length $\operatorname{codim}(I \cap J)$, note that $\operatorname{codim}(I \cap J) \leq \operatorname{codim} I$ and by Lemma 2.14, the free resolution of $S/(I \cap J)$ is a virtual resolution of length $\leq \operatorname{codim} I$ of S/I. □

Now that we can relate Cohen-Macaulayness to virtual Cohen-Macaulayness, we note the following result on Cohen-Macaulayness.

Theorem 4.7 (2.6, [HTT05])**.** Let I be a monomial ideal of the polynomial ring $S = \Bbbk[x_1, \dots, x_n]$ over a field $\Bbbk$. If S/I is Cohen-Macaulay, then $S/\operatorname{rad}(I)$ is also Cohen-Macaulay.

Combining the above, we obtain the following proposition:

Theorem 4.8. Let Δ be a simplicial complex on the product projective space $\mathbb{P}^{\vec{n}}$. If there exists J a monomial ideal with $V(J) = \varnothing$ such that $S/(I_\Delta \cap J)$ is Cohen-Macaulay, then there exists Δ' containing only irrelevant facets such that $\operatorname{rad}(J) = I_{\Delta'}$ and $S/(I_\Delta \cap I_{\Delta'})$ is Cohen-Macaulay. In particular, this implies $\Delta \cup \Delta'$ is Cohen-Macaulay and Δ is virtually Cohen-Macaulay.

Proof. Note that for any monomial ideal J, $\operatorname{rad}(J)$ is generated by linear monomials. Therefore by the Stanley-Reisner correspondence, there exists Δ' a simlicial complex with $I_{\Delta'} = \operatorname{rad}(J)$ and $V(I_{\Delta'}) = \varnothing$. Since $I_\Delta \cap I_{\Delta'} = \operatorname{rad}(I_\Delta \cap J)$, the rest of the statement follows from Lemma 4.6 and Theorem 4.7. □

Corollary 4.8.1. Let Δ be a simplicial complex on the product projective space $\mathbb{P}^{\vec{n}}$ with B its irrelevant ideal. If there exists $\vec{a} \in \mathbb{Z}^r$ such that $S/(I_\Delta \cap B^{\vec{a}})$ is Cohen-Macaulay, then $S/(I_\Delta \cap B^{\operatorname{supp} \vec{a}})$ is Cohen-Macaulay where $\operatorname{supp} \vec{a} \in \{0,1\}^r$ is the support of $\vec{a}$.

By Theorem 3.1, we can also conclude the following properties of virtually Cohen-Macaulay simplicial complexes.

Corollary 4.8.2. For a simplicial complex Δ, if there exists $\vec{a} \in \mathbb{Z}^r$ such that $I_\Delta \cap B^{\vec{a}}$ is Cohen-Macaulay:

- Consider $\operatorname{supp} \vec{a} \in \{0,1\}^r$. We have that $(\operatorname{supp} \vec{a})_i$ can be 1 only if $\dim \mathbb{P}^{n_i} = \dim \Delta$.
- Δ is pure and gallery-connected up to adding irrelevant facets.

5. Balanced Complexes

In this section, we will be considering balanced complexes in the product projective space $\mathbb{P}^{\vec{n}} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. We prove that every balanced complex is in fact virtually Cohen-Macaulay. First, we give some useful definitions.

Definition 5.1. Let Δ be a pure simplicial complex on the product of projective spaces $\mathbb{P}^{\vec{n}} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. A facet $F \in \Delta$ is *balanced* if it contains exactly one vertex from every component. A simplicial complex is *balanced* if all of its facets are balanced.

Now by Lemma 2.16, it suffices to show that for every pure and balanced simplicial complex Δ, there exists a simplicial complex Δ' obtained from adding irrelevant facets to Δ such that Δ' is Cohen-Macaulay. Recall the definition of shellability of a simplicial complex.

Definition 5.2 ([MS05], Definition 13.44)**.** A *shelling* of Δ is an ordered list $F_1, F_2, \dots, F_m$ of its facets such that for all $i = 2, \dots, m$, $(\bigcup_{k=1}^{i-1} F_k) \cap F_i$ is pure of codimension 1. If a simplicial complex is pure and has a shelling, then it is *shellable*.

Note that there is a stronger condition on simplicial complexes that also implies Cohen-Macaulayness stated in the following Theorem 5.3.

Theorem 5.3 (13.45, [MS05])**.** Let Δ be a simplicial complex. If Δ is pure and shellable, then Δ is Cohen-Macaulay.

In order to prove the main results in this section, we provide some notations and useful facts.

Definition 5.4. Let Δ be a simplicial complex and let $v \notin \Delta$ be a vertex. Then $\Delta * v := \{F \cup \{v\} : F \in \Delta\}$ is called the *cone* of Δ on v.

$$\Delta_k = \{A \in \Delta \mid \dim A \le k \text{ and } A \subset F \text{ for some facet } F \text{ with } \dim F \ge k\}$$

is the *k-skeleton* of Δ. We use $\delta\Delta$ to denote the top skeleton $\Delta_{\dim \Delta - 1}$.

Theorem 5.5 (2.9, [BW97])**.** Let Δ be a simplicial complex. If Δ is shellable, then its k-skeleton Δ_k is also shellable for all $k \le \dim \Delta$.

The following lemma is crucial for the proof of virtual Cohen-Macaulayness of balanced complexes.

Lemma 5.6. Let Δ be a simplicial complex and $n = \dim \Delta$. If Δ has a shelling that begins with the facet F_1, then for some vertex $v \notin \Delta$, $(\Delta * v)_n$ has a shelling that begins with the facet F_1.

Proof. Let $F_1, \dots, F_k$ be a shelling of Δ that begins with F_1. Notice that

$$(\Delta * v)_n = \Delta \sqcup (\Delta_{n-1} * v).$$

If we let $\sigma_{i1}, \dots, \sigma_{in_i}$ be some ordering of the facets of δF_i, then the facets of $(\Delta * v)_n$ are either of the form F_i or of the form $\sigma_{ij} \cup \{v\}$, where σ_{ij} is a facet of δF_i. Obtain an ordering of these facets as follows: add the F_i's first in the original shelling, then starting from the smallest i, in increasing order, add all facets of form $\sigma_{ij} \cup \{v\}$ that have not been added yet. From now on, we deonte the facets $\sigma_{ij} \cup \{v\}$ as $\sigma_1 \cup \{v\}, \dots, \sigma_m \cup \{v\}$. We now show that $F_1, \dots, F_k, \sigma_1 \cup \{v\}, \dots, \sigma_m \cup \{v\}$ is a shelling of $(\Delta * v)_n$.

Since $F_1, \dots, F_k$ is a shelling, it suffices to show that for all $1 \le x \le m$,

$$(\sigma_x \cup \{v\}) \cap \left(\bigcup_{i=1}^{k} F_i \cup \bigcup_{j=1}^{x-1} (\sigma_j \cup \{v\}) \right) = \left((\sigma_x \cup \{v\}) \cap \bigcup_{i=1}^{k} F_i \right) \cup \left(\{v\} \cup (\sigma_x \cap \bigcup_{j=1}^{x-1} \sigma_j) \right) \tag{2}$$

is non-empty and pure of codimension 1. For the first part of the intersection, let $\overline{F}$ be the first facet of Δ that contains σ_x . Since $\overline{F}$ is in $\bigcup_{i=1}^{k} F_j$,

$$(\sigma_x \cup \{v\}) \cap \bigcup_{i=1}^{k} F_j = \sigma_x,$$

which is pure of codimension 1 as desired. For the second part of the intersection, let $r \le x$ be the smallest positive integer such that $\overline{F}$ is the first facet of Δ that contains σ_r. Then by construction, for all $r \le j \le x$, $\overline{F}$ is the first facet of Δ that contains σ_j. We examine $1 \le j < r$ and $r \le j \le x-1$ separately. Let $\mathrm{ind}(\overline{F})$ denote the index of $\overline{F}$, i.e. $F_{\mathrm{ind}(\overline{F})} = \overline{F}$. Since $\overline{F} \cap \bigcup_{i=1}^{\mathrm{ind}(\overline{F})-1} F_i$ is pure of codimension one, there exists a non-emtpy set of indices $S \subseteq \{1, \dots, r-1\}$ such that

$$\bigcup_{j \in S} \sigma_j = \overline{F} \cap \bigcup_{i=1}^{\mathrm{ind}(\overline{F})-1} F_i = \overline{F} \cap \bigcup_{j=1}^{r-1} \sigma_j \subseteq \delta \overline{F}.$$

Since $\sigma_x \subset \overline{F}$, we have

$$\sigma_x \cap \bigcup_{j=1}^{r-1} \sigma_j = (\sigma_x \cap \overline{F}) \cap \bigcup_{j=1}^{r-1} \sigma_j = \sigma_x \cap (\overline{F} \cap \bigcup_{j=1}^{r-1} \sigma_j) = \sigma_x \cap \bigcup_{j \in S} \sigma_j.$$

However, σ_j is a facet of $\overline{F}$ for all $j \in S$. The intersection of two facets in the same simplex has codimension 1 in the simplex. Hence for all $j \in S$,

$$\mathrm{codim}\,(\sigma_x \cap \sigma_j) = 2, \quad \mathrm{codim}\,(v * (\sigma_x \cap \sigma_j)) = 1$$

as desired. On the other hand, for $r \le j \le x-1$, σ_j is obviously a facet of $\overline{F}$ by construction, it then follows that

$$v * (\sigma_x \cap \bigcup_{j=r}^{x-1} \sigma_j)$$

is also pure of codimension 1. We have now shown that Equation 2 is non-empty and pure of codimension 1, which concludes the proof. □

Definition 5.7. Given a vertex set V on product projective space $\mathbb{P}^{\vec{n}}$ on r components. The *irrelevant complex* supported on V is the simplicial complex consisting of all possible n-dimensional irrelevant facets on V

$$\Delta_{irr}(V) := \{\sigma \in 2^V \mid \dim \sigma = n-1, |\mathrm{comp}(\sigma)| < r\}.$$

Proposition 5.8. Let $\Delta_{irr}(V)$ be the irrelevant complex supported on V in project projective space $\mathbb{P}^{\vec{n}}$ of r components. Then there exists a balanced facet R such that $\Delta = \Delta_{irr}(V) \cup \{R\}$ is shellable.

Remark 5.9. Since all balanced facets are "isomorphic" up to permuting vertices within components with respect to $\Delta_{irr}(V)$, Proposition 5.8 is equivalent to the following statement: Let $\Delta_{irr}(V)$ be the simplicial complex formed by all $(n-1)$-dimensional irrelevant facets of vertex set V in the product projective space of r components. Then for any balanced facet R, $\Delta_{irr} \cup \{R\}$ is shellable.

Proof. We proceed by strong induction on the number of vertices in Δ. The base case is when Δ contains only one vertex, in which there is nothing to prove. For the inductive step, we assume that if V is a vertex set with $|V| \leq r$ and $\Delta_{irr}(V)$ is the irrelevant complex supported on V, then there exists R a balanced facet such that $\Delta_{irr}(V) \cup \{R\}$ is shellable.

Now let V' be a vertex set with $|V'| = n+1$ and choose any $v \in V'$. Let $V = V' \setminus \{v\}$. Suppose there are r components in V'. Let $\Delta_{irr}(V)$, $\Delta_{irr}(V')$ be irrelevant complex supported on V and V' respectively.

There are two possible cases:

(1) v is the only vertex in its component. V now has $r-1$ components.
(2) There exists $u \in V$ in the component of v. V still has r components.

Now we prove the statement for both cases.

Case 1: By the inductive hypothesis, there exists R a balanced facet of $r-1$ components such that $\Delta_{irr}(V) \cup \{R\}$ is shellable. Let $R' = R \cup \{v\}$, which is balanced because v is the only vertex in its component. If $\sigma \in \Delta_{irr}(V')$ contains the vertex v, then since there is only one vertex in the component of v, $\sigma \setminus v$ must be irrelevant. Hence $\sigma \setminus v \in \Delta_{irr}$. Let $\overline{V}$ be the unique $(|V|-1)$-dimensional simplex (simplicial complex with one facet) supported on V. Then the facets of $(\overline{V})_n$ and $\Delta_{irr}(V) * v$ are exactly the facets of $\Delta_{irr}(V')$. Hence

$$\Delta_{irr}(V') \cup R' = ((\Delta_{irr} * v) \cup (\overline{V})_n) \cup (R \cup \{v\}) = (\overline{V})_n \cup ((\Delta_{irr} \cup R) * v).$$

By Theorem 5.5, since $\overline{V}$ is shellable, $(\overline{V})_n$ is also shellable. Let $F_1, \ldots, F_p$ be a shelling of $(\overline{V})_n$. On the other hand, by the induction hypothesis, $\Delta_{irr} \cup R$ is shellable. Let $G'_1, \ldots, G'_q$ be one of its shelling, then notice that the ordering $G_i = G'_i \cup \{v\}$ is a shelling on $(\Delta_{irr}(V) \cup R) * v$.

Now we show that the ordering $F_1, \ldots, F_p, G_1, \ldots, G_q$ is a shelling of Δ'. Since $F_1, \ldots, F_p$ is a shelling, it suffices to show that adding the facets G_i maintains the conditions of a shelling, specifically we need to show that

$$G_i \cap ((\overline{V})_n \cup \bigcup_{k=1}^{i-1} G_k) = (G_i \cap (\overline{V})_n) \cup (G_i \cap \bigcup_{k=1}^{i-1} G_k)$$

is pure of codimension 1. But $G_i \cap \Delta_1 = G_i \setminus \{v\} \in (\overline{V})_n$ has codimension 1, and $G_i \cap (\bigcup_{k=1}^{i-1} G_k)$ is pure of codimension 1 since $G_1, \ldots, G_q$ is a shelling on $(\Delta_{irr}(V) \cup R) * v$. Therefore $F_1, \ldots, F_p, G_1, \ldots, G_q$ is in fact a shelling order of Δ'.

Case 2: By the inductive hypothesis, there exists R a balanced facet (which is balanced on both V and V') such that $\Delta_{irr} \cup \{R\}$ is shellable. Let $\Delta_v := \{\sigma \in 2^V \mid \dim \sigma = n-2, \sigma \cup \{v\} \in \Delta'_{irr}\}$. Let $\operatorname{comp}(v) = \{u_1, \ldots, u_m, v\}$ be all vertices in the component of v and let $S = \{u \in V : \operatorname{comp}(u) \neq \operatorname{comp}(v)\}$ denote the set of vertices in V in different components from $\{v\}$. Then $V = S \cup \{u_1, \ldots, u_m\}$.

Claim: Let $R' = \{R \setminus \{u \in V : \operatorname{comp}(v) = \operatorname{comp}(v)\}\}$ be R restricted on the vertices S, then $\Delta_v \cup R'$ is shellable.

Proof. Let $\Delta_v|S = \{F \in \Delta_v : F \subseteq S\}$ be the subset of Δ_V. Let

$$\Delta_v^{(0)} := \Delta_v|_S \cup \{R'\}$$

and let

$$\Delta_v^{(k)} := \Delta_v|_{S \cup \{u_1, \ldots, u_k\}} \cup \{R'\}.$$

Note that $\dim \Delta_v^{(k)} = r-2$, $\Delta_v^{(m)} = \Delta_v \cup \{R'\}$ and

$$\Delta_v^{(k)} = \Delta_v^{(k-1)} \cup ((\Delta_v^{(k-1)})_{r-3} * u_k) = (\Delta_v^{(k-1)} * u_k)_{r-2}.$$

By Remark 5.9, since $\Delta_v^{(0)} = \Delta_v|_S \cup \{R'\}$ is supported on at most $n-2$ vertices, $\Delta_v^{(0)}$ is shellable. If we suppose $\Delta_v^{(k-1)}$ is shellable, then by Theorem 5.5, $\Delta_v^{(k)} = (\Delta_v^{(k-1)} * u_k)_{r-2}$ is also shellable. Then it follows that $\Delta_v^{(m)} = \Delta_v \cup \{R|_S\}$ is shellable. □

By the above claim, $(\Delta_v \cup \{R'\}) * v = (\Delta_v * v) \cup \{R' * v\}$ is shellable. Let $F_1, \dots, F_p$ be a shelling of $\Delta_{irr}(V) \cup \{R\}$ with $F_1 = R$ and $G_1, \dots, G_q$ be a shelling order of $(\Delta_v * v) \cup \{R' * v\}$ with $G_1 = R' * v$. We show the following:

Claim: The ordering $G_1, \dots, G_q, F_2, \dots, F_p$ is a shelling of $\Delta_{irr}(V') \cup \{R' * v\}$.

Proof. Since $G_1, \dots, G_q$ is a shelling order, it suffices to show that adding the facets $F_2, \dots, F_p$ maintains a valid shelling. Notice that the only way that adding a facet F_i can fail is when there exists F_j with $j < i$ such that $\operatorname{codim}(F_j \cap F_i) > 1$ and $F_j \cap F_i$ is maximal among all the intersections $F_k \cap F_i$ with $k < i$. Therefore we show that there exists G_k such that $F_i \cap F_j \subseteq G_k \cap F_i$ and $\operatorname{codim}(F_i \cap G_k) = 1$. Notice that the problem is caused by missing the facet $F_1 = R$, therefore we must have $F_i \cap F_j \subseteq F_i \cap R$ and $\operatorname{codim}(F_i \cap R) = 1$.

First, suppose F_i contains no vertex in the same component as v, then $G_1 = \{R' * v\}$ is our desired G_k. Let $w \in F_1$ denote the vertex in the same component as v, then notice that $G_1 = F_1 \setminus \{w\} \cup \{v\}$ by definition. Then $w \notin F_1 \cap F_i$ and it follows that $F_1 \cap F_i = G_1 \cap F_i$.

Now suppose F_i contains a vertex in the same component as v, let $u \in F_i$ be a vertex in the same component as v. If $u \notin F_i \cap F_j$, then consider the facet $D = F_i \setminus \{u\} \cup \{v\}$ and by definition $D \cap F_i \supseteq F_i \cap F_j$ and $\operatorname{codim} D \cap F_i = 1$. Note that since u is in the same component of v, D is an irrelevant facet containing v already added in the G_i's and D is the desired G_k. On the other hand, if $u \in F_i \cap F_j$, then consider the facet $D = F_i \setminus \{w\} \cup \{v\}$ where $w \in F_i \setminus (F_i \cap F_j)$. By definition $D \cap F_i \supseteq F_i \cap F_j$ and $\operatorname{codim} D \cap F_i = 1$ and notice that D contains at least two vertices in the same component as v, hence D is an irrelevant facet containing v and D is the desired G_k.

Now we can conclude that the claim is proved. □

From the above claim, it follows that $\Delta_{irr}(V') \cup \{R' * v\}$ is shellable with a shelling where the first facet in the shelling is the relevant facet. This concludes the proof. □

Now we are ready to show the main result of this section.

Theorem 5.10. If Δ is a pure and balanced simplicial complex on vertex set V in the product projective space $\mathbb{P}^{\vec{n}}$ of r components, then Δ is virtually Cohen-Macaulay.

Proof. Again, let $\Delta_{irr}(V)$ denote the simplicial complex formed by all possible irrelevant facets of dimension $r-1$ on the vertex set V. We show that the simplicial complex $\Delta_{irr}(V) \cup \Delta$ is always shellable. We show that adding more balanced facets to the simplicial complex $\Delta_{irr}(V) \cup \{R\}$ where R is a balanced facet remains a shelling.

Case 1: When V has only one vertex for every component, then $\Delta \cup \Delta_{irr}(V) = \Delta_{irr}(V) \cup \{F\}$ where F is the only possible balanced facet on V. Then by Proposition 5.8, we are done.

Case 2: When V has only one component C with more than one vertex and other component all singletons. Let $\{x_1, \dots, x_{n-1}\}$ denote the vertices of the singleton components and $C = \{u_1, \dots, u_k\}$ where $k > 1$. By Proposition 5.8, let $F = \{x_1, \dots, x_{n-1}, u_1\}$, then $\Delta' = \Delta_{irr}(V) \cup \{F\}$ is shellable. Now we show that adding more balanced facets $\{x_1, \dots, x_{n-1}, u_i\}$ after the shelling of Δ' still preserves the conditions for a shelling. If any facet $S = \{x_1, \dots, x_{n-1}, u_i\}$ is added to Δ', notice that $S \cap \Delta'$ is exactly the $(n-2)$-skeleton of S since all the subsets of $n-1$ vertices in S are ridges already in Δ'. The ridge $\{x_1, \dots, x_{n-1}\}$ is contained in the initial balanced facet $\{x_1, \dots, x_{n-1}, u_1\} \in \Delta'$ and the ridges of form $\{x_1, \dots, x_{n-1}, u_i\} \setminus \{x_j\}$ for some $1 \le j \le n-1$ are contained in the irrelevant facets $\{x_1, \dots, x_{n-1}, u_i, u_1\} \setminus \{x_j\}$. Hence $S \cap \Delta'$ is pure of codimension 1.

Case 3: When V has multiple components with more than one vertex. In this case, for any F a balanced facet, all of its ridges are contained in $\Delta_{irr}(V)$. Let $F = \{v_1, \dots, v_r\}$ where

$v_1, \ldots, v_r$ are in the components $1, \ldots, r$ correspondingly. By pigeon hole principle, any $(r-1)$-subset R of F contains some u where $\mathrm{comp}(u)$ contains more than one vertex. Then consider the facet $R \cup \{w\}$ where $w \neq u \in \mathrm{comp}(u)$ is an irrelevant facet already added in $\Delta_{irr}(V)$.

By the above arguments, we can conclude that $\Delta' = \Delta \cup \Delta_{irr}(V)$ is pure and shellable. By Theorem 5.3, Δ' is Cohen-Macaulay, and by Lemma 2.16, this implies Δ is virtual Cohen-Macaulay.

□

6. Notes on minor ideals of free resolutions

During the course of trying to prove Conjecture 4.1, the following facts about free resolutions and minor ideals were observed. In the following, let I_ϕ denote the ideal generated by the determinants of the minors of full rank in the map ϕ. Let $P_\phi = (p_{ij})$ denote the matrix of the linear map ϕ, where each entry of P_ϕ is a monomial (not necessarily squarefree) in $x_1, \ldots, x_n$. Note that since we are working in a Noetherian ring, every ideal is primary decomposable.

First we state a useful fact on monomial ideals.

Claim 6.1. Let $r := [F_0 \leftarrow F_1 \cdots \leftarrow F_k \leftarrow 0]$ be the resolution of the Stanley-Reisner ideal I_Δ. Then the matrices of each map $F_{i-1} \leftarrow F_i : \varphi_i$ have square-free monomial entries.

Proof. First, use the fine grading for monomials, whereby a monomial $\prod_n x_i^{a_i}$ corresponds to the multidegree $(a_1, \ldots, a_n)$. Note that the entries in matrices of resolutions of monomial ideals are monomials, since there is only one monomial in a given multidegree. Hochster's formula Theorem 2.5 guarantees that all Betti numbers of S/I_Δ are squarefree, i.e. of the form $\{0,1\}^n$, where n is the number of variables in the Cox ring. This implies that each map $F_{i-1} \leftarrow F_i : \varphi_i$ increases the degree of each variable in the Cox ring by at most 1. This implies that each entry in the matrix of φ_i has squarefree entries. □

Claim 6.2. Let $r := [F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_k \leftarrow 0]$ be a minimal free resolution of the monomial ideal I_Δ. For any linear maps $\varphi_i : F_{j-1} \leftarrow F_i$, let l_i be the rank of φ_i, then the determinants of minors of rank l_i in φ_i are monomials.

Proof. Consider a fixed i and denote ϕ_i by ϕ, l_i by l. Given a rank l minor $M = [p_{i,j}]$ in ϕ,

$$\det M = \sum_{\sigma \in S_l} sgn(\sigma) \prod_{i=1}^{l} p_{i,\sigma(i)}. \tag{3}$$

But there exists $\vec{d} \in \mathbb{N}^r$ such that for all $\sigma \in S_l$,

$$\deg\left(\prod_{i=1}^{l} p_{i,\sigma(i)}\right) = \vec{d}.$$

To see this, write

$$F_i = \bigoplus_{k=1}^{\dim F_i} S(-\vec{a}_k), \qquad F_{i-1} = \bigoplus_{k=1}^{\dim F_{i-1}} S(-\vec{b}_k).$$

Because all maps in r are degree-preserving, we have either that $\vec{a}_j = \deg p_{i,j} + \vec{b}_i$ or $p_{i,j} = 0$. Hence

$$\begin{aligned}\deg\left(\prod_{i=1}^{l} p_{i,\sigma(i)}\right) &= \sum_{i=1}^{l} \deg p_{i,\sigma(i)} \\ &= \sum_{i=1}^{l} (\vec{a}_{\sigma(i)} - \vec{b}_i) \\ &= \sum_{i=1}^{l} \vec{a}_{\sigma(i)} - \sum_{i=1}^{l} \vec{b}_i \\ &= \sum_{i'=1}^{l} \vec{a}_{i'} - \sum_{i=1}^{l} \vec{b}_i.\end{aligned}$$

It follows by 3 that $\det M$ is a monomial as desired. □

Lemma 6.3. Let I be a monomial ideal in $S = k[x_1, \dots, x_n]$, then

1. If I is generated by pure powers of a subset of variables, then I is a primary ideal.
2. If r is a minimal generator of I such that $r = r_1 r_2$ where r_1 and r_2 are relatively prime, then $I = (I + \langle r_1 \rangle) \cap (I + \langle r_2 \rangle)$.

Claim 6.4. Let $J = \langle x_{i_1}, \dots, x_{i_m} \rangle$ be the ideal generated by a subset of the variables $\{x_1, \dots, x_n\}$ such that the every monomial in some column of P_ϕ contains exactly one of the generators of J, i.e. J is generated by choosing one variable from each monomial of some column of P_ϕ. Then the ideal J is a minimal prime associated to I_ϕ.

Proof. By Lemma 6.3, we can find a primary component Q of a monomial ideal $I = \langle p_1, \dots, p_m \rangle$ as follows: choose a generator $p_i = p_{i_1} \cdots p_{i_k}$ of I where p_{i_j}'s are relatively prime, add any p_{i_j} to the ideal Q as a generator. Delete all the generators $p_r \in I$ with $p_{i_j} | p_r$ and repeat until I is empty. In other words, we want to find a set of pure powers of variables such that every generator in I is divisible by at least one element in the set. The intersection of all such Q is a primary decomposition of the monomial ideal I (not necessarily irreducible).

Now note that by definition of I_ϕ and Lemma 6.3, the generators of I_ϕ are the monomials of the form $\prod_{i=1}^{\ell} m_{i\sigma(i)}$ for some $\sigma \in S_\ell$ and minor $M = (m_{ij})$ of P_ϕ where $\ell = \operatorname{rank} I_\phi$. Then all the monomials generating I_ϕ must be divisible by some element in every column of P_ϕ. Hence by construction of J in the claim, J is exactly generated by a subset of variables such that every generator in I_ϕ is divisible by at least one element in the J. Again by Lemma 6.3, it is clear that J is the radical of some primary component J' of I_ϕ formed by a subset of the variables with the lowest power among the generators of I_ϕ. Notice that such J' has a minimal number of generators by construction. Therefore J is a minimal prime of I_ϕ. □

Remark 6.5. The construction of J' is minimal since we always choose the variables with its lowest power in the ideal as generators at each step and for the next step we can restrict to all the generators not containing the variable. We cannot directly have J as a primary component since there might be cases where the lowest power of some x_i in the monomial ideal is > 1. Then the primary components of the monomial ideal should not contain x_i as a generator. However, by taking the radical, the monomial ideal generated by pure powers of the variables become the ideal generated by the variables, so the above argument holds.

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