QUIVER MUTATIONS

SHIYU LI

1. Introduction

In [2][3], the mathematicians Fomin and Zelevinsky described the mathematical object known as a quiver, and connected it with the theory of cluster algebras. In particular, each quiver can be represented by a seed of a cluster algebra, which couples a set of n variables with the adjacency matrix of the quiver. By performing a transformation on the quiver, we change accordingly the values of these variables in the cluster algebra.
In this paper, we will show the patterns generated by several such transformations, most notably those involving the particular quiver shown here. We will be demonstrating relations between our quiver mutations and the Fibonacci numbers.
We begin with a few simple definitions.

Definition 1. *A quiver is a directed graph where multiple edges are allowed between each pair of nodes. It may also contain loops; however, the quivers we will be examining will not. We will call the vertices of this graph nodes.*

Definition 2. *We define mutation of a quiver Q with nodes $\{1, ..., n\}$ in a similar way as seen in this paper:* [1]. *Thus, when we mutate a quiver at a node k, we perform the following steps:*
1. For every pair of nodes $i, j \neq k$, every edge from i to k, and every edge from k to a j, we create an edge from $i \to j$.
2. We reverse the direction of any edges touching k.
3. In the graph resulting from the above two steps we remove both edges of any 2-cycles we see.

Then, between each two nodes i and j, we may have a variable a_{ij} which is the number of directed arrows from i to j. If this variable is negative, it will mean there are only arrows from j to i. We notice $a_{ij} = -a_{ji}$.
One critical aspect of mutation that we will take note of is that mutating a quiver twice around the same node will always return us to the original quiver; that is, mutation is an involution transformation.

Definition 3. *A cluster algebra is a commutative ring consisting of elements called seeds. These seeds contain a element of $\mathbb{R}^n$ coupled with a $n \times n$ matrix; this is the matrix that represents the quiver.*

We attach a cluster variable x_k to each node k of the quiver, which gives us a cluster algebra seed. Then, when we mutate the quiver at a vertex k, we change

Date: July 28, 2011.

the variable x_k associated with k to

$$x'_k = \frac{\prod_{i\to k} x_i + \prod_{k\to j} x_j}{x_k}.$$

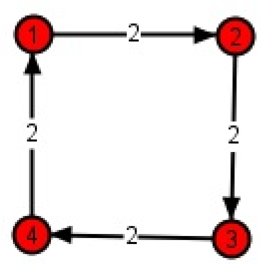
If we originally start with the cluster seed $C = \{[x_{01}, ..., x_{0n}], Q\}$, then, it can be shown [citation] that each variable of the cluster seed $C = \{[x_1, ..., x_n], Q\}$ obtained after any finite sequence of mutations is a Laurent polynomial. As we will see, this definition of cluster variable leads to many interesting patterns in the mutations.

Definition 4. *A Laurent polynomial in the variables $x_1, ..., x_n$ is an expression where each monomial is the product of integral (positive and negative) powers of $x_1, ..., x_n$.*

Fomin and Zelevinsky have shown that the polynomials obtained while mutating a quiver continue to be Laurent polynomials. [7] These two mathematicians have titled this the "'Laurent phenomenon"', as normally the composition of two such polynomials may well cease to be a Laurent polynomial. The values of such polynomials, with appropriate numbers substituted in for the variables, may take on various mathematically significant numbers.

2. Quiver Mutations

We consider the quiver $Q =$



and the cluster algebra seed $S = ([1, 1, 1, 1], Q)$. Then we mutate this in the directions $1, 3, 2, 3, 2, 3, ...$ Let Q_n denote the quiver obtained after n mutations in the sequence [1,3,2,3,2,3, ...], with $Q_0 = Q$ drawn above. Let a_n denote the number resulting from n mutations along this sequence, we then have the sequence of a_ns as

$$1, 2, 2, 8, 40, 208, 1088, 5696, 29824, 156160, ...$$

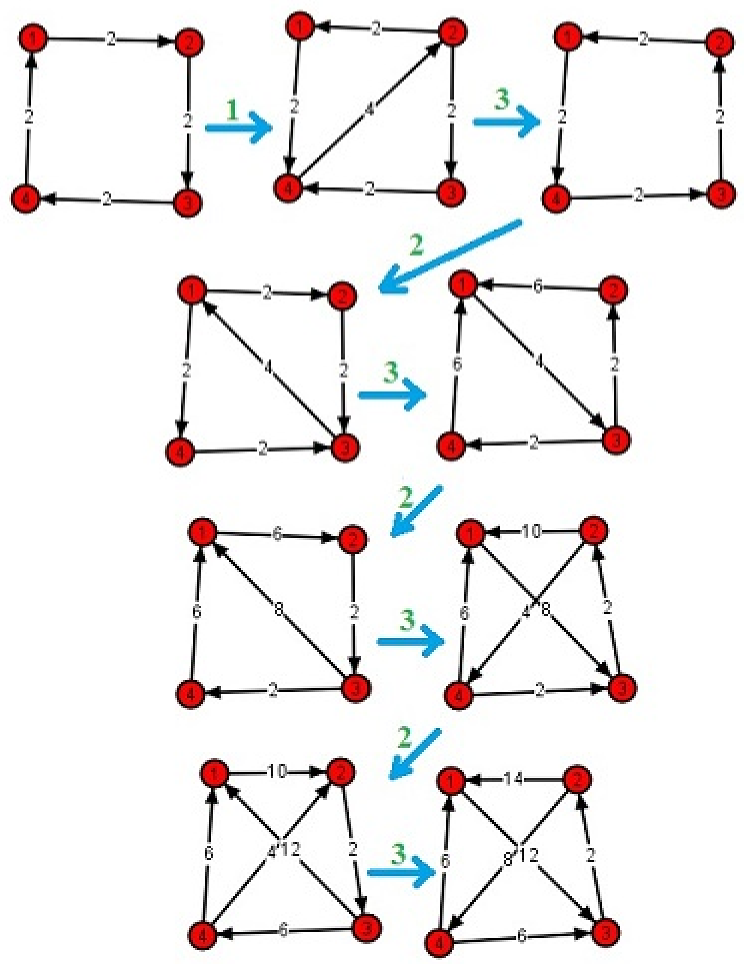
We thus conjecture that

Theorem 1. *When $n \geq 2$, $a_n = 2^{n-1} \cdot F_{2n-4}$, when we take F_0 as the start of the Fibonacci sequence.*

We shall first determine the form of the quiver associated with a_n. After finding the number of arrows between each two vertices in Q_n for all n, we will then mutate and apply induction to the formula to complete the proof of the theorem. Since the mutation pattern has a repeating block of 2 mutations, we will split each proof into odd and even cases. Define $(c_n)_{i,j}$ to be the number of arrows from $i \to j$ in the quiver Q_n. We claim that when $m \geq 3$, then

Lemma 1. $(c_{2m-1})_{0,1} = 6$, $(c_{2m-1})_{2,3} = 2$
$(c_{2m-1})_{3,0} = 4m - 10$, $(c_{2m-1})_{1,2} = 4m - 6$
$(c_{2m-1})_{3,1} = 4m - 4$, $(c_{2m-1})_{0,2} = 4m - 12$,
while
$(c_{2m})_{0,1} = 6$, $(c_{2m})_{3,2} = 2$
$(c_{2m})_{0,3} = 4m - 10$, $(c_{2m})_{2,1} = 4m - 2$
$(c_{2m})_{1,3} = 4m - 4$, $(c_{2m})_{2,0} = 4m - 8$.

The following diagram shows the first few mutations of the quiver according to the sequence.



Proof. The proof of this uses induction. We first see from the figure above that it holds in the case of a_5 and a_6.
Suppose it is true for a_{2m-1}. Then Q_{2m} follows from Q_{2m-1} by mutating at 3. Since Q_{2m-1} is given above, we get that Q_{2m} reverses directions of all arrows at 3. By this, we obtain that $(c_{2m})_{0,3} = 4m - 10$, $(c_{2m})_{3,2} = 2$, and $(c_{2m})_{1,3} = 4m - 4$, since the number of arrows from 3 to any other point will not change.

We also find that this mutation adds $2(4m-10)$ arrows from 2 to 0 and $2(4m-4)$ arrows from 2 to 1, while no arrows between 0 and 1 are changed. Thus, $(c_{2m})_{0,1} = 6$, $(c_{2m})_{2,1} = 4m-2$, and $(c_{2m})_{2,0} = 4m-8$.
Now, we will induct from a_{2m} to a_{2m+1}.
Then Q_{2m+1} follows from Q_{2m} by mutating at 2. Since Q_{2m} is given above, we get that Q_{2m+1} reverses directions of all arrows at 2. By this, we obtain that $(c_{2m+1})_{2,3} = 2$, $(c_{2m+1})_{0,2} = 4m-8$, and $(c_{2m+1})_{1,2} = 4m-2$, since the number of arrows between 3 and any other point will not change during this mutation process. Meanwhile, this mutation adds $2(4m-8)$ arrows from 3 to 0 and $2(4m-2)$ arrows from 3 to 1, leaving the number of arrows between 0 and 1 unchanged. Thus, $(c_{2m+1})_{0,1} = 6$, $(c_{2m+1})_{3,0} = 8m-16-(4m-10) = 4m-6$, and $(c_{2m+1})_{3,1} = 8m-4-(4m-4) = 4m$.
Therefore, the inductive hypothesis holds, and we have proved the lemma. □

Now, we want to show that the sequence generated by such a mutation is in accordance with what we conjectured.

Proof. We know that

$$x'_k = \frac{\prod_{i\to k} x_i + \prod_{k\to j} x_j}{x_k}.$$

To show that a_n is the number we desire, we have two cases we must confirm: that when n is odd and that when n is even. Our base case will be a_5 and a_6.

Case 1. *When n is even, then $n = 2m$ and a_n is obtained by mutating Q_{2m-1} at 3. Thus*

$$\begin{aligned} a_n &= \frac{x_2^2 + x_0^{4m-10}x_1^{4m-4}}{x_3} \\ &= \frac{a_{n-1}^2 + 2^{4m-4}}{a_{n-2}} \\ &= \frac{(2^{n-2}F_{2n-6})^2 + 2^{2n-4}}{2^{n-3}F_{2n-8}} \\ &= 2^{n-1}\frac{F_{2n-6}^2+1}{F_{2n-8}} \\ &= 2^{n-1}F_{2n-4} \end{aligned}$$

as desired.

Case 2. *When n is odd, then $n = 2m+1$ and a_n is obtained by mutating the quiver associated with a_{2m} at 2. Thus*

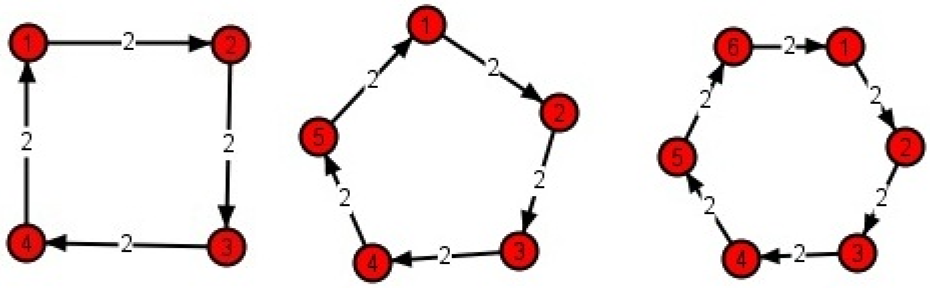
$$\begin{aligned} a_n &= \frac{x_3^2 + x_1^{4m-2}x_0^{4m-8}}{x_2} \\ &= \frac{a_{n-1}^2 + 2^{4m-2}}{a_{n-2}} \\ &= \frac{(2^{n-2}F_{2n-6})^2 + 2^{2n-4}}{2^{n-3}F_{2n-8}} \\ &= 2^{n-1}\frac{F_{2n-6}^2+1}{F_{2n-8}} \\ &= 2^{n-1}F_{2n-4} \end{aligned}$$

as desired.

□

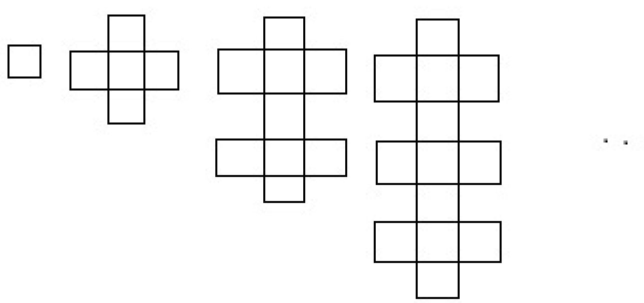
In fact, we may state that

Theorem 2. *Mutating in the directions* $1, 3, 2, 3, 2, 3, ...$ *generates the same cluster variable values as above for any quiver in this family:*



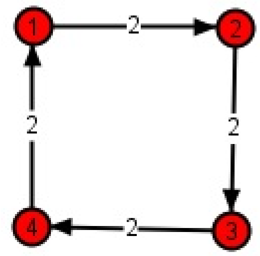
Proof. To show this, we consider what would happen if we combined the two vertices $n-1, n$ on the graph with n vertices into one, call it $*$. Then when we mutate the new quiver, we would get as many edges $i \to *$ as we would the sum of $i \to n-1$ and $i \to n$, by the distributive property. Similarly, we get the same edges $* \to j$. Since the original had $x_{n-1} = x_n = 1$ for the cluster variables, it follows that we get the same sequence. □

Since the sequence of a_n's come from mutating Q by the specific sequence $1, 3, 2, 3, 2, 3, ...$, by comparing the above results to InJee Jeong's verification of Musiker's conjecture [6], we get that the a_n's give the number of perfect matchings in the following sequence of figures.



3. Explorations of Other Mutation Sequences

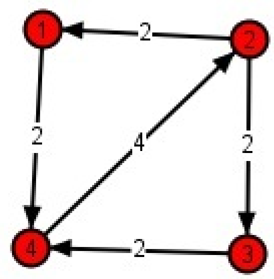
We remain using the following quiver:



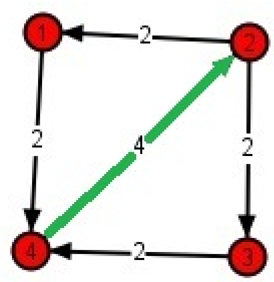
Let ω be a sequence of the numbers 2 and 4. Let $\bar{\omega}$ denote the reverse sequence of ω. We may assume ω has no two consecutive 2s or 4s, because mutation is an involution. Then for any such sequence ω, mutating the quiver by the sequence

$$1, \omega, 1, 3, 1, 3, \bar{\omega}, 3$$

will lead to the reverse quiver. This is because after we mutate by 1, we get the resulting quiver



and mutating by ω makes the resulting graph symmetric with respect to the diagonal shown in this figure.



This results in the mutations 1 and 3 commuting with respect to each other, and thus mutating 1,3,1,3 leaves the graph unchanged. Then $\bar{\omega}$ will return the quiver, leaving this sequence with a result the same as if we just mutated 1,3.

Acknowledgement 1. *This research was done in the University of Minnesota REU program, which was supported by NSF grant number DMS-1001933. I am grateful to Vic Reiner, Gregg Musiker, and Pasha Pylyavskyy for their guidance and support. I would like to show my gratitude to Gregg Musiker; he not only proposed the original idea and the numerous quivers used but also read through the* *drafts and gave me countless insightful suggestions throughout the REU program. In addition, I frequently used his Sage package with Christian Stump* [10] *to test conjectures.*

References

[1] Bousquet-Melou, M; Propp, J; West, J; *Perfect Matchings for three-term Gale-Robinson sequences*, 2009

[2] Sergey Fomin and Andrei Zelevinsky. *Cluster algebras. I. Foundations.* J. Amer. Math. Soc., 15(2):497529 (electronic), 2002.

[3] Sergey Fomin and Andrei Zelevinsky. *Cluster algebras. II. Finite type classification.* Invent. Math., 154(1):63121, 2003.

[4] Eric H. Kuo. *Applications of Graphical Condensation for Enumerating Matchings and Tilings.* arXiv:math/0304090v2

[5] David E Speyer, *Perfect Matchings and the Octahedron Recurrence.* arXiv:math/0402452v2

[6] InJee Jeong, *Bipartite Graphs, Quivers, and Cluster Variables.* http://www.math.umn.edu/ reiner/REU/Jeong2011.pdf

[7] S. Fomin and A. Zelevinsky, The Laurent phenomenon, arXiv:math.CO/0104241.

[8] Gregg Musiker and Ralf Schiffler. *Cluster expansion formulas and perfect matchings.* J. Algebraic Combin., 32(2):187209, 2010.

[9] Allan P. Fordy, Robert J. Marsh. *Cluster mutation-periodic quivers and associated Laurent sequences* http://arxiv.org/abs/0904.0200

[10] Christian Stump, Gregg Musiker. *A compendium on the cluster algebra and quiver package in sage.* arXiv:1102.4844v2, 2011.