

# Hecke Algebras, Representations, and Character Tables of Monoids

Jared Marx-Kuo, Vaughan McDonald, John M. O'Brien, Alexander Vetter

July 25 2018

## 1 Introduction

### 1.1 Motivation

As Solomon [8] aptly states: the theory of reductive monoids has an advertising problem. While this paper cannot rectify this situation, the below paragraph is an attempt to amend it

Behind every great reductive group lies a great reductive monoid. Indeed,  $GL_n(K)$  lies inside  $M_n(K)$ , the monoid of  $n \times n$  matrices over a field  $K$ . Within this monoid, we can consider  $S_n \hookrightarrow M_n(K)$  where the corresponding reductive monoid to  $S_n$  is in fact

$$R_n(K) := \{\text{Matrices which represent at most } n \text{ non-attacking rooks on an } n \times n \text{ chessboard}\}$$

where a rook is represented by a 1 and an empty space is represented by a 0. As an example,

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in R_3(\mathbb{R}), \quad N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \notin R_3(\mathbb{R})$$

Formally, let  $R$  be the set of all one-to-one maps  $\sigma$  with domain and range  $I(\sigma), J(\sigma) \subseteq \{1, \dots, n\}$ . Then we can define  $\sigma\tau : I(\sigma) \cap \tau^{-1}(J(\sigma)) \rightarrow J(\tau)$  by  $i(\sigma\tau) = (i\sigma)\tau \in J(\tau)$ . This forms a monoid isomorphic to a submonoid of  $M_n(F)$  such that

$$R \ni \sigma \mapsto \sum_{i \in I(\sigma)} E_{i, i\sigma}$$

It is easy to see that the the size of the rook monoid is:

$$\sum_{i=0}^n \binom{n}{i}^2 i! \tag{1}$$

More generally,

**Definition 1.1.** A *reductive monoid* is an algebraic monoid which is irreducible as an algebraic set and has a connected reductive group of units.

For further definitions, see Solomon [8].

Turning to applications, monoids are abundant in computer programming, and even implemented in languages like Haskell. Gondran and Minoux [5] provide an extensive listing of applications of monoids in data analysis and searching. Monoids (respectively reductive monoids) are extensions of groups (respectively reductive groups) and can shed new light on old groups that we know well. Solomon does so with his analysis of irreducible representations of  $S_n$  [9] by looking at those of the rook monoid.

## 1.2 Results

In this paper, we extend some well known results for groups to analogous results for monoids and then investigate the specific nature of the symplectic rook monoid, along with its character table and rules for decomposing its irreducible representations. In section 2, we prove the Borel Matsumoto theorem and Froebnius Reciprocity for subgroups of the group of units in a monoid using Godelle's [4] definition of convolution for monoids. In section 3 we introduce the symplectic rook monoid and provide an embedding of it into the rook monoid. In section 4, we provide the character table for the symplectic rook monoid,  $RSp_{2n}$ , when  $n = 3$ , as calculated according to results from Li, Li, Cao [7]. In section 5, we recall results from Solomon [9] about the structure of the character table of the rook monoid. Then using techniques from Geck and Pfeiffer's text on the characters of finite Coxeter Groups and Iwahori-Hecke algebras [3], we provide an analogous decomposition of the character table of the symplectic rook monoid.

## 1.3 Essential Definitions

- Let  $M$  be a monoid with group of units  $G(M)$ . We call  $M$  algebraic if it is a Zariski-closed subset of  $\text{Mat}_n(\mathbb{F})$  for some  $n \in \mathbb{Z}$  and  $\mathbb{F}$  a field. We call  $M$  reductive if  $M$  is an irreducible algebraic monoid with  $G(M)$  a reductive group under the usual definition. If  $M$  is reductive,  $M$  has the Renner decomposition

$$M = \bigsqcup_{r \in R} BrB \quad (2)$$

where  $B$  is a Borel subgroup of  $G(M)$ ,  $R$  is the Renner monoid of  $M$ , and  $\underline{r}$  is a choice of lift of  $r \in R$ .

- Symplectic rook monoid and symplectic renner monoid will be taken to be synonymous due to them being isomorphic as shown in [7]. We define the symplectic rook monoid  $RSp_{2n}$  in terms of the admissible sets definition.
- Let  $\mathcal{N} = \{1, 2, \dots, 2n-1, 2n\}$ . Let, as in [7],  $\theta : \mathcal{N} \rightarrow \mathcal{N}$  be the involution given by  $\theta(t) = 2n+1-t$ . A subset  $P$  of  $\mathcal{N}$  is admissible if, for all  $t \in \mathcal{N}$ , then  $\theta(t) \notin P$ . Define  $RSp_{2n}$  to be the monoid of all injective partial transformations on  $\mathcal{N}$  sending admissible sets to admissible sets. The  $B_n$  Weyl group embeds as  $G(RSp_{2n})$ , the group of full injective transformations on  $\mathcal{N}$ .
- Let  $\mathcal{Q}_n = SP_n \sqcup \bigcup_{r=0}^n S_r$  be the union of signed partitions of size  $n$  and the union of partitions of  $r$  for all  $0 \leq r \leq n$ . This is an indexing set for the collection of irreducible representations of the symplectic rook monoid, as well as its conjugacy classes. Furthermore, it has a standard ordering as described on p. 847 of [7].

## 2 The Borel-Matsumoto Theorem for Finite Monoids

Let  $M$  be a finite monoid,  $G(M)$  the group of units of  $M$ , and  $K$  a subgroup of  $G(M)$ , and  $\mathbb{F}$  a field of characteristic not dividing  $|K|$ .

For  $\phi, \psi : M \rightarrow \mathbb{F}$ , define  $\phi * \psi$  as in Godelle [4] by

$$\phi * \psi(m) = \sum_{yz=m} \phi(y)\psi(z) \quad (3)$$

Similarly, for  $(\pi, V)$  a representation of  $M$  and  $\phi$  as above define  $\pi(\phi)$  by

$$\pi(\phi)v = \sum_{x \in M} \phi(x)\pi(x)v \quad (4)$$

Note that  $\pi(\phi) \circ \pi(\psi) = \pi(\phi * \psi)$  as in Bump.

**Proposition 1.** For  $\phi, \psi \in \mathcal{H}$ ,  $\pi(\phi * \psi) = \pi(\phi) \circ \pi(\psi)$ .

*Proof.* Consider  $\pi(\phi) \circ \pi(\psi)$ . We have the following:

$$(\pi(\phi) \circ \pi(\psi))v = \sum_{x \in M} \phi(x)\pi(x) \sum_{y \in M} \psi(y)\pi(y)v \quad (5)$$

$$= \sum_{x, y \in M} \phi(x)\psi(y)\pi(x)\pi(y)v \quad (6)$$

$$= \sum_{x, y \in M} \phi(x)\psi(y)\pi(xy)v \quad (7)$$

$$= \sum_{z \in M} \sum_{xy=z} \phi(x)\psi(y)\pi(z)v \quad (8)$$

$$= \sum_{z \in M} (\phi * \psi)(z)\pi(z)v \quad (9)$$

$$= \pi(\phi * \psi)v \quad (10)$$

Thus  $\pi(\phi) \circ \pi(\psi) = \pi(\phi * \psi)$ .  $\square$

Let  $\mathcal{H}$  be the  $F$ -algebra of functions from  $M$  to  $F$  under addition and convolution. Define, for  $v \in V$ ,

$$\mathcal{H}v = \{\pi(\phi)v \mid \phi \in \mathcal{H}\}. \quad (11)$$

Define an action of  $M$  on  $\mathcal{H}v$  by  $m \cdot (\pi(\phi)v) = \pi(m)\pi(\phi)v$ . Notice that, since for  $f_m : M \rightarrow F$  defined by  $f_m(m) = 1, f_m(x) = 0$  for  $x \neq m$ ,  $\pi(f_m)v = \pi(m)v$ , that  $\mathcal{H}v$  is closed under action by  $M$ . Thus, it is a subrepresentation.

Similarly, let  $\mathcal{H}_K$  be the  $F$ -algebra of functions from  $M$  to  $F$  under convolution that are constant on double-cosets of  $K$ ; i.e.  $\phi$  such that  $\phi(m) = \phi(k_1mk_2)$  for all  $k_1, k_2 \in K$ . Also let

$$V^K = \{v \in V \mid \pi(k)v = v \quad \forall k \in K\}$$

as in Bump.

**Theorem 2.1.** Let  $(\pi, V)$  be an irreducible representation of  $M$  with  $V^K \neq \{0\}$ . Then  $V^K$  is irreducible as an  $\mathcal{H}_K$ -module.

*Proof.* We follow Bump's proof of the group case closely [1]. We claim that, for all nonzero  $u \in V^K$ , that  $\mathcal{H}_K u := \{\pi(\phi)u \mid \phi \in \mathcal{H}_K\}$  equals  $V^K$ . In other words, we wish to show that, for all  $v \in V^K$  there exists  $\phi \in \mathcal{H}_K$  such that  $\pi(\phi)u = v$ .

Since  $(\pi, V)$  is an irreducible representation of  $M$ , there are no proper non-trivial subrepresentations in  $V$ . Because there is an  $M$ -action on  $\mathcal{H}u \neq \{0\}$ , then  $\mathcal{H}u = V$ . Thus there exists  $\psi \in \mathcal{H}$  such that  $\pi(\psi)u = v$ .

Define  $\phi \in \mathcal{H}$  by, for  $x \in M$

$$\phi(x) = \frac{1}{|K|^2} \sum_{k_1, k_2 \in K} \psi(k_1 x k_2) \quad (12)$$

Since  $\phi$  must be invariant over left and right cosets of  $K$ ,  $\phi$  lies in  $\mathcal{H}_K$ . Now consider the following:

$$\pi(\phi)u = \frac{1}{|K|^2} \sum_{k_1, k_2 \in K} \sum_{x \in M} \psi(k_1 x k_2) \pi(x)u \quad (13)$$

Notice that  $x \mapsto k_1^{-1} x k_2^{-1}$  is a bijection from  $M$  to  $M$ , as it has an inverse  $x \mapsto k_1 x k_2$ . Thus we can make the following change of variables:

$$\pi(\phi)\mathbf{u} = \frac{1}{|\mathbb{K}|^2} \sum_{k_1, k_2 \in \mathbb{K}} \sum_{x \in \mathbb{M}} \psi(x) \pi(k_1^{-1} x k_2^{-1}) \mathbf{u} \quad (14)$$

$$= \frac{1}{|\mathbb{K}|^2} \sum_{k_1, k_2 \in \mathbb{K}} \sum_{x \in \mathbb{M}} \psi(x) \pi(k_1)^{-1} \pi(x) \pi(k_2)^{-1} \mathbf{u}. \quad (15)$$

Since  $\mathbf{u} \in V^{\mathbb{K}}$ , we have that  $\pi(k_2)^{-1} \mathbf{u} = \mathbf{u}$ . Thus,

$$\pi(\phi)\mathbf{u} = \frac{1}{|\mathbb{K}|} \sum_{k_1 \in \mathbb{K}} \sum_{x \in \mathbb{M}} \psi(x) \pi(k_1)^{-1} \pi(x) \mathbf{u} \quad (16)$$

$$= \frac{1}{|\mathbb{K}|} \sum_{k_1 \in \mathbb{K}} \pi(k_1)^{-1} \sum_{x \in \mathbb{M}} \psi(x) \pi(x) \mathbf{u} \quad (17)$$

$$= \frac{1}{|\mathbb{K}|} \sum_{k_1 \in \mathbb{K}} \pi(k_1)^{-1} \pi(\psi) \mathbf{u}. \quad (18)$$

Since  $\pi(\psi)\mathbf{u} = \mathbf{v}$  and  $\mathbf{v} \in V^{\mathbb{K}}$ ,

$$\pi(\phi)\mathbf{u} = \frac{1}{|\mathbb{K}|} \sum_{k_1 \in \mathbb{K}} \pi(k_1)^{-1} \mathbf{v} = \mathbf{v}. \quad (19)$$

Thus, for all  $\mathbf{v} \in V^{\mathbb{K}}$  there exists  $\phi \in \mathcal{H}_{\mathbb{K}}$  such that  $\pi(\phi)\mathbf{u} = \mathbf{v}$ . Thus,  $V^{\mathbb{K}}$  is irreducible as an  $\mathcal{H}_{\mathbb{K}}$ -module.  $\square$

Denote, for  $(\pi, V)$  a representation of  $M$ , let  $(\pi|_{\mathbb{G}}, V)$  be the restricted representation of  $G(M)$  defined by  $\pi|_{\mathbb{G}}(g) = \pi(g)$  for  $g \in G(M)$ . Define the contragredient representation of  $G(M)$   $(\hat{\pi}|_{\mathbb{G}}, \hat{V})$  by  $\langle \pi|_{\mathbb{G}}(g)v, \hat{v} \rangle = \langle v, \hat{\pi}|_{\mathbb{G}}(g^{-1})\hat{v} \rangle$  for all  $g \in G(M)$ .

We include three statements of Bump that are applicable unchanged here by simple restriction to  $\pi|_{\mathbb{G}}$ .

**Lemma 1** (Bump's Lemma 3). *Let  $l: V^{\mathbb{K}} \rightarrow F$  be a linear functional. Then there exists  $\hat{v} \in \hat{V}^{\mathbb{K}}$  such that for all  $v \in V^{\mathbb{K}}$ ,  $l(v) = \langle v, \hat{v} \rangle$ . [1]*

*Proof.* Let  $\hat{v}_0$  be a linear functional on  $V$  that restricts to  $l$  on  $V^{\mathbb{K}}$ .

Define  $\hat{v} = \frac{1}{|\mathbb{K}|} \sum_{k \in \mathbb{K}} \hat{\pi}|_{\mathbb{G}}(k) \hat{v}_0$ . For  $v \in V^{\mathbb{K}}$ , then, we have the following equality:

$$\langle v, \hat{v} \rangle = \frac{1}{|\mathbb{K}|} \sum_{k \in \mathbb{K}} \langle v, \hat{\pi}|_{\mathbb{G}}(k) \hat{v}_0 \rangle \quad (20)$$

$$= \frac{1}{|\mathbb{K}|} \sum_{k \in \mathbb{K}} \langle \pi|_{\mathbb{G}}(k)^{-1} v, \hat{v}_0 \rangle \quad (21)$$

$$= \frac{1}{|\mathbb{K}|} \sum_{k \in \mathbb{K}} \langle \pi(k)^{-1} v, \hat{v}_0 \rangle \quad (22)$$

$$= \frac{1}{|\mathbb{K}|} \sum_{k \in \mathbb{K}} \langle v, \hat{v}_0 \rangle \quad (23)$$

$$= l(v) \quad (24)$$

$\square$

**Lemma 2** (Bump's Proposition 1). *If  $V^K \neq 0$  then  $\hat{V}^K \neq 0$ . [1]*

**Lemma 3** (Bump's Proposition 2). *Let  $R$  be an algebra over  $F$ , and  $N_1, N_2$  simple  $R$ -modules that are finite-dimensional as vector spaces over  $F$ . If there exist linear functionals  $L_i : N_i \rightarrow F$  and  $\mathbf{n}_i \in N_i$  such that  $L_i(\mathbf{n}_i) \neq 0$  and  $L_1(r\mathbf{n}_1) = L_2(r\mathbf{n}_2)$  for all  $r \in R$ , then  $N_1 \cong N_2$  as  $R$ -modules. [1]*

We particularly care about the case when two representations  $(\pi_i, V_i)$  share matrix coefficients  $\langle \pi_i(\mathbf{m})v, \hat{v}_0 \rangle$  for all  $\mathbf{m} \in M$ .

**Lemma 4.** *Let  $(\pi, V)$  and  $(\sigma, W)$  be two irreducible representations of  $M$  with nonzero matrix coefficients  $\langle \pi(\mathbf{m})v, \hat{v}_0 \rangle = \langle \sigma(\mathbf{m})w, \hat{w}_0 \rangle$  for some  $v, v_0, w, w_0$ , and all  $\mathbf{m} \in M$ . Then  $(\pi, V) \cong (\sigma, W)$ .*

*Proof.* Define actions of  $F[M]$  on  $V$  and  $W$  by letting  $\mathbf{m}v = \pi(\mathbf{m})v$  and  $\mathbf{m}w = \sigma(\mathbf{m})w$  for all  $v \in V, w \in W$ , and  $\mathbf{m} \in M$  respectively and then extending by linearity. Thus  $V$  and  $W$  become  $F[M]$ -modules. Because the representations are each irreducible,  $V$  and  $W$  are simple as  $F[M]$ -modules. Since  $\langle \mathbf{m}v, \hat{v}_0 \rangle = \langle \mathbf{m}w, \hat{w}_0 \rangle$  for all  $\mathbf{m} \in M$  are two equal linear functionals on  $V$  and  $W$ , then  $V \cong W$  as  $F[M]$ -modules by Lemma 4. Equivalently,  $(\pi, V) \cong (\sigma, W)$ .  $\square$

Now we prove the second half of the Borel-Matsumoto Theorem.

**Theorem 2.2.** *If  $(\pi, V)$  and  $(\sigma, W)$  are two irreducible representations of  $M$  with  $V^K$  and  $W^K$  nonzero and isomorphic as  $\mathcal{H}_K$ -modules, then  $(\pi, V) \cong (\sigma, W)$ .*

*Proof.* Let  $\lambda : V^K \rightarrow W^K$  be an isomorphism of  $\mathcal{H}_K$ -modules and  $\iota : W^K \rightarrow F$  be a linear functional not equal to zero. Then by Bump's Lemma 2, there exist  $\hat{v} \in \hat{V}^K$  and  $\hat{w} \in \hat{W}^K$  such that  $(\iota \circ \lambda)(v) = \langle v, \hat{v} \rangle$  and  $\iota(w) = \langle w, \hat{w} \rangle$  for all  $v \in \hat{V}^K, w \in \hat{W}^K$ . Furthermore, there exist  $w_0 \in W^K, v_0 \in V^K$  such that  $\langle w_0, w \rangle \neq 0$  since  $\iota$  is nontrivial and  $v_0 = \lambda^{-1}(w_0)$  since  $\lambda$  is an isomorphism.

Then for  $\phi \in \mathcal{H}_K$ , we have that

$$\langle \sigma(\phi)w_0, \hat{w} \rangle = \langle \sigma(\phi)\lambda(v_0), \hat{w} \rangle = \langle \lambda(\pi(\phi)v_0), \hat{w} \rangle = (\iota \circ \lambda)(\pi(\phi)v_0) = \langle \pi(\phi)v_0, \hat{v} \rangle. \quad (25)$$

We show that equation 23 holds for all  $\phi \in \mathcal{H}$  as well as  $\mathcal{H}_K$ . For  $\phi \in \mathcal{H}$ , define  $\phi_K \in \mathcal{H}_K$  by

$$\phi_K(x) = \frac{1}{|K|^2} \sum_{k_1, k_2 \in K} \phi(k_1 x k_2) \quad (26)$$

for all  $x \in M$ . By equation 23, then  $\langle \pi(\phi_K)v_0, \hat{v} \rangle = \langle \sigma(\phi_K)w_0, \hat{w} \rangle$ . Furthermore, we have that

$$\langle \pi(\phi_K)v_0, \hat{v} \rangle = \langle \frac{1}{|K|^2} \sum_{k_1, k_2 \in K} \sum_{x \in M} \phi(k_1 x k_2) \pi(x) v_0, \hat{v} \rangle \quad (27)$$

$$= \frac{1}{|K|^2} \langle \sum_{k_1, k_2 \in K} \sum_{x \in M} \phi(x) \pi(k_1)^{-1} \pi(x) \pi(k_2)^{-1} v_0, \hat{v} \rangle \quad (28)$$

$$= \frac{1}{|K|^2} \langle \sum_{k_1, k_2 \in K} \pi(k_1)^{-1} \circ \left( \sum_{x \in M} \phi(x) \pi(x) \right) \circ \pi(k_2)^{-1} v_0, \hat{v} \rangle \quad (29)$$

$$= \frac{1}{|K|^2} \langle \sum_{k_1, k_2 \in K} \pi(k_1)^{-1} \pi(\phi) \pi(k_2)^{-1} v_0, \hat{v} \rangle \quad (30)$$

$$= \frac{1}{|K|^2} \sum_{k_1, k_2 \in K} \langle \pi(k_1)^{-1} \pi(\phi) \pi(k_2)^{-1} v_0, \hat{v} \rangle \quad (31)$$

$$= \frac{1}{|K|^2} \sum_{k_1, k_2 \in K} \langle \pi|_G(k_1)^{-1} \pi(\phi) \pi|_G(k_2)^{-1} v_0, \hat{v} \rangle \quad (32)$$

$$= \frac{1}{|K|^2} \sum_{k_1, k_2 \in K} \langle \pi(\phi) \pi|_G(k_2)^{-1} v_0, \hat{\pi}|_G(k_1) \hat{v} \rangle. \quad (33)$$

$$= \frac{1}{|K|^2} \sum_{k_1, k_2 \in K} \langle \pi(\phi) v_0, \hat{v} \rangle \quad (34)$$

$$= \langle \pi(\phi) v_0, \hat{v} \rangle. \quad (35)$$

since  $v_0 \in V^K$  and  $\hat{v} \in \hat{V}^K$ .

Thus  $\langle \pi(\phi_K)v_0, \hat{v} \rangle = \langle \pi(\phi)v_0, \hat{v} \rangle$  for all  $\phi \in \mathcal{H}$ . Similarly,  $\langle \sigma(\phi_K)w_0, \hat{w} \rangle = \langle \sigma(\phi)w_0, \hat{w} \rangle$ . With this information, then, we have that  $\langle \pi(\phi_K)v_0, \hat{v} \rangle = \langle \sigma(\phi_K)w_0, \hat{w} \rangle$  implies that  $\langle \pi(\phi)v_0, \hat{v} \rangle = \langle \sigma(\phi)w_0, \hat{w} \rangle$ .

Let  $\phi_m \in \mathcal{H}$  for all  $m \in M$  be the function that sends all  $x$  in  $M$  with  $x \neq m$  to 0 and  $m$  to 1. Then  $\pi(\phi_m)v = \pi(m)v$  and  $\sigma(\phi_m)w = \sigma(m)w$ .

Thus, we have that  $\langle \pi(m)v_0, \hat{v} \rangle = \langle \sigma(m)w_0, \hat{w} \rangle$  for all  $m \in M$ . By Lemma 5, then,  $(\pi, V)$  and  $(\sigma, W)$  are equivalent.  $\square$

Let  $M$  be a finite monoid,  $G(M)$  its group of units,  $N$  a submonoid of  $M$ ,  $G(N)$  its group of units, and  $(\pi, V)$  a representation of  $M$ . Define the vector space  $\text{Ind}_N^M V$  as follows:

$$\text{Ind}_N^M V = \{f : M \rightarrow V \mid f(nm) = \pi(n)f(m) \quad \forall n \in N, m \in M\} \quad (36)$$

Define  $(\pi^M, \text{Ind}_N^M V)$  by  $\pi^M(m)f(x) = f(xm)$  for all  $m$ .

**Lemma 5.** *The pair  $(\pi^M, \text{Ind}_N^M V)$  is a representation of  $M$ .*

*Proof.* First, we check that  $\text{Ind}_N^M V$  is closed under the action of  $\pi^M(m)$ . Trivially, if  $f(nx) = \pi(n)f(x)$  then  $\pi^M(m)f(nx) = f(nxm) = \pi(n)f(xm)$  for all  $m \in M, n \in N$ .

We check that  $\pi^M(m)$  is linear for all  $m$ .

$$\forall z \in F, \forall f, g \in \text{Ind}_N^M V \quad z\pi^M(m)f(x) = z f(xm) = \pi^M(m)(zf)(x)$$

$$\pi^M(m)(f + g)(x) = (f + g)(xm) = \pi^M(m)f(x) + \pi^M(m)g(x)$$

Now, we check that  $\pi^M$  is a homomorphism of monoids. Let  $m, x, y \in M$ . Then  $\pi^M(mx)f(y) = f(y mx) = \pi^M(x)f(y m) = \pi^M(m)\pi^M(x)f(y)$ . Finally,  $\pi^M(1)f(x) = f(x)$ , implying that  $\pi^M$  maps the identity to the identity. Clearly, then,  $\pi^M(mx) = \pi^M(m)\pi^M(x)$ , and  $(\pi^M, \text{Ind}_N^M V)$  is a representation of  $M$ .  $\square$

Thus we can call  $(\pi^M, \text{Ind}_N^M V)$  the induced representation of  $M$ .

**Theorem 2.3** (Frobenius Reciprocity). *If  $(\pi, V)$  is a representation of  $N$ , a submonoid of  $M$ , and  $(\sigma, W)$  a representation of  $M$ , then  $\text{Hom}_M(W, \text{Ind}_N^M V) \cong \text{Hom}_N(W, V)$  as vector spaces.*

*Proof.* For  $\phi \in \text{Hom}_M(W, \text{Ind}_N^M V)$ , define  $F : \text{Hom}_M(W, \text{Ind}_N^M V) \rightarrow \text{Hom}_N(W, V)$  by  $F(\phi)$ , such that  $F(\phi)(w) = \phi(w)(1)$ ,  $1$  being the identity element of  $M$ . We first show that  $F(\phi)$  is linear. Because  $\phi$  is linear,

$$F(\phi)(w + w_0) = \phi(w + w_0)(1) = \phi(w)(1) + \phi(w_0)(1) = F(\phi)(w) + F(\phi)(w_0)$$

and for  $z \in F$ ,

$$F(\phi)(zw) = \phi(zw)(1) = z\phi(w)(1) = zF(\phi)(w)$$

We now claim that  $F(\phi)$  is a morphism of  $N$ -modules For  $n \in N$ ,

$$\begin{aligned} F(\phi)(\sigma(n)w) &= \phi(\sigma(n)w)(1) = \pi^M(n)\phi(\sigma(1)w)(1) \\ &= \phi(w)(n) = \pi(n)\phi(w)(1) = \pi(n)F(\phi)(w) \end{aligned}$$

Thus  $F(\phi)$  is an  $N$ -module homomorphism from  $W$  to  $V$ . Since

$$F(\phi + \psi)(w) = (\phi + \psi)(w)(1) = \phi(w)(1) + \psi(w)(1) = F(\phi)(w) + F(\psi)(w)$$

and  $F(z \cdot \phi)(w) = (z\phi)(w)(1) = zF(\phi)(w)$ , then  $F$  is a vector space homomorphism.

. For  $\tau \in \text{Hom}_N(W, V)$ , let  $G : \text{Hom}_N(W, V) \rightarrow \text{Hom}_M(W, \text{Ind}_N^M V)$  such that

$$(G(\tau)(w))(m) = G(\tau)(w)(m) = \tau(\sigma(m)w)$$

then,  $\tau(\sigma(nm)w) = \tau(\sigma(n)\sigma(m)w) = \pi(n)\tau(\sigma(m)w)$ , so  $G(\tau)(w)$  is in  $\text{Ind}_N^M$ . We check that  $G(\tau)(-)(m)$  is linear. This follows from the definition:

$$\begin{aligned} G(\tau)(w + w_0)(m) &= \tau(\sigma(m)(w + w_0)) = \tau(\sigma(m)w + \sigma(m)w_0) \\ &= \tau(\sigma(m)w) + \tau(\sigma(m)w_0) = G(\tau)(w)(m) + G(\tau)(w_0)(m) \end{aligned}$$

and for  $z \in F$ , we have

$$G(\tau)(zw)(m) = \tau(\sigma(m)(zw)) = z\tau(\sigma(m)w)$$

Next, we check that  $G(\tau)$  respects  $M$ . We have that for  $x \in M$ ,

$$\begin{aligned} G(\tau)(\sigma(x)w)(m) &= \tau(\sigma(m)\sigma(x)w) = \tau(\sigma(mx)w) \\ &= (\pi^M(x) \circ \tau)(\sigma(m)w) = \pi^M(x)(G(\tau)(w)(m)) \end{aligned}$$

Thus  $G(\tau) \in \text{Hom}_M(W, \text{Ind}_N^M V)$ . Finally, we check that  $G$  itself is linear:

$$\begin{aligned} G(\tau + \eta)(w)(m) &= (\tau + \eta)(\sigma(m)w) \\ &= \tau(\sigma(m)w) + \eta(\sigma(m)w) = G(\tau)(w)(m) + G(\eta)(w)(m) \end{aligned}$$

and for  $k \in K$ ,  $G(k\tau)(w)(m) = k(\tau(\sigma(m)w)) = k \cdot G(\tau)(w)(m)$ . Thus  $G$  is a homomorphism of vector spaces.

Now, we show that  $F$  and  $G$  are inverses. First, we check the mapping  $G \circ F : \text{Hom}_M(W, \text{Ind}_N^M V) \rightarrow \text{Hom}_M(W, \text{Ind}_N^M V)$ . Let  $\phi \in \text{Hom}_M(W, \text{Ind}_N^M V)$ . Then  $G \circ F(\phi)$  works as follows. Since  $F(\phi)$  is the map sending  $w$  to  $\phi(w)(1)$ ,

$$\begin{aligned} (G \circ F)(\phi)(w)(m) &= G(F(\phi))(w)(m) = F(\phi)(\sigma(m)w) \\ &= F(\phi)(\sigma(1 * m)w) = \pi^M(m)F(\phi)(w) \\ &= \pi^M(m)\phi(w)(1) = \phi(w)(m) \end{aligned}$$

by definition of the induced representation. Since  $(G \circ F)(\phi)(w)(\mathfrak{m}) = \phi(w)(\mathfrak{m})$ ,  $G \circ F$  is the identity morphism on  $\text{Hom}_{\mathcal{M}}(W, \text{Ind}_{\mathcal{N}}^W)$ .

Next, we check  $F \circ G : \text{Hom}_{\mathcal{N}}(V, W) \rightarrow \text{Hom}_{\mathcal{N}}(V, W)$ . Let  $\tau \in \text{Hom}_{\mathcal{N}}(V, W)$ . Then

$$\begin{aligned} (F \circ G)(\tau)(w)(\mathfrak{n}) &= F(G(\tau))(w)(\mathfrak{n}) \\ &= G(\tau)(w)(1 \cdot \mathfrak{n}) = \tau(\pi(\mathfrak{n})w) = \sigma(\mathfrak{n})\tau(w) = \tau(w)(\mathfrak{n}) \end{aligned}$$

Thus  $F \circ G$  is the identity morphism on  $\text{Hom}_{\mathcal{N}}(V, W)$ . Since we have that both  $G \circ F$  and  $F \circ G$  are identity morphisms on their respective domains, they are inverses. Thus, we have that  $\text{Hom}_{\mathcal{M}}(W, \text{Ind}_{\mathcal{N}}^M) \cong \text{Hom}_{\mathcal{N}}(W, V)$  as vector spaces over  $F$ .

□

### 3 Presentation for the Symplectic Renner Monoid

In this section, we give a classification for the symplectic rook monoid as a replacement for the previously established classification (see Li, Li, and Cao [7] p. 843, Corollary 2.3):

$$\mathcal{RSp}_{\mathfrak{n}} = \{A \in \mathcal{R}_{\mathfrak{n}} \mid APA^t = A^tPA = 0 \text{ or } P\}$$

for

$$P = \begin{pmatrix} 0 & J_m \\ -J_m & 0 \end{pmatrix}, \quad J_m = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

where  $\mathfrak{n} = 2m$ ,  $J_m$  is the  $m \times m$  anti-diagonal matrix of 1's, and  $\mathcal{R}_{\mathfrak{n}}$  is the rook monoid, a submonoid of  $\text{Mat}_{\mathfrak{n}}$ . Yet note that for  $m = 1$ , the symplectic renner monoid is the entirety of the rook monoid (see p. 842 of [7]),

$$\mathcal{RSp}_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

However note that

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \implies APA^t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

indicating that the description given in the corollary is not accurate, and indeed, there is no immediate proof of this proposition in the same paper. Proposition 2.4 of the same paper gives the following classification

$$\mathcal{R} = \{A \in \mathcal{R}_{\mathfrak{n}} \mid APA^t = A^tPA = 0\} \cup W$$

and if we take [7]'s definition of the symplectic Weyl group as the collection of "injective partial transformations of  $\mathfrak{n}$  that map all admissible sets of  $\mathfrak{n}$  to admissible sets" ([7], p.841). With the knowledge that that  $\{1, \dots, \mathfrak{n}\}$  is an admissible set, we see that elements of the Weyl group are full rank, and also elements of the rook monoid, so that the Weyl group is a submonoid of  $S_{\mathfrak{n}} \subseteq \mathcal{R}_{\mathfrak{n}}$ . With this, we establish:

**Theorem 3.1.**

$$\mathcal{RSp}_{\mathfrak{n}} \cong \{A \in \mathcal{R}_{\mathfrak{n}} \mid A^t J_{\mathfrak{n}} A = A J_{\mathfrak{n}} A^t = 0 \text{ or } J_{\mathfrak{n}}\}$$

*Proof.* I first show that

$$M \in \{A \in \mathcal{R}_{\mathfrak{n}} \mid A^t J A = A J A^t = 0\} \iff M \in \{A \in \mathcal{R}_{\mathfrak{n}} \mid A^t J_{\mathfrak{n}} A = A J_{\mathfrak{n}} A^t = 0\}$$



Indeed, note that  $J_n \mathbf{A}^t = (\mathbf{b}_{ij})$  is still an element of  $\mathcal{R}_n$  by nature of having at most one entry in every row and column that is non-zero (and hence = 1), so that the condition that

$$AJ_n \mathbf{A}^t = 0 \iff \forall i, j, \quad (AJ_n \mathbf{A}^t)_{ij} = c_{ij} = \sum_{k=1}^n a_{ik} b_{kj} = 0$$

means that there is at most one non-zero summand in the evaluation of  $c_{ij}$ , forcing that summand to be 0. Clearly, if  $\mathbf{d}_{ij} := (AJ_n \mathbf{A}^t)_{ij}$ , then  $c_{ij} = \pm d_{ij}$ , so that  $c_{ij} = 0 \iff d_{ij} = 0$ . Thus  $AJ_n \mathbf{A}^t = 0 \iff AJ_n \mathbf{A}^t = 0$ , and the non-full rank components of these sets coincide.

It suffices to show that

$$W = \{A \in \mathcal{R}_n \mid A^t J_n A = AJ_n \mathbf{A}^t = J_n\}$$

Let  $\bar{i} = n + 1 - i$ . For  $A \in W$ , we have that

$$A(\mathbf{k}) = \mathbf{i}_k \implies A(\bar{\mathbf{k}}) = \bar{\mathbf{i}}_k$$

for if not, then using our first definition of the symplectic rook monoid,  $\{\mathbf{k}, A^{-1}(\bar{\mathbf{i}}_k)\}$  would be an admissible set mapped to a non-admissible set. Yet note that

$$A^t J_n A = AJ_n \mathbf{A}^t = J_n \iff AJ_n = J_n A \iff J_n A J_n = A$$

having used the fact that  $A^t = A^{-1}$  and  $J_n^{-1} = J_n$ , but  $J_n A J_n = A$  is exactly the condition that  $A(\mathbf{k}) = \mathbf{i}_k \iff A(\bar{\mathbf{k}}) = \bar{\mathbf{i}}_k$ , as  $J_n$  is the permutation corresponding to  $(1 \bar{1})(2 \bar{2}) \cdots (n \bar{n})$  and hence conjugating a permutation matrix by  $J_n$  makes it so that  $\mathbf{k} \mapsto \bar{A}(\bar{\mathbf{k}})$ , thus

$$A = J_n A J_n \iff A(\mathbf{k}) = \bar{A}(\bar{\mathbf{k}}) \iff \bar{A}(\bar{\mathbf{k}}) = A(\bar{\mathbf{k}})$$

so every  $A \in W$  satisfies  $A = J_n A J_n$ . Similarly, matrices  $A \in \mathcal{R}_n$  satisfying  $A = J_n A J_n$  are full rank and map admissible sets to admissible sets. For if not, then there would exist

$$\mathbf{k}, s \neq \bar{\mathbf{k}} \text{ s.t. } A(\mathbf{k}) = \mathbf{i}_k, \quad A(s) = \bar{\mathbf{i}}_k$$

contradicting the condition that  $J_n A J_n = A$ . □

With this presentation in mind, we obtain the same formula for the size of this monoid:

$$|\mathbf{RSp}_{2n}| = 2^n n! + \sum_{i=0}^n \binom{n}{i}^2 2^i \tag{37}$$

**Remark** This correction should not invalidate the rest of Li, Li, and Cao's results, as they make use of the symplectic Renner monoid in terms of admissible sets. That being said, this correction is useful for readers who would want to calculate the elements of the symplectic rook monoid as a submonoid of the rook monoid. Moreover, it has been difficult for the authors of this paper to find other characterizations of the symplectic rook monoid that describe the symplectic weyl group as a submonoid of  $\mathcal{R}_n$ , so the authors hope that this correction will be computationally useful.

## 4 Example: Character Table for $\mathbf{RSp}_6$

Below we present the character table for the symplectic rook monoid, as calculated via the method of Li, Li, Cao [7]. The matrix is presented in the transpose form as their convention dictates, with different row and

column labellings.

	W1	W2	W3	W4	W5	W6	W7	W8	W9	W10	(1,1,1)	(2,1)	(1,1,1)	(2)	(1,1)	(1)	(0)
C1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0
C2	1	1	1	-1	-1	-1	-1	1	1	-1	0	0	0	0	0	0	0
C3	1	1	-1	-1	1	-1	1	1	-1	1	0	0	0	0	0	0	0
C4	1	1	-1	1	-1	1	-1	1	-1	-1	0	0	0	0	0	0	0
C5	2	2	0	0	-2	0	1	-1	0	-2	0	0	0	0	0	0	0
C6	2	2	0	0	2	0	-1	-1	0	2	0	0	0	0	0	0	0
C7	3	-1	-1	1	-1	-1	0	0	1	3	0	0	0	0	0	0	0
C8	3	-1	1	1	1	-1	0	0	-1	-3	0	0	0	0	0	0	0
C9	3	-1	1	-1	-1	1	0	0	-1	3	0	0	0	0	0	0	0
C10	3	-1	-1	-1	1	1	0	0	1	-3	0	0	0	0	0	0	0
(1,1,1)	8	0	0	0	0	4	0	2	0	0	1	1	1	0	0	0	0
(2,1)	16	0	0	0	0	0	0	-2	0	0	2	0	-1	0	0	0	0
(3)	8	0	0	0	0	-4	0	2	0	0	1	-1	1	0	0	0	0
(1,1)	12	0	2	0	4	2	0	0	0	0	3	1	0	1	1	0	0
(2)	12	0	-2	0	4	-2	0	0	0	0	3	-1	0	1	-1	0	0
(1)	6	2	0	0	4	2	0	0	2	0	3	1	0	2	0	1	0
(0)	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

where  $C_i$  stands for the  $i$ th irreducible character of indexed by a signed partition,  $(\lambda, \mu)$ , such that  $|\lambda| + |\mu| = n$  and  $W_i$  stands for the  $i$ th conjugacy class of the Weyl group, with the same indexing convention. By [7], we know that the Munn classes and irreps of  $RSp_{2n}$  are both indexed by elements of  $\Omega_n$ . In this case, the conjugacy class representatives for the Weyl group are

$$\begin{aligned}
\{W_1, W_2, W_3\} &= \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\} \\
\{W_4, W_5, W_6\} &= \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \right\} \\
\{W_7, W_8, W_9\} &= \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \right\} \\
W_{10} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

$\{W_1, W_2, W_3, W_4, W_5, W_6, W_7, W_8, W_9, W_{10}\} =$

$$\{((1)^3 | \emptyset), ((1) | (1)^2), ((2) | (1)), (\emptyset | (1,2)), ((1)^2 | (1)), ((2,1) | \emptyset), (\emptyset | (3)), ((1) | (2)), (\emptyset | (1)^3)\}$$

where the correspondence between  $2n \times 2n$  matrices and signed partitions (also known as pairs of partitions) of  $n$  is explained in the subsequent section. Taking the transpose of the above yields

$$M := C(\text{RSp}_6) = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 8 & 16 & 8 & 12 & 12 & 6 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 1 & 1 & -1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 2 & -2 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 & -2 & 2 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 4 & 4 & 4 & 1 \\ 1 & -1 & -1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & 4 & 0 & -4 & 2 & -2 & 2 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 2 & -2 & 2 & 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 1 & -1 & 1 & -1 & -2 & 2 & 3 & -3 & 3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 3 & 3 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that we can write  $M = YB = AY$  where  $Y$  is the block diagonal matrix of the Weyl Group,  $B_3$ , and then  $S_3, S_2, S_1, S_0$ . From here,

$$Y = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & -2 & 0 & 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 2 & 0 & -1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -1 & -1 & 1 & -1 & -1 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -1 & 1 & 1 & 1 & -1 & 0 & 0 & -1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -1 & 1 & -1 & -1 & 1 & 0 & 0 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -1 & -1 & -1 & 1 & 1 & 0 & 0 & 1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned}
B = Y^{-1}M &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
A = MY^{-1} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 12 & 0 & 6 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 3 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

## 5 Finding the A matrix $C(\text{RSp}_{2n})$

Let  $M = C(\text{RSp}_{2n})$  as in the previous example. Let  $\lambda \vdash r$  for  $0 \leq r \leq n$ , let  $\sigma$  be a representative of the Munn class indexed by  $\alpha \in \mathcal{Q}_n$ , let  $\sigma_\kappa \in S_r$  be the “conjugate” of  $\sigma$ , and let  $C(r, \sigma) := C(r)$ , all as on p. 849 of [7], except note that here we make the dependence of  $C(r)$  on  $\sigma$  more explicit. The value of  $\chi_\lambda^*(\sigma)$  is independent of which representative of the Munn class is chosen by theorem 4.3 of [7].

$$(M)_{\lambda, \alpha} = \chi_\lambda^*(\sigma) = \sum_{\kappa \in C(r, \sigma)} \chi_\lambda(\sigma_\kappa)$$

Partition  $C(\tau, \sigma)$  into a disjoint union as follows

$$\begin{aligned} C(\tau, \sigma) &= \bigsqcup_{\mu \vdash \tau} \{K \in C(\tau, \sigma) \mid K = \{i_1, \dots, i_{m(1)}, j_1, \dots, j_{m(2)}, \dots, l_1, \dots, l_{m(e)}\} \\ &\quad \text{s.t. } \mu = (m(1), m(2), \dots, m(e))\} \\ &= \bigsqcup_{\mu \vdash \tau} S_{\mu, \sigma} \end{aligned}$$

i.e. we organize the  $K$  by the cycle types of  $S_\tau$  that they imitate (again, see p. 849 of [what?](#) for reference). Given a fixed  $\mu \vdash \tau$ , we know that

$$\forall K \in S_{\mu, \sigma}, \quad \sigma_\mu := \sigma_K = (1 \cdots m(1)) (m(1) + 1 \cdots m(1) + m(2)) \cdots \left( \left[ \sum_{i=1}^{e-1} m(i) \right] + 1 \cdots \left[ \sum_{i=1}^e m(i) \right] \right)$$

the independence of  $\sigma_\mu$  is immediate from having the same cycle type and Corollary 4.1 of [7]. From this, we can write

$$(M)_{\lambda, \alpha} = \sum_{K \in C(\tau, \sigma)} \chi_\lambda(\sigma_K) = \sum_{\mu \vdash \tau} \sum_{K \in S_{\mu, \sigma}} \chi_\lambda(\sigma_K) = \sum_{\mu \vdash \tau} |S_{\mu, \sigma}| \chi_\lambda(\sigma_K)$$

From here, we set

$$A_{\alpha, \mu} = \begin{cases} |S_{\mu, \alpha}| & |\mu| = |\alpha| \\ 0 & |\mu| \neq |\alpha| \end{cases}$$

where we adopt the convention that the rows of  $A$  are indexed by conjugacy classes and columns are indexed by the irreducible representations of  $\mathbf{RSp}_{2n}$ . To see that this is the correct definition of  $A$ , note that

$$(AY)_{\lambda, \delta} = \sum_{\beta \in \Omega_n} A_{\lambda, \beta} Y_{\beta, \delta} = \sum_{\beta \text{ s.t. } |\beta| = |\lambda|} |S_{\beta, \lambda}| Y_{\beta, \delta}$$

Because  $|\beta| \neq |\delta| \implies Y_{\beta, \delta} = 0$ . Let  $G$  be the corresponding group in which the Munn Class indexed by  $\lambda$  and  $\delta$  lies, and let  $\sigma$  be an element of that Munn class with  $r = |\lambda| = |\delta|$ . Then

$$(AY)_{\lambda, \delta} = \sum_{\beta \in G} |S_{\beta, \lambda}| Y_{\beta, \delta} = \sum_{\mu \vdash r} |S_{\mu, \sigma}| \chi_\lambda(\sigma_K) = (M)_{\lambda, (\sigma)} = (M)_{\lambda, \delta}$$

having noted that  $(\sigma) = \delta$  as Munn classes.

In the particular case that both  $\lambda$  and  $\delta$  are partitions of  $r \leq n$ , then any element  $\sigma \in C_\lambda$ , the Munn class corresponding to partition  $\lambda$ , has rank less than or equal to  $n$ . In particular, let  $M_{\sigma_K}$  represent the  $r \times r$  matrix corresponding to  $\sigma_K$  as an element of  $S_r$ . Then from our knowledge of the conjugacy classes of  $\mathbf{RSp}_{2n}$ , we know that

$$N = \begin{pmatrix} M_{\sigma_K} & 0 \\ 0 & 0 \end{pmatrix} \in C_\lambda \subseteq \mathbf{RSp}_{2n}$$

is a valid representative, and hence any set,  $S \subseteq I^\circ(N)$  with  $|S| = r$  consisting of some cycles of  $N^\circ$  (see p. 849 of Li, Li, Cao) *will automatically be admissible*. In particular, this means that

$$|S_{\beta, \lambda}| = \binom{\beta}{\lambda} = \prod_{i=1}^s \binom{\beta_i}{\lambda_i}$$

which is the same  $A$ -matrix entry as in Solomon's "Representations of the Rook Monoid," p. 321 [9]. This shows that the character table of the symplectic rook monoid has the following form

$$C(\mathbf{RSp}_{2n}) = \begin{pmatrix} C(B_n) & * \\ 0 & C(R_n) \end{pmatrix}$$

where  $C(M)$  denotes the character table of the inputted monoid as a block matrix in our larger matrix. Moreover,

$$A = \begin{pmatrix} \text{Id} & \mathbf{U} \\ 0 & \mathbf{T} \end{pmatrix}$$

where  $T_{\beta,\lambda} = \begin{pmatrix} \beta \\ \lambda \end{pmatrix} = |S_{\beta,\lambda}|$  for the  $\beta, \lambda$  described above.  $\mathbf{U}$  is given by the values of  $|S_{\mu,\lambda}|$  for  $\mu \in \text{SP}_m$  and  $\lambda \in \cup_{r=0}^m P_r$  under the standard ordering of  $\Omega_n$ .

## 5.1 Determining the $\mathbf{U}$ -block

To determine  $\mathbf{U}$ , we consider elements of the Weyl group as described in Geck and Pfeiffer [3] propositions 1.4.1 (p. 21), 3.4.2 (p. 92–93), and 3.4.7 (p. 96–97). In particular, we can determine  $\mathbf{U}$  by looking at the representatives

$$w_{\alpha,\beta} = b_{m_1,e_1}^- \cdots b_{m_l,e_l}^- \cdot b_{m_{l+1},e_{l+1}}^+ \cdots b_{m_r,e_r}^+$$

where  $\beta = (e_1, \dots, e_l)$  and  $\alpha = (e_{l+1}, \dots, e_r)$  consist of a decreasing partition and increasing partition of some  $r$  and  $n - r$  respectively, so that  $|\alpha| + |\beta| = n$ . We have that

$$b_{m,e}^+ = s_{m+1} s_{m+2} \cdots s_{m+e-1} \in W_n$$

$$b_{m,e}^- = t_m s_{m+1} s_{m+2} \cdots s_{m+e-1} \in W_n$$

where  $s_i$  is the matrix corresponding to the permutation  $(i, i+1) \in S_n$ , also realized as a matrix group, and  $t_m$  for  $0 \leq m \leq n-1$  is the identity matrix in  $\text{Mat}(n)$  except  $t_{m+1,m+1} = -1$ . We note that

$$\begin{aligned} b_{m,e}^+ &= s_{m+1} \cdots s_{m+e-1} \cong (m+1, m+2)_n \cdots (m+e-1, m+e)_n \in S_n \\ &\rightarrow (m+1, m+2, \dots, m+e)_{2n} \quad (\overline{m+1}, \overline{m+2}, \dots, \overline{m+e})_{2n} \in S_{2n} \end{aligned}$$

$$\begin{aligned} b_{m,e}^- &= t_m s_{m+1} \cdots s_{m+e-1} \cong t_m (m+1, m+2)_n \cdots (m+e-1, m+e)_n \\ &\rightarrow (m+1, m+2, \dots, m+e-1, m+e, \overline{m+1}, \overline{m+2}, \dots, \overline{m+e})_{2n} \in S_{2n} \end{aligned}$$

Here we've included the correspondence between signed  $n \times n$  matrices, which are demarcated by  $(\cdots)_n$  and unsigned matrices of size  $2n \times 2n$ , demarcated by  $(\cdots)_{2n}$ . This correspondence is as follows: given a signed matrix  $M_\sigma$ , then for the permutation matrix contained in  $\text{Mat}(2n)$ , call it  $M'$ , we have

$$(M_\sigma)_{ij} = 1 \implies (M')_{ij} = 1 = (M')_{\overline{i}\overline{j}} = 1$$

$$(M_\sigma)_{ij} = -1 \implies (M')_{\overline{i}\overline{j}} = 1 = (M')_{i\overline{j}}$$

With this, the image of  $b_{m,e}^+$  in  $\text{Mat}(2n)$  is justified. The image of  $b_{m,e}^-$  can be explained as follows: consider the image of  $b_{m,e}^-$  and that of  $b_{m,e}^+$  as matrices in  $\text{Mat}(2n)$ , denoted by  $M^-$  and  $M^+$ . These matrices are very similar, in particular, rows  $m+2$  through  $m+e-2$  coincide. However  $M_{m+1,m+e}^+ = 1 = M_{\overline{m+1},\overline{m+e}}^+$ , while  $M_{\overline{m+1},m+e}^- = 1 = M_{m+1,\overline{m+e}}^-$ , which comes from the effect of multiplying by  $t_m$ . This means that instead of mapping  $m+e$  to  $m+1$  under the action of  $b_{m,e}^-$ , we have that  $m+e$  is mapped to  $\overline{m+1}$ . From there  $\overline{m+1} \rightarrow \overline{m+2} \rightarrow \cdots \rightarrow \overline{m+e}$  normally, as

$$M_{\overline{m+2},\overline{m+1}}^- = \cdots = M_{\overline{m+e},\overline{m+e-1}}^- = M_{\overline{m+2},\overline{m+1}}^+ = \cdots = M_{\overline{m+e},\overline{m+e-1}}^+ = 1$$

holds. However  $b_{m,e}^- \overline{m+e} = m+1$ , as  $M_{m+1,\overline{m+e}}^- = 1$ , finishing the matrix description. As an example, consider

$$\begin{aligned} b_{0,3}^- &= t_0 s_1 s_2 = t_0 (1\ 2)(2\ 3) = t_0 (1\ 2\ 3) \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

$$\mapsto \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \rightarrow (1\ 2\ 3\ 6\ 5\ 4) = (1\ 2\ 3\ \bar{1}\ \bar{2}\ \bar{3})$$

With this, we note that the representatives  $\{w_{\alpha,\beta}\}$  consist of blocks of *disjoint* signed permutations, i.e. each block affects some distinct subset of  $n$ . In particular, this is because the subset in question is  $\{m_i + 1, \dots, m_i + e_i\}$  for a given  $b_{m_i, e_i}$ . Now consider the set

$$C(r) = C(r, \sigma) = \{K \subseteq I^\circ(\sigma) \mid K \text{ admissible consists of all the elements of some cycles of } \sigma^\circ \text{ with } |K| = r\}$$

for which we sum over in

$$\chi_\lambda^*(\sigma) = \sum_{K \in C(r, \sigma)} \chi_\lambda(\sigma_K)$$

from p. 848, 849 of [7]. We note that no admissible set in  $C(r)$  can contain a cycle induced by a block  $b_{m,e}^-$ , as such a cycle contains both  $m+i$  and  $\overline{m+i}$  for  $1 \leq i \leq e$ . Thus the cycles from which an admissible  $K \in C(r)$  can be created are those coming from the  $b_{m,e}^+$  blocks.

Now recall the  $S_{\mu,\alpha}$  notation. Let  $\{\mu_i\}$  denote the number of  $i$ -cycles in the partition  $\mu$ . Similarly, let  $\{\alpha_i\}$  denote the number of admissible  $i$ -cycles produced by  $w_{\alpha,\beta}$ , which is equal to the number of  $\{e_k\}_{k>1}$  such that  $e_k = i$ . Then

$$|S_{\mu,\sigma}| = \prod_{i=1}^{2n} \binom{\alpha_i}{\mu_i} \cdot 2^{\mu_i} = 2^{\sum_i \mu_i} \cdot \prod_{i=1}^{2n} \binom{\alpha_i}{\mu_i}$$

because if there are  $\alpha_i$   $i$ -cycles, then we have  $\binom{\alpha_i}{\mu_i} \cdot 2^{\mu_i}$  ways of choosing them, as we could consider choosing  $i$  cycles from the collection of cycles acting on  $\{1, \dots, n\}$  or from the disjoint collection of cycles acting  $\{n+1, \dots, 2n\} = \{\bar{n}, \dots, \bar{1}\}$ . Note that such cycles come in pairs as shown by the image of  $b_{m,e}^+$  in  $\text{Mat}(2n)$ , and so for every chosen  $i$  cycle, we can choose that cycle or its admissible conjugate. This determines the values of  $|S_{\mu,\sigma}| = |S_{\mu,\alpha}|$  and hence the value of  $U$ , completing the description of the  $A$  matrix.

## 6 Restricting Monoid Representations to Group Representations

Among many sources, Steinberg [10] tells us that the character table of any finite inverse semigroup is block upper-triangular. Using this, Solomon decomposes the character table of  $R_n$  into the product of a block diagonal matrix and a much simpler block-upper-triangular matrix [9]. In type  $A_n$ , he finds matrices  $A$  and  $B$  such that the character table  $M = AY = YB$  where  $Y = \text{diag}(X_n, \dots, X_0)$ .

To explain his result, we first define

**Definition 6.1.** *Given groups  $G, H$  and corresponding representations  $V_G$  and  $V_H$ , we define the box tensor representation  $V_G \boxtimes V_H$  to be the representation of  $G \times H$  with the action  $(g, h) \cdot (v_1 \boxtimes v_2) = gv_1 \boxtimes hv_2$ .*

We can now describe Solomon's restriction. He shows that given  $\chi^*$  an irreducible representation of  $R_n$  corresponding to a partition of  $k$ , the restriction  $\chi^*|_{S_n} = \text{Ind}_{S_k \times S_{n-k}}^{S_n} (\chi \boxtimes \eta_{n-k})$ , where  $\eta_{n-k}$  is the trivial representation. We show in general

**Theorem 6.2.** *Let  $W_n$  be a Weyl group of type  $A_n, B_n, C_n$ , or  $D_n$ , with corresponding Renner monoids  $RW_n$ . Let  $\chi$  be a character  $S_r$ , and  $\chi^*$  the associated character of  $W_n$ . Then*

$$\chi^*|_{W_n} = \text{Ind}_{S_k \times W_{n-k}}^{W_n} (\chi \boxtimes \eta_{n-k})$$

*Proof.* By setting  $e = e_r$  we have  $W^*(e)$  is generated by roots  $\{\alpha \in \Delta, s_\alpha e = es_\alpha \neq e\}$ . Thus,  $W^*(e)$  is generated by elements which involve the first  $r$  indices. We know what the Coxeter diagrams for these groups look like, and the only generators which involve the first  $r$  indices are the transpositions  $(k \ k+1)$  for  $k \leq r$ . However,  $(r \ r+1)e_r \neq e_r(r \ r+1)$ , so  $W^*(e) = (1 \ 2) \dots (r-1 \ r) = S_r$ . We now apply the general character formula of Li, Li, and Cao. Given an irreducible character  $\chi$  of  $W^*(e)$ , we have that

$$\chi^*(\sigma) = \sum_{K \in \mathcal{F}(e), K\sigma=K} \chi(\mu_K \sigma \mu_K^-)$$

where  $\mathcal{F}(e) = w \cdot [r], w \in W$ .

Because  $K \in \mathcal{F}(e)$ ,  $|K| = r$  and there exists a Weyl group element  $w$  that restricts to  $\mu_K$ , so that  $\mu_K \sigma \mu_K^-$  is the restriction of  $w\sigma w^{-1}$  to  $[r]$ . Therefore,  $\chi(\sigma) = (\chi \otimes \eta_{n-r})(w\sigma w^{-1})$ , so that

$$\chi^*(\sigma) = \sum (\chi \boxtimes \eta_{n-r})(w\sigma w^{-1})$$

summing over all  $|K| = r$  with  $w_K \sigma w_K^{-1} \in S_r \times W_{n-r}$ . The elements  $w_K$  are a set of coset representatives  $W_n/(S_r \times W_{n-r})$ , which implies

$$\chi^*(\sigma) = \sum (\chi \boxtimes \eta_{n-r})(w\sigma w^{-1}) = \text{Ind}_{S_r \times W_{n-r}}^{W_n} (\chi \boxtimes \eta_{n-r})(\sigma)$$

as desired. □

## 6.1 Characters of the Type A Renner Monoid

The above gives us an explicit combinatorial interpretation in the type A case. For an irreducible representation of  $S_k$  indexed by  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_j \geq 1)$  where  $\sum_{i=1}^j \lambda_i = k$ , the induced representation of  $\lambda$  in  $R_n$

$$\chi^* = \bigoplus_{\lambda \subseteq \mu} \chi_\mu \tag{38}$$

where  $\mu$  is an irreducible representation of  $S_n$  and  $\mu$  differs from  $\lambda$  by a horizontal strip. This is the Pieri rule in the type A case, and it allows us to more easily compute the restriction of irreducible representations of the Rook monoid to  $S_n$ . Further, we use this as motivation in the type B case.

## 6.2 Type B Case

We'll now determine an analogue of the Pieri rule for  $B_n$ . We first highlight the two facts from Geck and Pfeiffer.

**Proposition 2.** *Geck and Pfeiffer Page 178, Lemma 6.1.3 Let  $n \geq 1$  and  $k, l \geq 0$  be integers such that  $n = k + l$ . Let  $(\lambda_1, \lambda_2)$  and  $(\mu_1, \mu_2)$  be pairs of partitions with  $|\lambda_1| + |\lambda_2| = k$  and  $|\mu_1| + |\mu_2| = l$ . Then, using the diagonal embedding  $W_k \times W_l \subseteq W_n$ , we have*

$$\text{Ind}_{B_k \times B_{n-k}}^{B_n} (\chi_{a_1, a_2} \boxtimes \chi_{b_1, b_2}) = \sum_{(\nu_1, \nu_2)} c_{a_1, b_1}^{\nu_1} c_{a_2, b_2}^{\nu_2} \chi_{\nu_1, \nu_2}$$

where the sum runs over all pairs of partitions  $(\nu_1, \nu_2)$  for which  $|\nu_i| = |\lambda_i| + |\mu_i|$  for  $i = 1, 2$ .

**Proposition 3** (Geck and Pfeiffer Page 179, Lemma 6.1.4). *Let  $n \geq 1$  and consider the subgroup  $S_n \subset B_n$ . Let  $\nu \vdash n$  and  $\chi_\nu \in \text{Irr}(S_n)$  be the corresponding irreducible character. Then*

$$\text{Ind}_{S_n}^{B_n} = \sum_{\lambda, \mu} c_{\lambda, \mu}^\nu \chi_{(\lambda, \mu)}$$



We can now derive a more explicit formula for  $\text{Ind}_{S_k \times B_{n-k}}^{B_n} (\chi_\nu \boxtimes \eta_{n-k})$  for a fixed partition  $\nu \vdash k$ . Using transitivity of induction, we have

$$\begin{aligned}
\text{Ind}_{S_k \times B_{n-k}}^{B_n} (\chi_\nu \boxtimes \eta_{n-k}) &= \text{Ind}_{B_k \times B_{n-k}}^{B_n} \text{Ind}_{S_k \times B_{n-k}}^{B_k \times B_{n-k}} (\chi_\nu \boxtimes \eta_{n-k}) \\
&= \text{Ind}_{B_k \times B_{n-k}}^{B_n} \left( \sum_{\lambda, \mu} c_{\lambda, \mu}^\nu \chi_{\lambda, \mu} \boxtimes \eta_{n-k} \right) \\
&= \sum_{\lambda, \mu} c_{\lambda, \mu}^\nu \text{Ind}_{B_k \times B_{n-k}}^{B_n} (\chi_{\lambda, \mu} \boxtimes \chi_{[n-k], \emptyset}) \\
&= \sum_{\substack{\lambda, \mu \\ \lambda + \mu \vdash k}} c_{\lambda, \mu}^\nu \sum_{\substack{\nu_1, \nu_2 \\ |\nu_i| = |\lambda_i| + |\mu_i| \\ \nu_1 + \nu_2 \vdash n}} c_{\lambda, [n-k]}^{\nu_1} c_{\mu, \emptyset}^{\nu_2} \chi_{\nu_1, \nu_2}
\end{aligned}$$

Note that  $c_{\mu, \emptyset}^{\nu_2} = 0$  unless  $\mu = \nu_2$  in which case it is equal to 1. This sum reduces to

$$\begin{aligned}
\text{Ind}_{S_k \times B_{n-k}}^{B_n} (\chi_\nu \boxtimes \eta_{n-k}) &= \sum_{\substack{\lambda, \mu \\ \lambda + \mu \vdash k}} c_{\lambda, \mu}^\nu \sum_{\substack{\nu_1 \\ \nu_1 + \mu \vdash n}} c_{\lambda, [n-k]}^{\nu_1} \chi_{\nu_1, \mu} \\
&= \sum_{\substack{\gamma, \mu \\ \gamma + \mu \vdash n}} \left( \sum_{\substack{\lambda, \lambda + \mu \vdash k \\ \gamma - \lambda \text{ is} \\ \text{n-k horiz. strip}}} c_{\lambda, \mu}^\nu c_{\lambda, [n-k]}^\gamma \right) \chi_{\gamma, \mu} \\
&= \sum_{\substack{\gamma, \mu \\ \gamma + \mu \vdash n}} \left( \sum_{\substack{\lambda \\ \gamma - \lambda \text{ is} \\ \text{n-k horiz. strip}}} c_{\lambda, \mu}^\nu \right) \chi_{\gamma, \mu}
\end{aligned}$$

The line comes from swapping the order of summation and noting that  $c_{\lambda, [n-k]}^\gamma$  is an order type-A Pieri coefficient, which is 1 if  $\gamma - \lambda$  is a horizontal strip and 0 otherwise. These explicit formulas yield a nicer way to determine the character table.

### 6.3 A and B matrices in type A case

Solomon showed that the character table,  $M$ , of the rook monoid can be written as

$$M = AY = YB \tag{39}$$

where  $Y$  is the block diagonal matrix, where the blocks are the character tables of  $S_n$  down to  $S_0$ . It turns out that the  $B$  matrix is simple to compute and is completely determined by the Pieri rule. We will show in the Hecke algebra case,  $\mathcal{H}(\mathcal{R}_n)$ , that if  $M$  is the Hecke algebra character table and  $Y$  is the block diagonal matrix where the blocks are the character tables of  $\mathcal{H}(S_n)$  down to  $\mathcal{H}(S_0)$ , then  $M = YB$  where  $B$  is the same matrix as in the regular monoid case. The  $A$  matrix can be described using binomial coefficients. However, from the smallest cases, the  $A$  matrix is dependent upon  $q$  in the Hecke Algebra case, thus we do not compute this here.

### 6.4 B matrix for $\mathcal{H}(\mathcal{R}_n)$

Let  $\mathcal{H}(\mathcal{R}_n)$  be the Hecke algebra of the Rook monoid. Let  $M_n$  be the character table of  $\mathcal{H}(\mathcal{R}_n)$  and let  $H_n$  be the character table of the Hecke algebra  $\mathcal{H}(S_n)$ . Let  $Y_n$  be the block diagonal matrix whose blocks are  $H_i$  for  $i = 0, \dots, n$  where  $H_n$  is the top left block and  $H_0$  is the bottom right block.

**Theorem 6.3.** *We can compute the character table as  $M_n = Y_n B_n$  where  $B_n$  is the B matrix computed in Solomon [9].*

*Proof.* In [2], the authors determine the character table is

$$M_n = \begin{bmatrix} H_n & * \\ 0 & M_{n-1} \end{bmatrix} \quad (40)$$

Thus, by induction, assume  $M_{n-1} = Y_{n-1} B_{n-1}$ . Then it is clear that

$$B_n = \begin{bmatrix} \text{Id} & P \\ 0 & B_{n-1} \end{bmatrix} \quad (41)$$

Thus, we need to show that P is determined in exactly the same way as the upper right part of the matrix in [9]. Note that we already know what this P would be if we let  $q = 1$ . Further, let  $\lambda$  index an irreducible representation of  $\mathcal{H}(\mathbb{R}_n)$ , then it is clear that the matrix P encodes the restriction of  $V^\lambda$  to  $\mathcal{H}(S_n)$ . In particular,

$$V^\lambda \downarrow \mathcal{H}(S_n) = \bigoplus_{\mu \vdash n} \alpha_\mu W^\mu \quad (42)$$

where  $\alpha_\mu \in \mathbb{N}$ . We see that P encodes these  $\alpha_\mu$ , and the  $\alpha_\mu$  are independent of  $q$ . Thus, we can set  $q = 1$ , and our P matrix is determined by the Pieri rule for  $S_n$ . As a result,  $B_n = B$ .  $\square$

Here we see that computing the character table of  $\mathcal{H}(\text{Mat}_n(\mathbb{F}_q), B)$  is reduced to knowing the character tables of the group case, and the character table of  $q = 1$  case. We now extend this to the Hecke algebra  $\mathcal{H}(\text{RSp}_{2n})$ .

## 6.5 B Matrix in type B

Using inspiration of Solomon, we find the B matrix associated to  $\text{RSp}_{2n}$  and the B matrix of the Hecke algebra  $\mathcal{H}(\text{RSp}_{2n})$ . Note that the A matrix was computed in section 5. We show that the same B matrix determines the character table for  $\mathcal{H}(\text{RSp}_{2n})$ . However, the A matrix is not the same when we extend to the Hecke algebra.

**Theorem 6.4.** *The B matrix of  $\text{RSp}_{2n}$  is the same for  $\mathcal{H}(\text{RSp}_{2n})$ .*

*Proof.* Note that we have an isomorphic copy of  $\mathcal{H}(\mathbb{R}_n)$  sitting inside of  $\mathcal{H}(\text{RSp}_{2n})$ . We claim that a subset of the irreducible representations of  $\mathcal{H}(\text{RSp}_{2n})$  restrict to irreducible representations of  $\mathcal{H}(\mathbb{R}_n)$ , or in other words, the irreducible representations of  $\mathcal{H}(\mathbb{R}_n)$  extend to irreducible representations of  $\mathcal{H}(\text{RSp}_{2n})$ . The details of that still need to be worked out. Given that this is the case, we have the character table of  $\mathcal{H}(\text{RSp}_{2n})$  is

$$M_{2n} = \begin{bmatrix} \mathcal{H}(B_n) & P \\ 0 & \mathcal{H}(\mathbb{R}_n) \end{bmatrix} \quad (43)$$

Thus, we clearly see the B matrix of  $\mathcal{H}(\text{RSp}_{2n})$  is

$$B = \begin{bmatrix} \text{Id} & * \\ 0 & B_n \end{bmatrix} \quad (44)$$

where  $B_n$  was determined in the type A case. However, just like in the type A case, we know exactly how to compute P. In particular, the section of P in the matrix are sums of elements from  $\mathcal{H}(\text{Sp}_n)$ . We determined that these must be integers because if  $\lambda$  indexes an irreducible representation of  $\mathcal{H}(\text{RSp}_{2n})$ , then we can compute its restriction to  $\mathcal{H}(\text{Sp}_n)$  as

$$V^\lambda \downarrow \mathcal{H}(B_n, B) = \bigoplus_{\mu + \gamma \vdash n} \alpha_{\mu, \gamma} V^{\mu, \gamma} \quad (45)$$

As in the previous section, we know that  $\alpha_{\mu,\gamma}$  are all positive integers or 0. In particular, they do not depend on  $q$ . Since we know what these values are when  $q = 1$ , we also know these values for all  $q$ . Hence, the  $*$  section of our B matrix is determined by our  $B_n$  Pieri rules. Thus, the B matrix for  $\mathcal{H}(\mathbf{RSp}_{2n})$  is the same as the B matrix for  $\mathbf{RSp}_{2n}$ .  $\square$

We now justify the form of  $M_{2n}$

**Lemma 6.** *The lower right section of the character table of  $\mathcal{H}(\mathbf{RSp}_{2n})$  is the character table of  $\mathcal{H}(\mathbf{R}_n)$*

*Proof.* Suppose  $V^\lambda$  is an irreducible representation of  $\mathcal{H}(\mathbf{RSp}_{2n})$ . Further, suppose  $V^\lambda$  corresponds to a representation that is not 0 on  $\mathcal{H}(\mathbf{R}_n)$ , i.e. according to our index, it is a column of the right half of our matrix. Then, since  $\mathcal{H}(\mathbf{R}_n) \subseteq \mathcal{H}(\mathbf{RSp}_{2n})$ , we can compute the restriction of  $V^\lambda$  to  $\mathcal{H}(\mathbf{R}_n)$ . If  $\mu_i$  index the irreducible representations of  $\mathcal{H}(\mathbf{R}_n)$  for  $i = 1, \dots, k$ , then

$$V^\lambda \downarrow \mathcal{H}(\mathbf{R}_n, B) = \bigoplus_{i=1}^k \alpha_i W^{\mu_i} \quad (46)$$

where,  $\alpha_{\mu_i} \geq 0$ . Let  $\chi_\lambda$  be the character of  $\lambda$  when restricted to  $\mathcal{H}(\mathbf{R}_n)$ , and let  $\chi_{\mu_i}$  be the character of  $\mu_i$  in  $\mathcal{H}(\mathbf{R}_n)$ . From the computation of our character table in the  $\mathbf{RSp}_{2n}$  case, we know that when  $q = 1$ ,  $\chi_\lambda = \chi_{\mu_j}$  for some  $j$ . Thus, we know

$$\chi_\lambda(q) = \sum_{i=1}^k \alpha_i \chi_{\mu_i}(q) \xrightarrow{q=1} \chi_\lambda = \sum_{i=1}^k \alpha_i \chi_{\mu_i} = \chi_{\mu_j} \quad (47)$$

Thus, this implies that  $\alpha_j = 1$  and  $\alpha_i = 0$  for  $i \neq j$ . In particular, this tells us the lower half of the character table of  $\mathcal{H}(\mathbf{RSp}_{2n}, B)$  is just the character table of  $\mathcal{H}(\mathbf{R}_n, B)$ .  $\square$

We note that it turns out the A matrix is dependent upon  $q$  when we extend our monoid to the hecke algebra case. Thus, we provide no further results for the A matrix.

## 7 The Representations of $\mathcal{H}(\mathbf{RSp}_{2n})$

Consider an irreducible representation of the Hecke Algebra of the Weyl Group,  $\rho_{(\lambda,\mu)}$ . Then the corresponding irreducible representation of  $\mathcal{H}(\mathbf{RSp}_{2n})$  with the same index, call this  $\tau_{(\lambda,\mu)}$  can be defined as follows:

$$\tau_{(\lambda,\mu)}(g) := \begin{cases} \rho_{(\lambda,\mu)}(g) & g \in B_n \\ 0 & g \in \mathbf{RSp}_{2n} - B_n \end{cases}$$

This representation is clearly irreducible and has the same character as those of the Weyl Group. Thus the representations coincide and the  $\{C_i\}$  are ordered under the same indexing given by the characters of the irreducible representations of the Weyl Group. Note that we're using the fact that the character table of the Hecke algebra is upper block diagonal. For the representations of just the irreducible representations of the Symplectic Rook Monoid, see Li, Li, and Cao [7].

## References

- [1] Daniel Bump. *Hecke Algebras*. 2011. URL: <http://sporadic.stanford.edu/bump/math263/hecke.pdf>. (accessed: 06.28.2018).
- [2] Momar Dieng, Tom Halverson, and Vahe Poladian. “Character Formulas for q-Rook Monoid Algebras”. In: *Journal of Algebraic Combinatorics* 17.2 (2003), pp. 99–123.
- [3] Meinolf Geck, Götz Pfeiffer, et al. *Characters of finite Coxeter groups and Iwahori-Hecke algebras*. 21. Oxford University Press, 2000.
- [4] Eddy Godelle. *Generic Hecke algebra for Renner monoids*. 2010. URL: <https://arxiv.org/abs/1002.1236>. (accessed: 06.28.2018).
- [5] Michel Gondran and Michel Minoux. *Graphs, dioids and semirings: new models and algorithms*. Vol. 41. Springer Science & Business Media, 2008.
- [6] Tom Halverson. “Representations of the q-rook monoid”. In: *Journal of Algebra* 273.1 (2004), pp. 227–251. ISSN: 0021-8693. DOI: <https://doi.org/10.1016/j.jalgebra.2003.11.002>. URL: <http://www.sciencedirect.com/science/article/pii/S0021869303007610>.
- [7] Zhenheng Li, Zhuo Li, and You’An Cao. “Representations of the symplectic rook monoid”. In: *International Journal of Algebra and Computation* 18.05 (2008), pp. 837–852.
- [8] Louis Solomon. “An introduction to reductive monoids”. In: *NATO ASI Series C Mathematical and Physical Sciences-Advanced Study Institute* 466 (1995), pp. 295–352.
- [9] Louis Solomon. “Representations of the rook monoid”. In: *Journal of Algebra* 256.2 (2002), pp. 309–342.
- [10] Benjamin Steinberg. “Möbius Functions and Semigroup Representation theory II: Character formulas and multiplicities”. In: *Advances in Mathematics* 217.4 (2008), pp. 1521–1557.