THE SMITH NORMAL FORM FOR THE UP-DOWN MAP IN YOUNG'S LATTICE

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Abstract. In this report we explore the Smith normal form for the incidence matrix between two adjacent ranks in Young's lattice, and for the product of this matrix with its transpose.

1. Introduction

Our main goal is to present a conjecture for the Smith normal form of the up-down map in Young's lattice. We start with definitions of the subject, and then prove some basic results. Along with the main conjecture, we present another conjecture that supports the main conjecture. We also present some potential paths for a proof, along with some data.

For integer partitions λ_1 and λ_2, by $\lambda_1 \subseteq \lambda_2$ we mean the Ferrers diagram of λ_2 contains the Ferrers diagram of λ_1.

Definition 1. For non-negative integer t, let $\mathbb{R}Y_t$ be the real vector space with basis Y_t, the partitions of t. For non-negative integers $i \leq j$, the *up* map $U_{i,j} : \mathbb{R}Y_i \to \mathbb{R}Y_j$ is a linear transformation from the partitions of i to those of j in Young's lattice defined by

$$U_{i,j}(\lambda) = \sum_{\substack{\mu \vdash j \\ \lambda \subseteq \mu}} \mu,$$

for all $\lambda \vdash i$.

The *down* map $D_{j,i} : \mathbb{R}Y_j \to \mathbb{R}Y_i$ is a linear transformation from the partitions of j to those of i in Young's lattice defined by

$$D_{j,i}(\lambda) = \sum_{\substack{\mu \vdash i \\ \lambda \supseteq \mu}} \mu,$$

for all $\lambda \vdash j$.

For the rest of the paper, when we write $D_{a,b}$ or $U_{a,b}$, we will assume that a and b are such that the map make sense. Let $[U_{a,b}]$ (resp. $[D_{a,b}]$) denote the matrix of the *up* map (resp. *down* map) with respect to the bases Y_b and Y_a. A key observation is that $[D_{j,i}] = [U_{i,j}]^t$.

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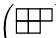
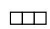
Example 2.

$$[D_{5,3}] = \begin{array}{c} \\ 3 \\ 21 \\ 111 \end{array} \begin{array}{c} \begin{array}{ccccccc} 5 & 41 & 32 & 311 & 221 & 2111 & 11111 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

So, for example, $D_{5,3}\Big($$\Big) =$  $+$ .

Definition 3. Given $|\lambda| = |\mu|$, define $\mu \leq \lambda$ if either $\mu = \lambda$, or else for some i, $\mu_1 = \lambda_1, \ldots, \mu_i = \lambda_i$, and $\mu_{i+1} < \lambda_{i+1}$, where $\mu = (\mu_1, \mu_2, \ldots)$ and $\lambda = (\lambda_1, \lambda_2, \ldots)$. The relation $\leq$ is called reverse lexicographic order.

As an example,

$$5 > 41 > 32 > 311 > 221 > 2111 > 11111.$$

Definition 4. A unimodular matrix is a square integral matrix that has an integral inverse.

It is a standard exercise to show that an integral matrix is unimodular if and only if it has determinant ± 1, the units of $\mathbb{Z}$.

Definition 5. A (possibly rectangular) diagonal matrix D is a diagonal form for a matrix A if there exist unimodular matrices R and C such that $D = RAC$. It is called the Smith normal form of A if the diagonal entries $d_{11}, d_{22}, \ldots$ of D are non-negative and $d_{ii} \mid d_{jj}$ for all $i \leq j$.

2. Conjectures and Observations

The *up-down* map DU_n is the map $D_{n+1,n} \circ U_{n,n+1}$. One can also generalize this to go up more than one step before coming back down, but we don't explore this as extensively; see Example 13 below. Given $[DU_n]$, let λ_k be the partition of n that indexes the kth row of $[DU_n]$. An easy observation can be made that $[DU_n]$ is symmetric by simply using $[D_{j,i}] = [U_{i,j}]^t$. The following proposition gives this result in a more combinatorial fashion.

Proposition 6. *$[DU_n]$ is symmetric.*

Proof. Since $[DU_n] = [D_{n+1,n} \circ U_{n,n+1}] = [D_{n+1,n}][U_{n,n+1}]$, we have

$$\begin{aligned} [DU_n]_{i,j} &= \sum_{i_1 \geq 1} [D_{n+1,n}]_{i,i_1} [U_{n,n+1}]_{i_1,j} \\ &= \#\{\lambda \vdash n+1 | \lambda \supset \lambda_i \text{ and } \lambda \supset \lambda_j\}, \end{aligned}$$

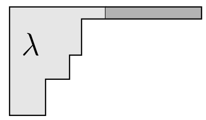
whence the result. $\square$

Another observation is of the Smith normal form for $[D_{i,j}]$ $(i \geq j)$.

Theorem 7. *Let i and j be integers such that $i \geq j \geq 0$. Then the Smith normal form of $[D_{i,j}]$ is*

$$\left(\begin{array}{cccc|c} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \Large 0 \\ & & & 1 & \end{array}\right).$$

Proof. Let the ordering of the rows and columns be decreasing reverse lexicographic. Given a partition λ of j, let $\tilde{\lambda}$ be obtained from λ by simply increasing the first part of λ until it is a partition of i.

$\tilde{\lambda} =$ 

It suffices to show that the square $p(j) \times p(j)$ submatrix of $[D_{i,j}]$ with columns indexed by $\{\tilde{\lambda} | \lambda \vdash j\}$ is upper unitriangular. To see this, let $\mu < \lambda$. By the definition of $<$, either μ has more parts than λ, or μ has a part, other than the first, of larger size than the corresponding part in λ. In either case, μ is not contained in $\tilde{\lambda}$. □

The above result also gives us the Smith normal form of $[U_{i,j}]$, since $[D_{i,j}] = [U_{i,j}]^t$. That is, the Smith normal form of $[U_{i,j}]$ is the transpose of the Smith normal form for $[D_{i,j}]$.

Furthermore, the Smith normal form of $[UD_n]$ isn't too hard to see. As above, it will be simple, consisting of only ones and zeros.

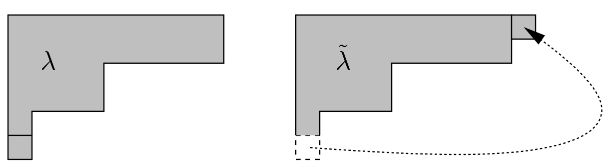
Proposition 8. *The map UD_n in Young's lattice has rank $p(n-1)$ and $p(n-1)$ $1's$ in its Smith normal form.*

Proof. Because UD_n and DU_{n-1} have the same nonzero eigenvalues, counting multiplicity (Prop. 9), and the latter has full rank (Thm. 11), it follows UD_n has rank $p(n-1)$.

Consider the set

$$P = \{\lambda \vdash n \mid \lambda \text{ has a 1-part}\},$$

and suppose $\lambda, \hat{\lambda} \in P$. Moreover, let the rows and columns of $[UD_n]$ be ordered from top to bottom and from left to right in decreasing reverse lexicographic order. The largest partition $\tilde{\lambda}$ reached by λ under UD_n is obtained from λ by taking off its lowest one-part and adjoining it to its first largest part.



It's clear $[\tilde{\lambda}]UD_n(\lambda) = 1$ since $\lambda \neq \tilde{\lambda}$ $(n > 1)$. Moreover, noting the aforementioned fact regarding the largest partition reached by another under UD_n, if $\lambda > \hat{\lambda}$ then $[\tilde{\lambda}]UD_n(\hat{\lambda}) = 0$. Thus, since $|P| = p(n-1)$, there is a lower triangular submatrix of size $p(n-1)$ with $1's$ down its diagonal. This gives the result. □

We now turn our attention to the Smith normal form of $[DU_n]$. For non-negative integer n, we let $p(n)$ be the number of partitions of n. For proofs[1] of Propositions 9 and 10 see [1] and [3], respectively.

Proposition 9. *Let A and B be $m \times n$ and $n \times m$ complex matrices, respectively. Then AB and BA have the same nonzero eigenvalues, counting multiplicity.*

[1]In fact, [1] gives 4 nice proofs!

Proposition 10. *For any $i \geq 0$ we have*

$$D_{i+1,i}U_{i,i+1} = U_{i-1,i}D_{i,i-1} + I_{p(i)}.$$

Using Propositions 9 and 10, the next theorem is easily proved by induction.

Theorem 11. *For $n \geq 0$, the eigenvalues of $[DU_n]$ are*

eigenvalue	*multiplicity*
$n+1$	1
$n-k$	$p(k+1)-p(k)$,

where $1 \leq k \leq n-1$.

Remark 12. Unlike $[DU_n]$, we don't necessarily get integer eigenvalues if we generalize by going up more than one step before coming back down, that is, for $[D_{b,a} \circ U_{a,b}]$ when $a < b$.

Example 13. $[D_{5,3} \circ U_{3,5}] = \begin{array}{c} \\ 3 \\ 21 \\ 111 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} 3 & 21 & 111 \end{array} \\ \begin{pmatrix} 4 & 3 & 1 \\ 3 & 5 & 3 \\ 1 & 3 & 4 \end{pmatrix} \end{array}$, which has eigenvalues 3, $5+3\sqrt{2}$, and $5-3\sqrt{2}$.

Conjecture 14 (Main Conjecture)**.** *For an integer $n \geq 1$, we have that the Smith normal form entries of $[DU_n]$ are*

entry	*multiplicity*
$(n+1)[(n-1)!]$	1
$(n-k)!$	$p(k+1)-2p(k)+p(k-1)$
1	$p(n)-p(n-1)+p(n-2)$,

where $3 \leq k \leq n-2$.

Remark 15. We were led to make the above conjecture by comparing the Smith normal form entries of $[DU_n]$ with its eigenvalues, as we now explain.

Let $n \geq 1$ be given and E be the multiset of eigenvalues for $[DU_n]$. Define E_0 to be the largest subset of E that is not a multiset, and E_i to be the largest subset of $E - E_0 - \cdots - E_{i-1}$ that is not a multiset, for $i \geq 1$. We observed that the Smith normal form entries $(s_1, \ldots, s_{p(n)})$ of $[DU_n]$ are given by

$$s_{p(n)-k} = \prod_{e \in E_k} e,$$

where $0 \leq k \leq p(n)-1$, and we take $s_{p(n)-j}$ to be 1 if $E_j = \emptyset$.

Example 16. The eigenvalues and Smith normal form entries of $[DU_6]$ are $\{7^{(1)}, 5^{(1)}, 4^{(1)}, 3^{(2)}, 2^{(2)}, 1^{(4)}\}$ and $(1^{(9)}, 6^{(1)}, 840^{(1)})$, respectively, where superscripts give the multiplicity. One can easily check the validity of the previous observation for this example.

Conjecture 14 has been tested up to $n = 20$, as has the following conjecture.

Conjecture 17. *Let*

$$\begin{aligned} P_n &= \{\lambda \mid \lambda \vdash n\}, \\ P_{1,n} &= \{\lambda \mid \lambda \vdash n, \lambda \textit{ has no 1-part, and } \lambda \textit{ has exactly one largest part}\}, \textit{ and} \\ P_{1,n}^{\text{conj}} &= \{\lambda \mid \lambda \textit{ is the conjugate of some } \mu \textit{ in } P_{1,n}\}. \end{aligned}$$

Then restricting $[DU_n]$ *to rows indexed by* $P_n \backslash P_{1,n}$ *and columns indexed by* $P_n \backslash P_{1,n}^{\text{conj}}$ *gives a square matrix* $\overline{[DU_n]}$ *of determinant* ± 1.

Proposition 18. $|P_n \backslash P_{1,n}| = |P_n \backslash P_{1,n}^{\text{conj}}| = p(n) - p(n-1) + p(n-2)$.

Proof. Since there is an obvious bijection (conjugation) between $P_{1,n}$ and $P_{1,n}^{\text{conj}}$, we have $|P_n \backslash P_{1,n}| = |P_n \backslash P_{1,n}^{\text{conj}}|$. Thus, it suffices to prove

$$|P_n \backslash P_{1,n}| = p(n) - p(n-1) + p(n-2).$$

Since $|P_n| = p(n)$, the former equates to showing $|P_{1,n}| = p(n-1) - p(n-2)$.

Let $S = \{\lambda | \lambda \vdash n-1, \text{ and } \lambda \text{ has a 1-part}\}$. There is a bijection between S and P_{n-2}. Let $\varphi : P_{n-2} \to S$ be defined by $\varphi(\lambda) = (\lambda, 1)$. This is obviously surjective, and easily seen to be injective. Thus $|P_{n-1} \backslash S| = p(n-1) - p(n-2)$.

We now show a bijection between $P_{n-1} \backslash S$ and $P_{1,n}$. Note that

$$P_{n-1} \backslash S = \{\lambda | \lambda \vdash n-1, \text{ and } \lambda \text{ has no 1-part}\}.$$

Let $\phi : P_{n-1} \backslash S \to P_{1,n}$ be defined by $\phi(\lambda) = (\lambda_1 + 1, \lambda_2, \ldots)$, where $\lambda = (\lambda_1, \lambda_2, \ldots)$. Then ϕ is easily seen to be both surjective and injective. We therefore have $|P_{1,n}| = p(n-1) - p(n-2)$, ending the proof. $\square$

The following theorem shows why Conjecture 17 is an important one in proving the main conjecture. For an integral matrix A, let $d_i(A)$ be the g.c.d of the determinants of all the $i \times i$ minors of A, where $d_i(A) = 0$ if all such $i \times i$ determinants are zero. $d_k(A)$ is called the kth *determinantal divisor* of A.

Theorem 19. *The Smith normal form entries* $(s_1, s_2, \ldots)$ *of a matrix* A *are given by the equation*

$$s_j(A) = \frac{d_j(A)}{d_{j-1}(A)},$$

where $d_0(A)$ *is taken to be 1.*

With Theorem 19 and Proposition 18 in mind, Conjecture 17 would show that

$$p(n) - p(n-1) + p(n-2)$$

is a lower bound for the number of ones in the Smith normal form of $[DU_n]$.

3. Possible Directions for Conjecture 2

The following is a list of possible directions for proving Conjecture 2.

a) Find an integral inverse for $\overline{[DU_n]}$. This shows that, as mentioned before, $\det \overline{[DU_n]} = \pm 1$, as desired.
b) A useful thing to have would be a formula relating $\overline{[DU_n]}$ and $\overline{[DU_{n-1}]}$. This would be useful for, say, an inductive proof.
c) Another possible proof is to induct and show when $\overline{[DU_n]}$ is restricted to its first j rows, the resulting matrix has Smith normal form

$$\left(\begin{array}{cccc|c} 1 & & & & \\ & 1 & & & \\ & & \ddots & & 0 \\ & & & 1 & \end{array}\right),$$

for $1 \leq j \leq p(n) - p(n-1) + p(n-2)$. This has been tested up to $n = 14$.

d) Finally, one could look at $\det \overline{[DU_n]} = \pm 1$ via the Binet-Cauchy formula. We now present our data from exploring direction (d).

Theorem 20. *(Binet-Cauchy Theorem) Let A be an $m \times n$ matrix and B an $n \times m$ matrix, where $m \leq n$. Then*

$$\det AB = \sum_{\substack{S \subset [n] \\ |S| = m}} (\det A[S])(\det B[S]),$$

where $A[S]$ (resp. $B[S]$) is the $m \times m$ matrix obtained by taking the columns of A (resp. rows of B) indexed by S.

Therefore we may write $\det \overline{[DU_n]}$ as

$$(1) \qquad \sum_{\substack{S \subset \{\lambda | \lambda \vdash n+1\} \\ |S| = p(n) - p(n-1) + p(n-2)}} (\det[D_{n+1,n}][P \backslash P_{1,n}][S])(\det[U_{n,n+1}][S][P_n \backslash P_{1,n}^{\text{conj}}]).$$

Let $V = (\det[D_{n+1,n}][P \backslash P_{1,n}][S])(\det[U_{n,n+1}][S][P_n \backslash P_{1,n}^{\text{conj}}])$. The following table gives the non-zero terms in (1), along with their value.

n for $[DU_n]$	S	V
3	$\{31, 211\}$	+1
4	$\{41, 32, 311, 2111\}$	-1
4	$\{41, 32, 221, 2111\}$	+1
4	$\{41, 311, 221, 2111\}$	-1
5	$\{51, 411, 321, 3111, 21111\}$	+1
6	$\{61, 511, 43, 421, 4111, 322, 3211, 31111, 211111\}$	+1
6	$\{61, 511, 43, 421, 4111, 322, 31111, 2221, 211111\}$	-1
6	$\{61, 511, 43, 421, 4111, 3211, 31111, 2221, 211111\}$	+1
6	$\{61, 511, 43, 4111, 331, 322, 3211, 31111, 211111\}$	-1
6	$\{61, 511, 43, 4111, 331, 322, 31111, 2221, 211111\}$	+1
6	$\{61, 511, 43, 4111, 331, 3211, 31111, 2221, 211111\}$	-1
6	$\{61, 511, 421, 4111, 331, 322, 3211, 31111, 211111\}$	+1
6	$\{61, 511, 421, 4111, 331, 322, 31111, 2221, 211111\}$	-1
6	$\{61, 511, 421, 4111, 331, 3211, 31111, 2221, 211111\}$	+1
7	$\{71, 611, 521, 5111, 431, 4211, 41111, 3221, 32111, 311111, 2111111\}$	+1

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