ENUMERATION OF 1- AND 2-CROSSING PARTITIONS WITH REFINEMENTS

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Abstract. An enumeration of the 1-crossing partitions of $[n]$ into k blocks by bijection with ordered trees with n edges, k internal nodes, and root degree $j = 4$ is presented. A semi-bijection of these ordered trees to Dyck paths of semilength n, k peaks, and $j = 4$ last peak height is used to derive a conjectured formula for the number of 1-crossing partitions of $[n]$ with k blocks.

We also explore some natural q-analogues and the Cyclic Sieving Phenomenon [6]. Moreover, enumeration of 1- and 2-crossing partitions via generating functions is presented.

1. Main theorem

Definition 1. Let $P = (P_1, P_2, \ldots, P_k)$ be a partition of the set $[n] = \{1, 2, \ldots, n\}$. A *crossing* of P is a 4-tuple $a < b < c < d$ in $[n]$ such that $a, c \in P_i$ and $b, d \in P_j$ with $i \neq j$. An *adjacent crossing* is a crossing in which c is the smallest element of P_i greater than a, and d is the smallest element of P_j greater than b (that is, the pairs are adjacent in their blocks).

A partition with m crossings is called a m-crossing partition; after Bóna [1], let $S_m(n)$ denote the set of m-crossing partitions of $[n]$, and let $S_m(n, k)$ denote the set of m-crossing partitions of $[n]$ into k blocks. The partitions with $m = 0$ are commonly called noncrossing partitions. Noncrossing partitions of $[n]$ are enumerated by the Catalan numbers: $|S_0(n)| = \frac{1}{n+1}\binom{2n}{n}$, while the k-block refinement is given by the Narayana numbers $N(n, k)$: $|S_0(n, k)| = N(n, k) = \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1}$.

Bóna has proven that $|S_1(n)| = \binom{2n-5}{n-4}$ by bijection to near-triangulations of an n-gon [1]. Our aim in this section is to prove his result and its block refinement by a semi-bijection with certain ordered trees.

Theorem 2 (Main Theorem)**.**

$$|S_1(n)| = \binom{2n-5}{n-1}$$

$$|S_1(n, k)| = \binom{n}{k-2}\binom{n-5}{k-3}$$

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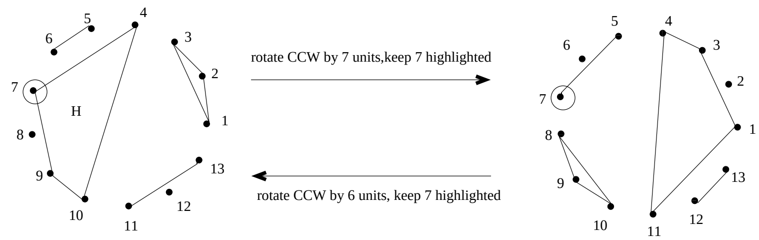


FIGURE 1. An illustration of the bijection from Prop. 5.

Note that this is consistent with the following instance of the Vandermonde convolution formula:

$$\binom{2n-5}{n-1} = \sum_k \binom{n}{k-2}\binom{n-5}{k-3} \tag{1}$$

To prove Theorem 2, we first relate 1-crossing partitions to noncrossing partitions in a way that preserves the k-block refinement.

Proposition 3. *Let $S_0^j(n,k)$ be the set of noncrossing partitions with at least 1 block of size j (a "j-block") and exactly 1 distinguished ("highlighted") j-block. $S_1(n,k)$ bijects to $S_0^4(n,k-1)$.*

Proof. Let $P \in S_1(n,k)$ have the crossing 4-tuplet $a < b < c < d$. Map P to a noncrossing partition $P' \in S_0^4(n,k-1)$ identical to P except that a,b,c,d are now in the same block and this block is highlighted. The inverse map sends the highlighted 4-block to a 1-crossing. □

Example 4. The partition of [8] into $\{\{1,2,3,4\},\{5,6,7,8\}\}$ can be highlighted in two ways; either the first or the second 4-block is highlighted. The corresponding 1-crossing partitions are $\{\{1,2,3,4\},\{5,7\},\{6,8\}\}$ and $\{\{1,3\},\{2,4\},\{5,6,7,8\}\}$.

Next, we rotate the partitions' cyclic diagrams so that the highlighted blocks are connected to 1.

Proposition 5. *Let $S_0^{j,1}(n,k)$ denote the number of noncrossing partitions in which 1 is inside a block of size j. Then $[j] \times S_0^j(n,k)$ bijects to $[n] \times S_0^{j,1}(n,k)$.*

Proof. A member $x' \in [j] \times S_0^j(n,k)$ corresponds to a set $x \in S_0^j(n,k)$ in which one of the elements of the highlighted j-block is itself highlighted. A member $y' \in [n] \times S_0^{j,1}(n,k)$ corresponds to a set $y \in S_0^{j,1}(n,k)$ in which any one of the elements of $[n]$ are highlighted.

Now let H_j be the highlighted j-block of some $x \in S_0^j(n,k)$ and let $m \in H_j$ be its distinguished member. To map x to its counterpart in $[n] \times S_0^{j,1}(n,k)$, rotate the cyclic diagram of x so that m is now labeled by 1, and in this rotated cyclic diagram let the element now labeled m be the distinguished element of $[n]$. To go the other way, suppose that $y \in S_0^{j,1}(n,k)$ has $p \in [n]$ as its highlighted member. Rotate the cyclic diagram of y so that 1 is now labeled by p, and let this element now labeled by p be the distinguished element. It is not difficult to verify that these maps are inverse to each other. See Fig. 1 for an illustration of the bijection. □

Therefore,

$$|S_0^{j,1}(n)| = \frac{j}{n}|S_0^j(n)| \tag{2}$$

and

$$|S_0^{j,1}(n,k)| = \frac{j}{n}|S_0^j(n,k)|. \tag{3}$$

Next, we relate $S_0^{j,1}(n,k)$ to a subclass of ordered trees by restricting a bijection of Dershowitz and Zaks [3] which we will describe. An *ordered tree* is a tree with a designated root (i.e. a rooted tree) whose (possibly empty) set of sons is the set of root nodes of a sequence of ordered subtrees – that is, an ordered tree is a rooted tree where every node has a linear order on its sons. The $n+1$ nodes of an ordered tree with n edges are either leaves (nodes of degree 1) or internal nodes (nodes of degree greater than 1).

The bijection in [3] is from $S_0(n,k)$ to $T(n,k)$ – the set of ordered trees with n edges and k internal nodes (or $n+1-k$ leaves). We construct the counterpart of $P \in S_0(n,k)$ in $T(n,k)$ as follows. Let $B_1 \in P$ be the block containing 1 where $B_1 = \{x_1 = 1, x_2, ...x_j\}$. Place a root node and give it j sons, labeled x_1 through x_j. Then for each $1 \leq i \leq j$, the interval $[x_i+1, x_{i+1}-1]$ is *closed* – it does not connect to anything else in the partition – and since P is noncrossing it is itself a noncrossing partition. Thus for each i we now treat x_i as the root of an ordered tree corresponding to the noncrossing partition of $[x_i+1, x_{i+1}-1]$ and proceed recursively. When this process finishes, we have a labeled tree which can be interpreted as an ordered tree in which the node labels are the record of a *preorder* traversal of the tree (preorder is the recursive traversal of an ordered tree's subtrees from left to right).

To go from an ordered tree to its noncrossing partition, traverse the ordered tree in preorder and label the nonroot vertices 1 through n. Then the descendants of a given internal node form a block in the corresponding noncrossing partition, so the partition can be "read off" from the ordered tree. Since a node corresponds to a block if and only if it is an internal node, we have a bijection from $S_0(n,k)$ to $T(n,k)$.

Corollary 6. *Let $T_j(n)$ denote the set of ordered trees with n edges and root degree j, and let $T_j(n,k)$ denote the refinement to k internal nodes. Then $S_0^{j,1}(n)$ bijects to $T_j(n)$ and $S_0^{j,1}(n,k)$ bijects to $T_j(n,k)$.*

Proof. The above bijection sends a member of $S_0^{j,1}(n)$ to an ordered tree with root degree j, since the block connected to 1 has j elements. The bijection naturally restricts to the k-refinement. □

It has been proven via generating functions [4][2] that

$$|T_j(n)| = \frac{j}{2n-j}\binom{2n-j}{n} = \frac{j}{n}\binom{2n-j-1}{n-1}. \tag{4}$$

and

$$|T_j(n,k)| = \frac{j}{n}\binom{n}{k-1}\binom{n-1-j}{k-2} = \frac{j}{k-1}\binom{n-1}{k-2}\binom{n-1-j}{k-2}. \tag{5}$$

Combining the previous results we can now derive the results of Theorem 2:

$$|S_1(n)| = |S_0^4(n)| = \frac{n}{4}|S_0^{4,1}(n)| = \frac{n}{4}|T_4(n)| = \binom{2n-5}{n-1}$$

$$|S_1(n,k)| = |S_0^4(n,k-1)| = \frac{n}{4}|S_0^{4,1}(n,k-1)| = \frac{n}{4}|T_4(n,k-1)| = \binom{n}{k-2}\binom{n-5}{k-3}$$

The rest of this section is devoted to bijective proofs of (4) and (5).

A *Dyck path* of semilength n is a lattice path from $(0,0)$ to (n,n) consisting only of moves that go a step up or a step to the right, with the restriction that the paths may not cross the $y=x$ line. A Dyck path can be thought of as a word composed of U's (up moves) and R's (right moves) where the number of R's never exceeds the number of U's at any point in the word. A peak of a Dyck path is a move up followed by a move to the right. For example, $URUURR$ is a Dyck path of semilength 3 and 2 peaks.

The following definition is needed for the next proposition.

Definition 7. Let T be an ordered tree, and x be a node in the tree. x is on the i^{th} *level* of T if the shortest path to the root contains i edges.

Proposition 8. *Let $D_j^f(n,k)$ denote the set of Dyck paths with semilength n, k peaks, and first peak height j. Then $T_j(n,k)$ bijects to $D_j^f(n,k)$ and, a fortiori, $T_j(n)$ bijects to $D_j^f(n)$.*

Proof. We will construct a Dyck path corresponding to an ordered tree $T \in T_j(n,k)$ as follows: begin at the root (the "0^{th} level" of T) and traverse the immediate sons of the root from left to right. Write a U for each son crossed (j U's total). Then move to the leftmost son of the root and move right on the Dyck path. We are now on the leftmost node of the 1^{st} level. In general, if one is on the i^{th} level's leftmost node, one proceeds to the $(i+1)^{th}$ level's leftmost node as follows: write a number of U's equal to the number of descendants of the node you are currently on, then move to the node to the right (if there is one) and write an R. If you at the end of the level, move to the leftmost node of the $(i+1)^{th}$ level and write an R. Using this rule one traverses all of T and writes the corresponding Dyck path.

This path will have n U's and n R's, since each U and each R corresponds to a distinct nonroot node. Further, it is a Dyck path; there are at least as many U's as R's at any given time in the traversal because each node is counted as a descendent (U move) before it is moved to in the traversal (R move). Each peak corresponds to an internal node because during the traversal one leaves an internal node if and only if a UR sequence was written.

We construct the inverse map: given a word $S \in D_j^f(n,k)$, write the root of an ordered tree and a number of descendants equal to the number of U's. Then move to the 1^{st} level and go to the m^{th} node from the left, where m is the number of R's after the U's. Its descendants are equal to the number of U's following, and so forth. By the same arguments as above in reverse, all ordered trees thus constructed are in $D_j^f(n,k)$. It is not difficult to see that this is indeed the inverse map. □

We now enumerate the sets $D_j^f(n)$ and the k-restriction $D_j^f(n,k)$. Here it will be convenient to consider instead the Dyck paths $D_j(n)$ with *last* peak height j, which biject to $D_j^f(n)$ by a reflection through the $y=-x$ axis (or equivalently, writing

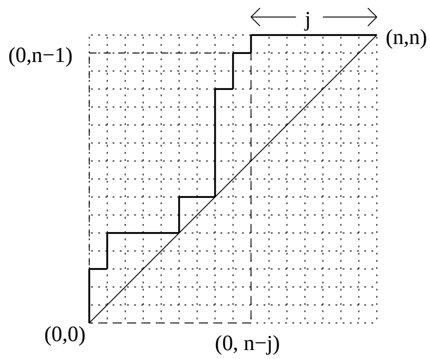


FIGURE 2. A path with last peak height j and the subpath in $L(n-1, n-j)$.

the word backwards and switching the identities of the R's and U's). Let $L(m, n)$ be the lattice paths from $(0, 0)$ to (m, n) consisting of up and right moves and staying above the $y = x$ diagonal; we call these *generalized Dyck paths*. One bijects $D_j(n)$ with the paths $L(n-1, n-j)$ by removing the last URRR... sequence from a member of $D_j(n)$. We will also need to consider *dominating* sequences – paths staying strictly above the diagonal – so we must also biject $L(m, n)$ to $L'(m+1, n)$ (paths staying strictly above the diagonal without touching it) by prefixing a U to each word in $L(m, n)$. The dominating sequences can now be counted by the following lemma [5].

Lemma 9 (Cycle Lemma). *For any sequence $p_1, p_2, \cdots, p_{m+n}$ of m U's and n R's, $m \geq n$, there exist exactly $m - n$ cyclic permutations of the sequence that are dominating.*

Proposition 10. *Let $m > n$. Then*

$$|L'(m, n)| = \frac{m-n}{m+n}\binom{m+n}{n}.$$

Proof. Let a *circular word* be a sequence of U's and R's under a cyclic rather than linear order. A circular word can be turned into a regular word by designating a starting letter for the linear order – that is, by "cutting" the circle after a given letter.

By the Cycle Lemma, a circular word consisting of m U's and n R's has $m - n$ distinct places we can cut the word to produce a member of $L'(m, n)$. One can show that there is only one way in which these $m-n$ cuts can fail to correspond to $m-n$ distinct members of $L'(m, n)$. Namely, if the circular word has s-fold rotational symmetry, there will be only $(m - n)/s$ distinct cuts. A given circular word also corresponds to $(m + n)/s$ distinct linear sequences that result from cutting the word in the $m+n$ possible places and again compensating for rotational symmetry. There are therefore $\frac{(m-n)/s}{(m+n)/s} = \frac{m-n}{m+n}$ distinct members of $L'(m, n)$ corresponding to each linear word of m U's and n R's. There are $\binom{m+n}{n}$ such words, so the total number of dominating sequences is given by the proposition. □

Then (4) follows from

$$\begin{aligned}|T_j(n)| &= |D_j^f(n)| = |D_j(n)| = |L(n-1,n-j)| \\ &= |L'(n,n-j)| = \frac{j}{2n-j}\binom{2n-j}{n}.\end{aligned}$$

The discussion above this last proposition is still valid if one appends k's to the sets to denote the k-refinement: replace $D_j(n)$ with $D_j(n,k)$, $L(n-1,n-j)$ with $L(n-1,n-j,k-1)$ (one peak is lost when the URRR... sequence is removed), and $L'(m+1,n)$ with $L'(m+1,n,k)$. A similar strategy then dispenses with the k-refinement:

Proposition 11. *Let $m > n$. Then*

$$|L'(m,n,k)| = \frac{m-n}{k}\binom{m-1}{k-1}\binom{n-1}{k-1}.$$

Proof. Again by the Cycle Lemma, given a circular word consisting of m U's and n R's there are $m-n$ distinct places we can "cut" the word to produce a member of $L'(m,n,k)$. If the circular word has s-fold rotational symmetry, there will be only $(m-n)/s$ distinct cuts. A given circular word also corresponds to several pairs of k-part compositions representing the ways the U's and R's may be arranged in the circular word to give k peaks. There are k places to start where the composition will begin (since it must start at a peak), and k/s distinct compositions that can result. There are then $\frac{(m-n)/s}{k/s} = \frac{m-n}{k}$ distinct members of $L'(m,n,k)$ corresponding to a pair of k-part compositions. There are $\binom{m-1}{k-1}\binom{n-1}{k-1}$ such pairs, so the total number of dominating sequences is given by the proposition. $\square$

Then (5) follows from

$$\begin{aligned}|T_j(n,k)| &= |D_j^f(n,k)| = |D_j(n,k)| = |L(n-1,n-j,k-1)| \\ &= |L'(n,n-j,k-1)| = \frac{j}{k-1}\binom{n-1}{k-2}\binom{n-j-1}{k-2}.\end{aligned}$$

2. q-analogues & the Cyclic Sieving Phenomenon

For background on the *cyclic sieving phenomenon*, see [6]. We begin by presenting a standard lemma which we will use throughout the section. First, however, we recall the definition of $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

Definition 12. We define the q-analogue of $\binom{n}{k}$ to be

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[n-k]!_q[k]!_q},$$

where $[j]_q = 1 + q + \cdots + q^{j-1}$ and $[n]!_q = [1]_q[2]_q\cdots[n]_q$.

Lemma 13. *If ω is a primitive dth root of unity,*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q=\omega} = \binom{n_0}{k_0}\begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_{q=\omega},$$

where $n = n_0 d + n_1$ and $k = k_0 d + k_1$, with $0 \le n_1 < d$ and $0 \le k_1 < d$.

An obvious q-analogue of Bóna's result [1] yields a system which exhibits the cyclic sieving phenomenon, as we now see.

Theorem 14.

$$\begin{bmatrix} 2n-5 \\ n-4 \end{bmatrix}_{q=\omega} = \#\left\{\begin{matrix}\text{\textit{1-crossing partitions of} } [n] \text{ \textit{invariant under} } d\text{\textit{-fold}} \\ \text{\textit{rotational symmetry.}}\end{matrix}\right\},$$

where ω is a primitive dth root of unity, and $d \mid n$.

Proof. The proof is straightforward; we simply count each side, and show they are equal. We begin with the RHS by showing

$$\#\left\{\begin{matrix}\text{1-crossing partitions of} \\ [n] \text{ invariant under } d\text{-fold} \\ \text{rotational symmetry.}\end{matrix}\right\} = \begin{cases} \binom{2n-5}{n-4} & \text{if } d=1; \\ \binom{n-3}{\frac{n}{2}-2} & \text{if } d=2; \\ \binom{\frac{n-4}{2}}{\frac{n-4}{4}} & \text{if } d=4; \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that if $d \neq 1, 2$, or 4, then the RHS in our theorem will be 0; after all, we have only one crossing. Also, the case when $d = 1$ is valid by Bóna's paper [1]. We now show the $d = 2, 4$ cases enumerate as claimed.

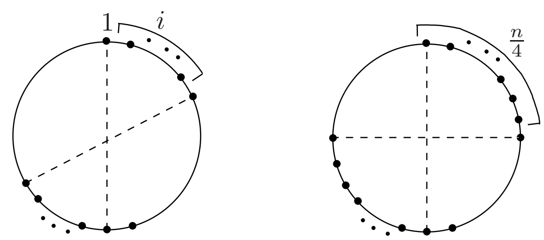


FIGURE 3. Cases $d = 2$ and $d = 4$, respectively.

Case $d = 2$. Let C_i denote the ith Catalan number. We'll use the well-known fact C_i enumerates the number of noncrossing partitions of $\{1, \ldots, i\}$ [9]. Thus, keeping the $d = 2$ picture of Figure 3 in mind, we have

$$\#\left\{\begin{matrix}\text{1-crossing partitions of } [n] \text{ invariant under} \\ \text{2-fold rotational symmetry.}\end{matrix}\right\} = \sum_{i\geq 0}\left(\frac{n}{2} - i - 1\right) C_i C_{\frac{n}{2}-2-i}.$$

One can sum this in a number of ways; using the Chu-Vandermonde identity [7] quickly does the job, however. Leaving the intermediate steps to the reader, we have

$$\begin{aligned}\sum_{i\geq 0}\frac{1}{i+1}\binom{2i}{i}\binom{2(n-i)}{n-i} &= \frac{(2n)!}{n!n!}\frac{\frac{-2n-1}{2}}{(-n-1)(-\frac{1}{2})}\left[{}_2F_1\left(\begin{matrix}-(n+1), & -\frac{1}{2} \\ & \frac{-2n-1}{2}\end{matrix}\right) - 1\right] \\ &= \frac{(2n)!}{n!n!}\frac{-2n-1}{n+1}[0-1] \\ &= \binom{2n+1}{n},\end{aligned}$$

and replacing n by $\frac{n}{2} - 2$ gives the claimed enumeration in this case.

Case $d = 4$. Looking at the $d = 4$ case of Figure 3, one can easily read off the enumeration. We simply have

$$\#\left\{\begin{matrix}\text{1-crossing partitions of } [n] \text{ invariant under} \\ \text{2-fold rotational symmetry.}\end{matrix}\right\} = \frac{n}{4}C_{\frac{n}{4}-1} = \binom{\frac{n-4}{2}}{\frac{n-4}{4}},$$

finishing the claimed enumeration for the RHS in the theorem.

For the LHS we break our work into 4 cases, depending on which of $2n-5$ and $n-4$ is divisible by d, and using Lemma 13 in each case.

Case 1: $d \mid 2n-5$ and $d \mid n-4$. Then $d=1$ since n and $n-1$ are coprime. With $d=1$, one then has

$$\begin{bmatrix} 2n-5 \\ n-4 \end{bmatrix}_{q=\omega} = \binom{2n-5}{n-4}.$$

Case 2: $d \mid 2n-5$ and $d \nmid n-4$. Then $d \neq 1, 2$ or 4. By Lemma 13, it is easy to see we have the LHS being 0.

Case 3: $d \nmid 2n-5$ and $d \mid n-4$. Then $d=2$ or 4, and by Lemma 13, we have

$$\begin{bmatrix} 2n-5 \\ n-4 \end{bmatrix}_{q=\omega} = \binom{\lfloor \frac{2n-5}{d} \rfloor}{\frac{n-4}{d}}.$$

This gives

$$\begin{bmatrix} 2n-5 \\ n-4 \end{bmatrix}_{q=\omega} = \begin{cases} \binom{n-3}{\frac{n}{2}-2} & \text{if } d=2; \\ \binom{\frac{n-4}{2}}{\frac{n-4}{4}} & \text{if } d=4. \end{cases}$$

Case 4: $d \nmid 2n-5$ and $d \nmid n-4$. Then $d \neq 1, 2$ or 4, and since $d \mid n$,

$$2n-5 \mod d = d-5 = n_1 \quad \text{and} \quad n-4 \mod d = d-4 = k_1.$$

Thus, $k_1 > n_1$, giving 0 here also.

One can then check these enumerations match up on both sides, finishing the proof. □

One also has the natural q-analogue of the refinement found in our main theorem exhibiting the cyclic sieving phenomenon!

Theorem 15.

$$\begin{bmatrix} n \\ k-2 \end{bmatrix} \begin{bmatrix} n-5 \\ k-3 \end{bmatrix}_{q=\omega} = \# \left\{ \begin{array}{l} \textit{1-crossing partitions of } [n] \textit{ with } k \textit{ blocks invariant} \\ \textit{under } d\textit{-fold rotational symmetry.} \end{array} \right\},$$

where ω is a primitive dth root of unity, and $d \mid n$.

Proof. Just as in Theorem 14, we simply evaluate both sides, and see they agree.

Case $d=1$. In this case, the LHS agrees with the RHS by Lemma 13 and Theorem 2.

Case $d=2$. Again, using the $d=2$ case of Figure 3, one can see the RHS is zero when $d \nmid k-2$. Certainly, the LHS matches this, since $n=0 \mod d$ while $k-2 \neq 0 \mod d$.

If $d \mid k-2$, the LHS becomes

$$\binom{\frac{n}{d}}{\frac{k-2}{d}} \binom{\lfloor \frac{n-5}{d} \rfloor}{\lfloor \frac{k-3}{d} \rfloor} = \binom{\frac{n}{2}}{\frac{k-2}{2}} \binom{\frac{n}{2}-3}{\frac{k-2}{2}-1}.$$

Similar reasoning as in the corresponding case above gives the RHS enumeration to be

$$\sum_{i,j \geq 0} \left(\frac{n}{2} - i - 1\right) N(i,j) N\left(\frac{n}{2} - 2 - i, \frac{k-2}{2} - j\right),$$

where here one uses the Narayana number $N(n,k)$ enumerates the number of noncrossing partitions of $[n]$ with exactly k blocks [9].

Case $d = 4$. Here, if $d \nmid k-2$, the RHS is obviously 0. On the other hand, the LHS is easily seen to agree by Lemma 13, since $d \mid n$.

If, however, we have $d \mid k-2$, the RHS is

$$\frac{n}{4} N\left(\frac{n}{4}-1, \frac{k-2}{4}\right).$$

On the other hand, with $d \mid k-2$, the LHS becomes

$$\binom{\frac{n}{4}}{\frac{k-2}{4}}\binom{\lfloor\frac{n-5}{4}\rfloor}{\lfloor\frac{k-3}{4}\rfloor}\begin{bmatrix}3\\3\end{bmatrix}_{q=\omega} = \binom{\frac{n}{4}}{\frac{k-2}{4}}\binom{\frac{n}{4}-2}{\frac{k-2}{4}-1} = \frac{n}{4}N\left(\frac{n}{4}-1, \frac{k-2}{4}\right).$$

Case $d \neq 1, 2$ *or* 4. As in Theorem 14, it is not hard to see the RHS evaluates to zero. Also, d cannot divide both $k-2$ and $k-3$. If $d \mid k-3$, then the LHS is also 0, since $n \mod d = 0$ but $k-2 \mod d \neq 0$. If, on the other hand, we have $d \mid k-2$, then the LHS evaluates to

$$\binom{\frac{n}{d}}{\frac{k-2}{d}}\begin{bmatrix}0\\0\end{bmatrix}\binom{\lfloor\frac{n-5}{d}\rfloor}{\lfloor\frac{k-3}{d}\rfloor}\begin{bmatrix}d-5\\d-1\end{bmatrix}_{q=\omega} = 0.$$

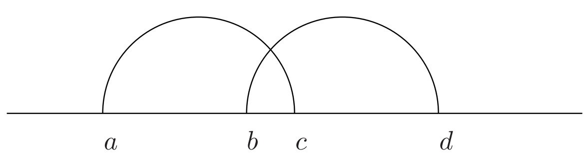
□

3. Enumeration of 1- and 2-crossing partitions via generating functions

We will now use generating functions to find expressions for the respective numbers of 1-crossing and 2-crossing partitions of $[n]$. A 1-crossing partition of $[n]$ has three parts:

- A crossing: $a < b < c < d$
- Noncrossing partitions of the three intervals enclosed by the crossing: $[a+1, b-1]$, $[b+1, c-1]$, and $[c+1, d-1]$
- A noncrossing partition of the "arc" $[d+1, n] \cup [1, a-1]$

Such a crossing looks like this:



Since elements on one side of the arc can share blocks with elements on the other side, the arc is not to be considered to be two separate intervals. Rather, it is to be treated as one interval, with the understanding that the only additional information needed is the number of elements that go on each side. If the arc is to contain m elements, we have $m+1$ choices for how the elements are to be partitioned into the arc's two halves; we can pick any number from 0 to m elements to place on the left side, and the remaining elements end up on the right. Once this is performed, we have C_m noncrossing partitions of the arc. Noncrossing partitions are enumerated by the Catalan numbers, which have generating function

$C(x) = \frac{1-\sqrt{1-4x}}{2x}$. Therefore, the generating function for the arc is

$$\begin{aligned}\sum_{m=0}^{\infty}(m+1)C_m x^m &= \frac{d}{dx}\sum_{m=0}^{\infty}C_m x^{m+1}\\ &= \frac{d}{dx}x\sum_{m=0}^{\infty}C_m x^m\\ &= \frac{d}{dx}[xC(x)].\end{aligned}$$

The generating functions for the other components of a 1-crossing partition are simpler. Since the noncrossing partitions of the three intervals subtended by the crossing are independent of one another, their generating function is $[C(x)]^3$. Finally, the generating function for the four elements that make up the crossing is simply x^4.

Hence the generating function $S_1(x)$ for the number of 1-crossing partitions of $[n]$ is the product of these three functions:

$$\begin{aligned}S_1(x) &= x^4[C(x)]^3\frac{d}{dx}[xC(x)]\\ &= x^4\left(\frac{1-\sqrt{1-4x}}{2x}\right)^3\frac{d}{dx}\left(\frac{1-\sqrt{1-4x}}{2}\right)\\ &= \frac{1}{2}x^4\frac{(1-\sqrt{1-4x})^3}{8x^3}\frac{d}{dx}(1-\sqrt{1-4x})\\ &= \frac{1}{2}\frac{x(1-\sqrt{1-4x})^3}{8}\frac{2}{\sqrt{1-4x}}\\ &= \frac{x(1-\sqrt{1-4x})^3}{8\sqrt{1-4x}}.\end{aligned}$$

Let us use the binomial theorem to rewrite $S_1(x)$:

$$\begin{aligned}S_1(x) &= \frac{x(1-\sqrt{1-4x})^3}{8\sqrt{1-4x}}\\ &= \frac{1}{8}x(1-\sqrt{1-4x})^3(\sqrt{1-4x})^{-1}\\ &= \frac{1}{8}x\sum_{j=0}^{3}\binom{3}{j}(-1)^j(\sqrt{1-4x})^{j-1}\\ &= \frac{1}{8}x\sum_{j=0}^{3}(-1)^j\binom{3}{j}\sum_{k=0}^{\infty}\binom{\frac{j-1}{2}}{k}(-4x)^k.\end{aligned}$$

where the last step used the "generalized" binomial theorem to expand

$$(\sqrt{1-4x})^{j-1} = (1-4x)^{\frac{j-1}{2}}.$$

Since we seek the coefficient of x^n in this series, we can neglect the factor of x that appears outside the sum, and then easily obtain the desired coefficient by substituting $n-1$ for k, yielding

$$\frac{1}{8}(-4)^{n-1}\sum_{j=0}^{3}(-1)^j\binom{3}{j}\binom{\frac{j-1}{2}}{n-1}.$$

Since this is the coefficient of x^n in the series expansion of $S_1(x)$, this is the number of 1-crossing partitions of $[n]$. However, the expression remains to be simplified.

Since we are restricting ourselves to values of n greater than or equal to 4, we can assume that the terms corresponding to $j = 1$ and $j = 3$ vanish, since for any of these admissible values of n, we have $\binom{0}{n-1}$ and $\binom{1}{n-1}$ both equal to zero. Hence, only two terms of the sum remain; the coefficient is now

$$\frac{1}{8}(-4)^{n-1}\left[\binom{-\frac{1}{2}}{n-1}+3\binom{\frac{1}{2}}{n-1}\right].$$

Judicious manipulation of factorials will now help us simplify our result further.

We have:

$$\begin{aligned}
&\frac{1}{8}(-4)^{n-1}\left[\binom{\frac{-1}{2}}{n-1}+3\binom{\frac{1}{2}}{n-1}\right]\\
= \quad & \frac{(-1)^{n-1}4^{n-1}}{8}\left[\frac{(\frac{-1}{2})(\frac{-3}{2})\cdots(\frac{-2n+3}{2})}{(n-1)!}+3\frac{(\frac{1}{2})(\frac{-1}{2})\cdots(\frac{-2n+5}{2})}{(n-1)!}\right]\\
= \quad & \frac{2^{n-1}}{8(n-1)!}\left[(1)(3)\cdots(2n-3)+3(-1)(1)\cdots(2n-5)\right]\\
= \quad & \frac{2^{n-1}}{8(n-1)!}\left[(1)(3)\cdots(2n-3)\left(1-\frac{3}{2n-3}\right)\right]\\
= \quad & \frac{2^{n-1}}{8(n-1)!}(1)(3)\cdots(2n-3)\left(\frac{2n-6}{2n-3}\right).
\end{aligned}$$

The product $(1)(3)\cdots(2n-3)$ is called a *semifactorial*; multiplying top and bottom by $2^{n-1}(n-1)!$ will convert it to a full-fledged factorial. Doing this yields the following (let us also write 8 as 2^3 for convenience):

$$\begin{aligned}
&\frac{2^{n-1}}{2^3(n-1)!}\frac{(2n-2)!}{(n-1)!2^{n-1}}\frac{2n-6}{2n-3}\\
= \quad & \frac{(2n-2)!(2n-6)}{2^3(n-1)!(n-1)!(2n-3)}\\
= \quad & \frac{(2n-5)!(2n-2)(2n-3)(2n-4)(2n-6)}{2^3(n-1)!(n-4)!(n-3)(n-2)(n-1)(2n-3)}\\
= \quad & \frac{(2n-5)!2^3(n-1)(n-2)(n-3)(2n-3)}{(n-1)!(n-4)!2^3(n-1)(n-2)(n-3)(2n-3)}\\
= \quad & \frac{(2n-5)!}{(n-1)!(n-4)!}\\
= \quad & \binom{2n-5}{n-4}.
\end{aligned}$$

So we see that 1-crossing partitions of $[n]$ are enumerated by a single binomial coefficient!

We now turn our attention to the enumeration of 2-crossing partitions of $[n]$. Here the problem is somewhat thornier. When we considered 1-crossing partitions, there was exactly one form they could take: two blocks containing two elements apiece constituted the crossing. However, with 2-crossing partitions, we have to consider more than one case.

It is clear that no fewer than five and no more than eight vertices can constitute the two crossings of such a partition. We consider two cases, broken down by the number of *adjacent* crossings that the partition contains:

- One adjacent crossing and one nonadjacent crossing
- Two adjacent crossings

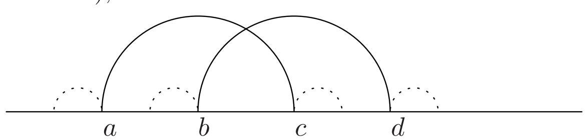
(Of course, "no adjacent crossings" is not a possibility, so this list is exhaustive)

We will call a partition's configuration of vertices that form crossings, noncrossing partitions of intervals, and noncrossing partitions of arcs the *trace type* of the partition. The possible trace types for 2-crossing partitions, described below, will each yield different generating functions; in the end, their sum will be our desired generating function for the overall case.

In the first case, the partition is formed by attaching a "tail" (a fifth vertex) to an adjacent crossing. Such a pair of crossings subtends four intervals and one arc. Using the same reasoning that was employed in the 1-crossing case, we find the following generating function for this first trace type:

$$4x^5[C(x)]^4\frac{d}{dx}[xC(x)],$$

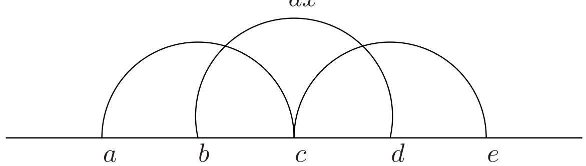
where the 4 appears because there are four places to attach the tail (the dotted lines in the figure below), and therefore four distinct subcases of the first case.



The second case is not disposed of so quickly. Five, six, or eight vertices can form the pair of adjacent crossings. We will list the trace types, following each with a diagram.

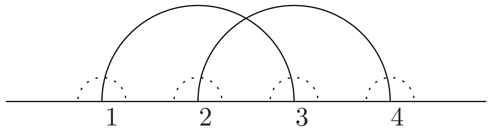
If five vertices are used, the crossings subtend four intervals and one arc, yielding the generating function

$$x^5[C(x)]^4\frac{d}{dx}[xC(x)].$$



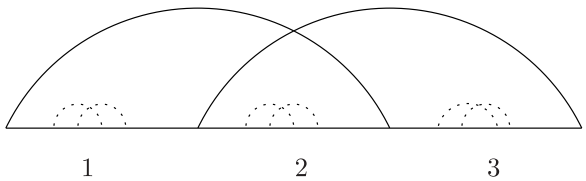
If six vertices are used, the crossings subtend five intervals and one arc. This can be done in three distinct ways, as depicted below; adding the dotted line at positions 1, 2, or 3 yield different types of partitions. (Adding the dotted line at position 4 produces, by symmetry, an identical trace type to the position 1 case, so we have three cases, not four.) This yields the generating function

$$3x^6[C(x)]^5\frac{d}{dx}[xC(x)].$$

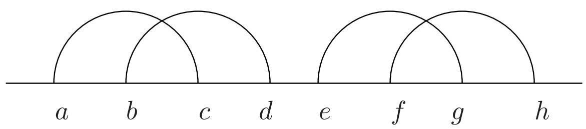


If eight vertices are used, then the two adjacent crossings do not share any vertices. Thus they either lie side-by-side, or are nested one in the other, as shown below:

(nested trace type)



(side-by-side trace type)



In the nested trace type, the crossings subtend five intervals and *two* separate arcs. Since there are three distinct ways to do this (by nesting the second crossing inside any of the three intervals, marked above by "1", "2" and "3", subtended by the first crossing) and two arcs (one inside the larger crossing, one outside), the generating function for this trace type is

$$3x^8[C(x)]^5\left\{\frac{d}{dx}[xC(x)]\right\}^2.$$

Finally, in the side-by-side trace type, the crossings subtend six intervals, but the "arc" in this case is not what we have become accustomed to. If the vertices are labeled a through h, the arc is a union of three, not two, disjoint intervals: $[1,a-1]\cup[d+1,e-1]\cup[h+1,n]$. However, once again, we can treat the arc as one large interval to be partitioned without crossings, after we have chosen how many vertices go where.

The number of ways to form a three-part arc with m elements is simply the number of compositions of m into 3 parts, which is $\binom{m+2}{2}$. And again, we have C_m noncrossing partitions of the arc after this division is performed. So the generating function for the arc becomes

$$\begin{aligned}
\sum_{m=0}^{\infty} \binom{m+2}{2} C_m x^m &= \frac{1}{2} \sum_{m=0}^{\infty} (m+1)(m+2) C_m x^m \\
&= \frac{1}{2} \sum_{m=0}^{\infty} \frac{d}{dx} (m+2) C_m x^{m+1} \\
&= \frac{1}{2} \sum_{m=0}^{\infty} \frac{d^2}{dx^2} C_m x^{m+2} \\
&= \frac{1}{2} \frac{d^2}{dx^2} x^2 \sum_{m=0}^{\infty} C_m x^m \\
&= \frac{1}{2} \frac{d^2}{dx^2} [x^2 C(x)]
\end{aligned}$$

yielding an overall generating function of

$$\frac{1}{2} x^8 (C(x))^6 \frac{d^2}{dx^2} [x^2 C(x)]$$

for the side-by-side case.

Hence the generating function for the number of 2-crossing partitions of $[n]$, or $S_2(x)$, is the sum of these generating functions:

$$\begin{aligned}
5x^5 [C(x)]^4 \frac{d}{dx} [xC(x)] &+ 3x^6 [C(x)]^5 \frac{d}{dx} [xC(x)] \\
&+ 3x^8 [C(x)]^5 \left\{ \frac{d}{dx} [xC(x)] \right\}^2 + \frac{1}{2} x^8 [C(x)]^6 \frac{d^2}{dx^2} [x^2 C(x)].
\end{aligned}$$

The coefficient of x^n in the power series expansion of $S_2(x)$ is the sum of the four respective coefficients of x^n in the power series expansions of its four summands. It is therefore easier to find the coefficient of each one separately, using methods identical to those employed in the 1-crossing case, and then add them to find the overall number of 2-crossing partitions of $[n]$.

The coefficient of x^n in the power series expansion of $5x^5 [C(x)]^4 \frac{d}{dx} [xC(x)]$ is

$$\frac{5}{4} \binom{2n-2}{n-1} \frac{(n-4)(n-3)}{(2n-3)(2n-5)} = 5 \binom{2n-6}{n-5}.$$

The coefficient of x^n in the power series expansion of

$$3x^6 [C(x)]^5 \frac{d}{dx} [xC(x)] \text{ is } \frac{3}{8} \binom{2n-2}{n-1} \frac{(n-5)(n-4)}{(2n-3)(2n-5)} = 3 \binom{2n-7}{n-6}. \tag{6}$$

The coefficient of x^n in the power series expansion of $3x^8 [C(x)]^5 \left\{ \frac{d}{dx} [xC(x)] \right\}^2$ is

$$6 \cdot 4^{n-6} - \frac{3}{8} \binom{2n-6}{n-3} \frac{5n^2 - 45n + 102}{(2n-7)(2n-9)}.$$

Finally, the coefficient of x^n in the power series expansion of $\frac{1}{2} x^8 [C(x)]^6 \frac{d^2}{dx^2} [x^2 C(x)]$ is

$$\frac{1}{16} \binom{2n-4}{n-2} \frac{n^4 - 3n^3 - 58n^2 + 366n - 612}{(2n-5)(2n-7)(2n-9)} - 6 \cdot 4^{n-6}.$$

Collating these terms yields the coefficient of x^n in $S_2(x)$, and therefore the number of 2-crossing partitions of $[n]$:

$$\binom{2n-2}{n-1}\frac{(n-1)(n-2)(n^5+189n^4-3091n^3+18255n^2-47274n+45360)}{8(2n-2)(2n-3)(2n-4)(2n-5)(2n-7)(2n-9)}.$$

4. The 2-crossings seem not to exhibit the sieving

With Theorem 14 in hand, there were high hopes that a similar property would hold for the different trace types of two crossings. In particular, we investigated q-analogues of the six vertex trace types. We expected a q-analogue such that when $q=\omega$, where ω is a primitive dth root of unity and $d|n$, it would count the number of two crossing partitions of $[n]$ with a given trace type invariant under d-fold rotational symmetry.

Let's focus on the 6-vertex trace type. We know that for $q=1$ a correct analogue should reduce to (6). When $q=-1$, the analogue should count the number of circular representations of partitions of $[n]$ with the six vertex trace type with 180° rotational symmetry. For all other dth roots of unity where $d|n$, a correct q-analogue should evaluate to zero since for the six vertex trace type there is only rotational symmetry by 180°. By examining the possible pictures of 180° rotational symmetry for small n (starting with $n=6$), one can verify that the correct values are

n	partitions with 180° rotational symmetry
6	3
7	0
8	4
9	0
10	15
11	0
12	36

Taking one possible q-analogue of the LHS of (6) gives

$$\frac{3}{8}\begin{bmatrix}2n-2\\ n-1\end{bmatrix}_q\frac{[n-5]_q[n-4]_q}{[2n-3]_q[2n-5]_q}. \tag{7}$$

This incorrectly gives zeroes for all $n\geq 6$ when $q=-1$. We then tried different possibilities by taking the q-analogues of the 3 and 8 out in front. However, this also led to incorrect values when $q=-1$ for various $n\geq 6$.

If one then looks at a q-analogue of the RHS of (6), one can also rewrite it as

$$3\begin{bmatrix}2n-7\\ n-6\end{bmatrix}_q. \tag{8}$$

The values for this and the expected values, for small n, are

n	q-analogue values with $q = -1$	values expected
6	3	3
7	3	0
8	12	4
9	15	0
10	45	15
11	63	0
12	168	36

A factor of three can be taken out by taking the q-analogue of the three to give the correct values for the $n = 8$ and $n = 10$ cases. This throws off the value for $n = 6$, but the value for $n = 12$ is more worrisome. This term is worrisome because there is a factor of 7 in the 168. There seems to be no way to get the 7 out in order to get the correct value of 36 for the $n = 12$ case and also get the correct value of 0 for the $n = 11$ case. This leads us to believe that a q-analogue taken along the same lines as (8) will not give a related property as Theorem 14 did. Thus if a q-analogue exists with the properties we are looking for, it would probably arise from taking a q-analogue along the lines of (7), but we could not find one.

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